A New Minimax Design for 2-D FIR Filters with Low Group Delay

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***Abstract*— A new minimax design for 2-D FIR filters with low group delay is proposed. The design method is based on sequential quadratic programming (SQP). The main reason to formulate and solve the design problem in a SQP formulation is that the complementarity conditions associated with the SQP lead to a very small number of nonzero Lagrange multipliers that need to be updated in a given iteration. This in turn improves design efficiency as well as the algorithm's numerical stability which is of critical importance as both the number of design variables and the constraints involved in a 2-D design are much higher than a 1-D design. Design examples with comparisons are presented to illustrate the effectiveness of the proposed method.**

I. Introduction

The design of optimal one-dimensional (1-D) linear-phase FIR filters in the minimax sense can be accomplished using the well-known Parks-McClellan algorithm [1] and its variants [2]. These algorithms, however, do not directly apply to the class of FIR filters with low (passband) group delay. In the two dimension (2-D) case, 2-D FIR filters (particularly with reduced passband group delay) are of use in image-processing related applications. The minimax design of 2-D linear-phase FIR filters were investigated in the literature [3][4][5]. For minimax FIR filters with low delay, in principle the problem can be formulated as constrained optimization problem, but there are technical difficulties that are more challenging than their 1-D counterparts. First, the number of design variables increases from $O(N)$ for the 1-D case to $O(N^2)$ for the 2-D case where $N-1$ denotes the filter order. Second, the number of constraints also increases significantly because the frequency baseband is now two-dimensional, thus the number of grid points involved are far more than that of the 1-D case.

In this paper, we present a new method for the minimax design of 2-D FIR filters with low delay based on sequential quadratic programming (SQP). The main reason that motivates us to use SQP techniques is that the complementarity conditions in an SQP-based design lead to a very small number of nonzero Lagrange multipliers compared to the number of constraints imposed in the design. As a result, numerical difficulties that would otherwise occur are largely eliminated and the amount of computation involved in a given iteration becomes rather moderate even for high-order filters. Simulations are presented to demonstrate the performance of the proposed algorithm.

II. Problem Formulation

Consider a 2-D FIR transfer function

$$H(z_1, z_2) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} h_{ij} z_1^{-i} z_2^{-j} \tag{1}$$

If we denote

$$\begin{aligned} H(\omega_1, \omega_2) &= H(e^{j\omega_1}, e^{j\omega_2}). \\ \boldsymbol{c}(\omega) &= [1 \quad \cos\omega \quad \cdots \quad \cos(N-1)\omega]^T, \\ \boldsymbol{s}(\omega) &= [0 \quad \sin\omega \quad \cdots \quad \sin(N-1)\omega]^T, \end{aligned}$$

and

$$\boldsymbol{H} = \{h_{ij}\}$$

then the frequency response of the 2-D FIR filter can be expressed as

$$\begin{aligned} H(\omega_1, \omega_2) = &[\boldsymbol{c}^T(\omega_1)\boldsymbol{H}\boldsymbol{c}(\omega_2) - \boldsymbol{s}^T(\omega_1)\boldsymbol{H}\boldsymbol{s}(\omega_2)] \\ &-j[\boldsymbol{s}^T(\omega_1)\boldsymbol{H}\boldsymbol{c}(\omega_2) + \boldsymbol{c}^T(\omega_1)\boldsymbol{H}\boldsymbol{s}(\omega_2)] \end{aligned}$$

To simplify the notation, we let $\boldsymbol{c}_i = \boldsymbol{c}(\omega_i)$ and $\boldsymbol{s}_i = \boldsymbol{s}(\omega_i)$ for $i = 1,\ 2$ and write

$$\begin{aligned} H(\omega_1, \omega_2) &= \mathrm{tr}[(\boldsymbol{c}_2\boldsymbol{c}_1^T - \boldsymbol{s}_2\boldsymbol{s}_1^T)\boldsymbol{H}] - j\mathrm{tr}[(\boldsymbol{c}_2\boldsymbol{s}_1^T + \boldsymbol{s}_2\boldsymbol{c}_1^T)\boldsymbol{H})] \\ &= \mathrm{tr}[\boldsymbol{P}(\omega_1, \omega_2)\boldsymbol{H}] - j\mathrm{tr}[\boldsymbol{Q}(\omega_1, \omega_2)\boldsymbol{H}] \end{aligned} \tag{2}$$

where $\mathrm{tr}(\cdot)$ denotes matrix trace, $\boldsymbol{P}(\omega_1, \omega_2) = \boldsymbol{c}_2\boldsymbol{c}_1^T - \boldsymbol{s}_2\boldsymbol{s}_1^T$, and $\boldsymbol{Q}(\omega_1, \omega_2) = \boldsymbol{c}_2\boldsymbol{s}_1^T + \boldsymbol{s}_2\boldsymbol{c}_1^T$. If we use $\boldsymbol{p}(\omega_1, \omega_2)$, $\boldsymbol{q}(\omega_1, \omega_2)$, and $\boldsymbol{h}$ to denote the column vectors generated by stacking the transposed row vectors of $\boldsymbol{P}(\omega_1, \omega_2)$, $\boldsymbol{Q}(\omega_1, \omega_2)$, and $\boldsymbol{H}$, respectively, then we have

$$H(\omega_1, \omega_2) = \boldsymbol{p}^T(\omega_1, \omega_2)\boldsymbol{h} - j\boldsymbol{q}^T(\omega_1, \omega_2)\boldsymbol{h} \tag{3}$$

We are interested in optimizing the transfer function $H(z_1, z_2)$ in (1) such that its frequency response best approximates a desired frequency response $H_d(\omega_1, \omega_2)$ in the minimax sense. This leads to the optimization problem

$$\underset{\boldsymbol{h}}{\text{minimize}} \ \underset{-\pi \le \omega_1, \omega_2 \le \pi}{\text{maximize}} \ |H(\omega_1, \omega_2) - H_d(\omega_1, \omega_2)| \tag{4}$$

which may be converted to the constrained optimization problem

$$\text{minimize} \quad \eta \tag{5a}$$

$$\begin{aligned} \text{subject to} \quad &[\boldsymbol{p}^T(\omega_1, \omega_2)\boldsymbol{h} - H_{dr}(\omega_1, \omega_2)]^2 \\ &+[\boldsymbol{q}^T(\omega_1, \omega_2)\boldsymbol{h} - H_{dj}(\omega_1, \omega_2)]^2 \le \eta \end{aligned} \tag{5b}$$

where $H_{dr}(\omega_1, \omega_2)$ and $-H_{dj}(\omega_1, \omega_2)$ are the real and imaginary parts of $H_d(\omega_1, \omega_2)$, i.e.,

$$H_d(\omega_1, \omega_2) = H_{dr}(\omega_1, \omega_2) - jH_{dj}(\omega_1, \omega_2) \tag{6}$$

Typically, the desired frequency response assumes the form

$$H_d(\omega_1, \omega_2) = A_d(\omega_1, \omega_2)e^{-jd(\omega_1+\omega_2)} \tag{7}$$

where $A_d(\omega_1, \omega_2)$ represents the desired amplitude response, and d is the desired group delay. For the design of FIR filter with low group delay, $d < (N-1)/2$. It follows from (6) and (7) that $H_{dr}(\omega_1, \omega_2) = A_d(\omega_1, \omega_2)\cos[d(\omega_1 + \omega_2)]$ and $H_{dj}(\omega_1, \omega_2) = A_d(\omega_1, \omega_2)\sin[d(\omega_1 + \omega_2)]$.

For feasible exercise of SQP algorithms, the constraints in (5b) are imposed on a dense grid of frequencies $\Omega_d = \{(\omega_{1i}, \omega_{2i}), i = 1, \ldots, K\} \subseteq \{(\omega_1, \omega_2), -\pi \le \omega_1, \omega_2 \le \pi\}$. In this way, the problem in (5) become

$$\begin{aligned} &\text{minimize} && \boldsymbol{e}^T\boldsymbol{x} && \text{(8a)}\\ &\text{subject to:} && a_i(\boldsymbol{x}) \ge 0 \text{ for } 1 \le i \le K && \text{(8b)} \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{e} &= [1 \ \ 0 \ \ \cdots \ \ 0]^T\\ \boldsymbol{x} &= [\eta \ \ \boldsymbol{h}^T]^T\\ a_i(\boldsymbol{x}) &= \eta - [\boldsymbol{p}^T(\omega_{1i}, \omega_{2i})\boldsymbol{h} - H_{dr}(\omega_{1i}, \omega_{2i})]^2\\ &\quad -[\boldsymbol{q}^T(\omega_{1i}, \omega_{2i})\boldsymbol{h} - H_{dj}(\omega_{1i}, \omega_{2i})]^2 \end{aligned}$$

III. An SQP-Based Design Algorithm

A. The KKT conditions and optimization problem

By defining the Lagrangian of (8) as

$$L(\boldsymbol{x}, \boldsymbol{\mu}) = \boldsymbol{e}^T\boldsymbol{x} - \sum_{i=1}^{K}\mu_i a_i(\boldsymbol{x})$$

where $\boldsymbol{\mu} = [\mu_1 \ \ \ldots \ \ \mu_K]$ collects the K Lagrange multipliers, the Karush-Kuhn-Tucker (KKT) conditions of (8) are given as follows [6]

$$\begin{aligned} \boldsymbol{e} - \sum_{i=1}^{K}\mu_i\nabla a_i(\boldsymbol{x}) &= 0 && \text{(9a)}\\ \boldsymbol{a}(\boldsymbol{x}) &\ge \boldsymbol{0} && \text{(9b)}\\ \boldsymbol{\mu} &\ge \boldsymbol{0} && \text{(9c)}\\ \mu_i a_i(\boldsymbol{x}) &= 0 \qquad 1 \le i \le K && \text{(9d)} \end{aligned}$$

where $\boldsymbol{a}(\boldsymbol{x}) = [a_1(\boldsymbol{x}) \cdots a_K(\boldsymbol{x})]^T$. The K conditions in (9d) are known as the *complementarity conditions* which in conjunction with (9b) imply that if $a_i(\boldsymbol{x}) > 0$, then the Lagrange multiplier μ_i must be zero. In the context of filter design, this means that if at the frequency $(\omega_{1i}, \omega_{2i})$ the approximation error of the current design is strictly smaller than the current bound η, then the associated Lagrange multiplier can be simply set to zero without computing it. An interesting observation made in our simulation studies is that the number of nonzero Lagrange multipliers, say $\hat{K}$, is typically only a small fraction of the total number of constraints K. As a result, the amount of computation involved in updating the vector $\boldsymbol{\mu}$ becomes insignificant even for high-order filters.

In the kth iteration of the SQP-based algorithm, the vector pair $\{\boldsymbol{x}_k, \boldsymbol{\mu}_k\}$ is updated to $\{\boldsymbol{x}_{k+1}, \boldsymbol{\mu}_{k+1}\} = \{\boldsymbol{x}_k + \boldsymbol{\delta}_x, \boldsymbol{\mu}_k + \boldsymbol{\delta}_\mu\}$ such that the KKT conditions in (9) are approximately satisfied up to the first order. The first-order approximation of (9) is found to be

$$\begin{aligned} \boldsymbol{Y}_k\boldsymbol{\delta}_x + \boldsymbol{e} - \boldsymbol{A}_k^T\boldsymbol{\mu}_{k+1} &= \boldsymbol{0} && \text{(10a)}\\ \boldsymbol{A}_k\boldsymbol{\delta}_x &\ge -\boldsymbol{a}_k && \text{(10b)}\\ \boldsymbol{\mu}_{k+1} &\ge \boldsymbol{0} && \text{(10c)}\\ (\boldsymbol{\mu}_{k+1})_i(\boldsymbol{A}_k\boldsymbol{\delta}_x + \boldsymbol{a}_k)_i &= 0 \qquad 1 \le i \le K && \text{(10d)} \end{aligned}$$

where $\boldsymbol{Y}_k = \nabla^2 L(\boldsymbol{x}_k, \boldsymbol{\mu}_k)$, and

$$\boldsymbol{A}_k = \begin{bmatrix} \nabla^T a_1(\boldsymbol{x}_k)\\ \vdots\\ \nabla^T a_K(\boldsymbol{x}_k) \end{bmatrix}_{K\times(N+1)} \tag{10e}$$

By the definition of $a_k(\boldsymbol{x})$ given in (8), we see that the ith row of $\boldsymbol{A}_k$ has the form $[1 \ \ 2\{H_{dr}\boldsymbol{p}^T + H_{dj}\boldsymbol{q}^T - (\boldsymbol{p}\boldsymbol{p}^T + \boldsymbol{q}\boldsymbol{q}^T)\boldsymbol{h}_k^T\}]$ where the frequency dependence of $\boldsymbol{p}$, $\boldsymbol{q}$, H_{dr}, and H_{dj} has been omitted. Also note that Eqs. (10a)–(10d) are the *exact* KKT conditions of the quadratic programming (QP) problem

$$\begin{aligned} &\text{minimize} && \frac{1}{2}\boldsymbol{\delta}^T\boldsymbol{Y}_k\boldsymbol{\delta} + \boldsymbol{\delta}^T\boldsymbol{e} && \text{(11a)}\\ &\text{subject to:} && \boldsymbol{A}_k\boldsymbol{\delta} \ge -\boldsymbol{a}_k && \text{(11b)} \end{aligned}$$

Now if we denote the solution of (11) by $\boldsymbol{\delta}_x$, then $\boldsymbol{x}_k$ can be updated to $x_{k+1} = x_k + \delta_x$. Next, the nonzero Langrange multiplies can be updated using (10a) as

$$\hat{\boldsymbol{\mu}}_{k+1} = (\boldsymbol{A}_{ak}\boldsymbol{A}_{ak}^T)^{-1}\boldsymbol{A}_{ak}(\boldsymbol{Y}_k\boldsymbol{\delta}_k + \boldsymbol{e}) \tag{12}$$

where the rows of $\boldsymbol{A}_{ak}$ are those of $\boldsymbol{A}_k$ satisfying $(\boldsymbol{A}_k\boldsymbol{\delta}_x + \boldsymbol{a}_k)_i = 0$. Once $\hat{\boldsymbol{\mu}}_{k+1}$ is calculated, $\boldsymbol{\mu}_{k+1}$ is obtained by inserting zeros into $\hat{\boldsymbol{\mu}}_{k+1}$ so that the zero components of $\boldsymbol{\mu}_{k+1}$ are in the place where the corresponding $(\boldsymbol{A}_k\boldsymbol{\delta}_x + \boldsymbol{a}_k)_k$ are nonzero. The fact that only a small number of constraints in (11) are active implies that the computation involved in (12) is very light and the QP problem in (11) can be solved efficiently using for example an active-set method [6].

The iteration continues until a convergence criterion in terms of $\|\boldsymbol{\delta}_k\|_2$ or the number of iterations performed is met. The optimal impulse response of the $N \times N$ 2-D FIR filter can then be readily constructed from the solution vector $\boldsymbol{x}^*$.

B. Convex relaxation of problem (11)

The optimization problem in (11) is not strictly convex. A strictly convex relaxation of (11) can be made by replacing $\boldsymbol{Y}_k$ in (11a) with a positive definite matrix, still denoted by $\boldsymbol{Y}_k$, as follows: with $\boldsymbol{Y}_0 = \boldsymbol{I}$, the Broyden-Fletcher-Goldfarb-Shanno

(BFGS) recursive formula [6] updates $\boldsymbol{Y}_k$ to

$$\begin{aligned} \boldsymbol{Y}_{k+1} &= \boldsymbol{Y}_k + \frac{\boldsymbol{\eta}_k \boldsymbol{\eta}_k^T}{\boldsymbol{\delta}_x^T \boldsymbol{\eta}_k} - \frac{\boldsymbol{\nu}_k \boldsymbol{\nu}_k^T}{\boldsymbol{\delta}_x^T \boldsymbol{\nu}_k} && \text{(13a)} \\ \boldsymbol{\nu}_k &= \boldsymbol{Y}_k \boldsymbol{\delta}_x && \text{(13b)} \\ \boldsymbol{\eta}_k &= \theta \boldsymbol{\gamma}_k + (1-\theta)\boldsymbol{\nu}_k && \text{(13c)} \\ \boldsymbol{\gamma}_k &= -(\boldsymbol{A}_{k+1} - \boldsymbol{A}_k)^T \boldsymbol{\mu}_{k+1} && \text{(13d)} \\ \theta &= \begin{cases} 1 & \text{if } \boldsymbol{\delta}_x^T(\boldsymbol{\gamma}_k - 0.2\boldsymbol{\nu}_k) \geq 0 \\ \frac{0.8\delta_x^T \nu_k}{\boldsymbol{\delta}_x^T(\boldsymbol{\nu}_k - \boldsymbol{\gamma}_k)} & \text{otherwise} \end{cases} && \text{(13e)} \end{aligned}$$

A desirable feature of the BFGS updates is that if $\boldsymbol{Y}_k$ is positive definite, then $\boldsymbol{Y}_{k+1}$ is guaranteed to be positive definite. Consequently, with $\boldsymbol{Y}_0 = \boldsymbol{I}$, every QP subproblem involved in the design process is a *convex* QP problem which can be solved using an interior-point or active set algorithm.

C. Line search

A further enhancement in the SQP algorithm described above is made by including a line search step in the algorithm that yields a positive scalar α_k to minimize a potential function $\Psi(\boldsymbol{x}_k + \alpha\boldsymbol{\delta}_x)$ over $\alpha \in [0,\ 1]$ where $\boldsymbol{\delta}_x$ is the solution of problem (11) with $\boldsymbol{Y}_k$ obtained using (13). The potential function adopted here assumes the form

$$\Psi(\boldsymbol{x}) = \boldsymbol{e}^T\boldsymbol{x} - \sum_{i=1}^{K} (\boldsymbol{\mu}_k)_i a_i(\boldsymbol{x})$$

Because $\boldsymbol{\mu}_k \geq \boldsymbol{0}$, minimizing $\Psi(\boldsymbol{x}_k + \alpha\boldsymbol{\delta}_x)$ helps reduce the objective function in (8a) and, in case some $a_i(\boldsymbol{x}_k + \boldsymbol{\delta}_x)$ fail to hold the constrains in (8b), reduce the degree of violation of these constraints. Therefore, the inclusion of a line search step turns out to be of great benefit at the cost of modest increase in the computational complexity. Since $\Psi(\boldsymbol{x}_k + \alpha\boldsymbol{d}_k)$ is a second-order polynomial in α, we can write $\Psi(\boldsymbol{x}_k + \alpha\boldsymbol{d}_k) = a_2^{(k)}\alpha^2 + a_1^{(k)}\alpha + a_0^{(k)}$ with $a_2^{(k)} > 0$ which achieves its unique minimum at $\alpha^* = -a_1^{(k)}/2a_2^{(k)}$. The value of α_k can then be determined as $\alpha_k = \min\{\alpha^*,\ 1\}$.

IV. Design examples

The proposed method was applied to design a circularly symmetric lowpass and a diamond-shaped lowpass 2-D FIR filters with low and approximately constant passband group delay.

Example 1 A minimax design of circularly symmetric lowpass 2-D FIR filter of size 27×27 with group delay $d = 11$, passband edge $\omega_p = 0.5\pi$, and stopband edge $\omega_a = 0.66\pi$ was performed using the proposed algorithm. Because of the symmetry of the filter, only the upper half of the baseband was involved in the optimization, where a total of 1479 grid points were placed uniformly in the region of interest. It took the algorithm 80 iterations to converge to a solution FIR filter whose impulse response, amplitude response, and group delay in the passband are depicted in Fig. 1a, b, and c, respectively. It is interesting to note that the average number of active constraints per iteration was $\hat{K} = 108$, about 7.3% of the total number of constrains $K = 1479$. The numerical evaluation of the filter designed can be described in terms of the maximum passband ripple $e_p = 0.0093$; minimum stopband attenuation $e_a = 40.9383$ dB, and maximum relative group delay deviation in the passband $e_{gd} = 0.0574$. For comparison purposes, the semidefinite programming (SDP)

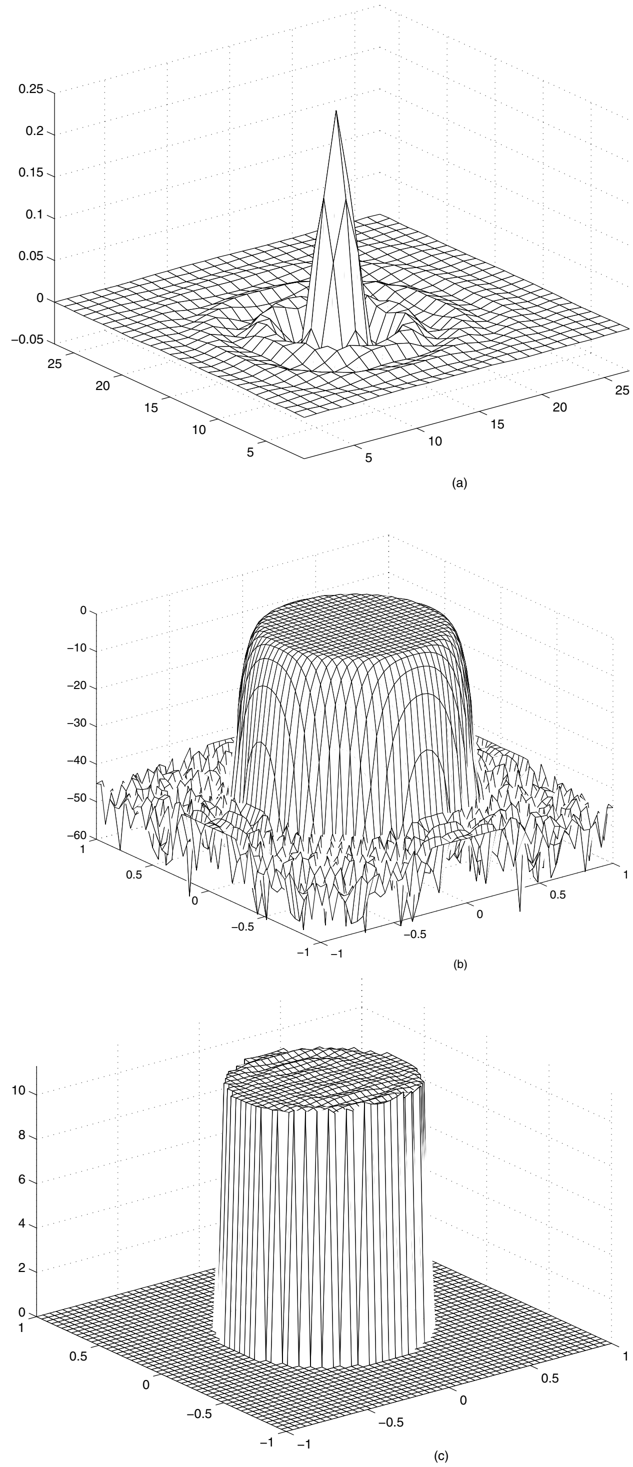

Figure 1: Circularly symmetric lowpass FIR filter of size 27×27 in Example 1: (a) impulse response, (b) amplitude response; and (c) group delay in passband.

based algorithm [7] and the recursive least pth algorithm [4] were also used to design the filter. As expected, the design methods were able to achieve practically the same (and optimal) design, but at the cost of higher computational complexity. The CPU time required by the proposed algorithm (normalized to unity), the algorithm in [7] and the algorithm in [4] were found to be 1.0, 1.2970, and 14.3091, respectively, and the actual CPU time required by the proposed algorithm implemented using MATLAB on a 3.1 GHz Pentium PC was 1045.6 seconds.

Example 2 A minimax design of diamond-shaped lowpass 2-D FIR filter of size 31×31 with group delay $d = 13$, passband edge $\omega_p = 0.8\pi$, and stopband edge $\omega_a = 0.96\pi$ was performed using the proposed algorithm. A total of 1524 grid points were placed in the upper half of the baseband. It took the algorithm 120 iterations to converge to a solution FIR filter with maximum passband ripple $e_p = 0.0107$, minimum stopband attenuation $e_a = 35.6299$ dB, and maximum relative group delay deviation $e_{gd} = 0.0731$. The average number of active constraints was found to be $\hat{K} = 199$, about 13% of the total number of constrains $\hat{K} = 1524$. The impulse response, amplitude response, and group delay in the passband are shown in Fig. 2a, b, and c, respectively. For comparisons, the CPU time consumed by the proposed algorithm (normalized to unity), the SDP-based algorithm [7] and the method of [4] were found to be 1.0, 1.3415, and 17.9177, respectively, and the actual CPU time required by the proposed algorithm was 2762.3 seconds.

V. Conclusion

We have proposed a new method for the design of minimax 2-D FIR filters based on SQP. Design examples presented in the paper have indicated that the method can be used to design relatively high order and nonlinear phase FIR filters that are optimal in the minimax sense.

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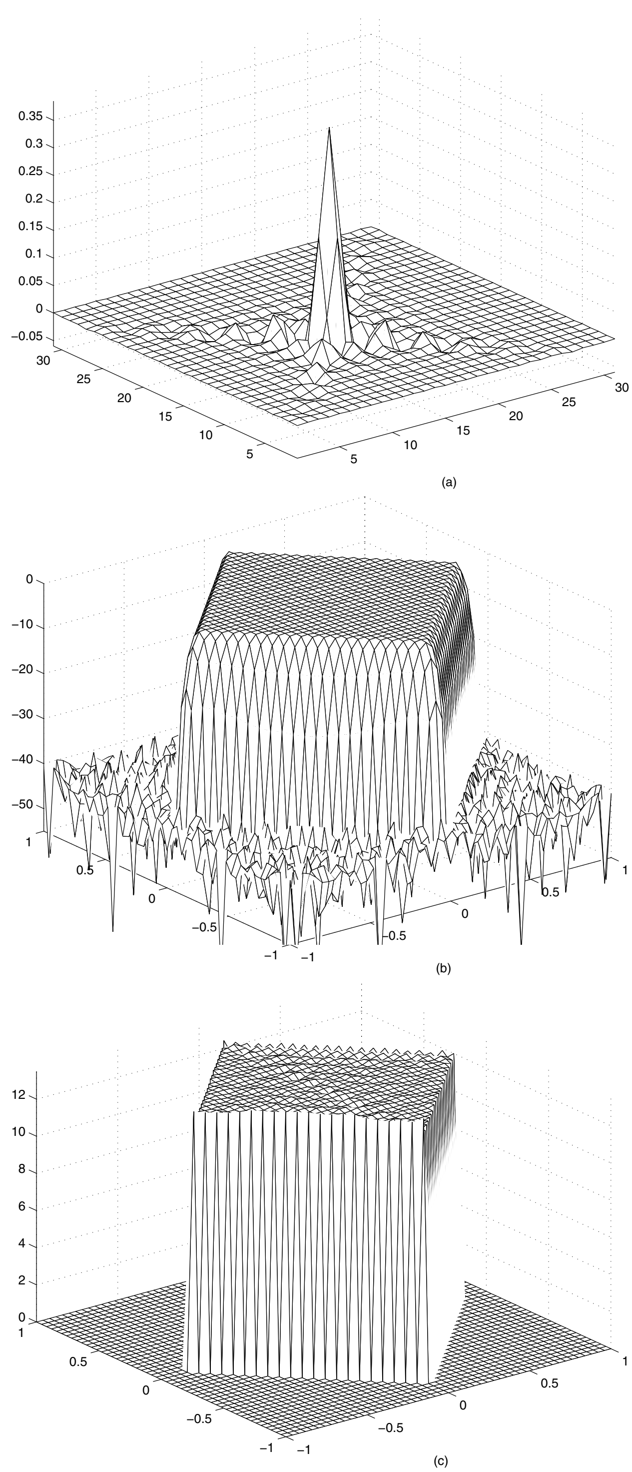

Figure 2: Diamond-shaped lowpass FIR filter of size 31×31 in Example 2: (a) impulse response, (b) amplitude response, and (c) group delay in passband.