Peak-to-Average Power-Ratio Reduction Algorithms for OFDM Systems via Constellation Extension

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Abstract— **New peak-to-average power-ratio (PAPR) reduction algorithms for orthogonal frequency-division multiplexing (OFDM) systems are investigated in a probabilistic framework. Specifically, de-randomization algorithms based on the Chernoff bound for PAPR reduction are developed by applying the so-called conditional probability method. Our simulations demonstrate that the proposed algorithms outperform several existing algorithms and the computational complexity of the proposed algorithms is found to be significantly less than that of existing algorithms.**

I. INTRODUCTION

Orthogonal frequency-division multiplexing (OFDM) has recently been used for data transmission in a number of communication systems [1][2]. A major problem of multicarrier modulation is its large peak-to-average power-ratio (PAPR) which makes system performance sensitive to distortion introduced by nonlinear devices, e.g., power amplifiers. In [3][4], a selective mapping (SLM) algorithm was proposed where a set of multicarrier signals are generated and the transmit signal with the lowest peak power is transmitted. This algorithm reduces the PAPR at the cost of an increase in computational complexity. In [5], a constellation extension technique for PAPR reduction was proposed where the subsymbol signs are determined by using a de-randomization algorithm. This algorithm reduces the PAPR at the cost of a slight reduction in bandwidth efficiency.

In this paper, PAPR reduction is investigated in a probabilistic framework [5]-[7]. New de-randomization algorithms are proposed based on a constellation extension technique. It is also shown that the PAPR-reduction performance can be further improved by combining the proposed algorithm with subset-by-subset optimization and selective rotation schemes. Design examples are presented which demonstrate that the solutions obtained by the proposed algorithms outperform solutions obtained by several existing algorithms [4][5].

II. PROBLEM FORMULATION

A. System Model

Consider an N-subcarrier OFDM transmitter as illustrated in Fig. 1, where $\mathrm{S/P}$, $\mathrm{P/S}$, and $\mathrm{D/A}$ represent serial-to-parallel, parallel-to-serial, and digital-to-analog converter, respectively, and the block labeled as "$\mathrm{Amp.}$" represents a power amplifier. Each of the subcarriers is independently modulated using phase-shift keying (PSK) or quadrature amplitude modulation (QAM).

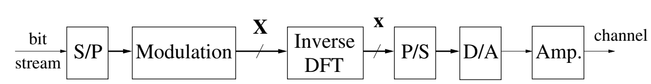

Fig. 1. Block diagram of a typical multicarrier transmitter.

B. Modulation Scheme

A $2M$-point modulation is adopted for each subcarrier. The modulation constellation is partitioned into two disjoint M-point groups $M^{(1)}$ and $M^{(2)}$. For a constellation point C in group $M^{(1)}$, point $-C$ falls into group $M^{(2)}$. For each subcarrier, the $\log_2 M$ input data bits can be mapped to either C or $-C$. Since C and $-C$ are in different groups, the data can be correctly recovered at the receiver without any side information. See Fig. 2 for the case of 16-QAM constellation.

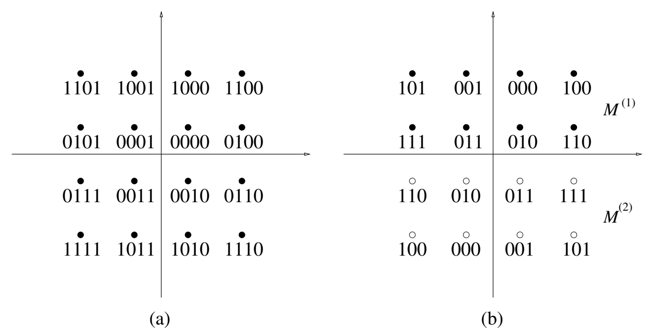

Fig. 2. (a) 16-QAM constellation with Gray code bit mapping. (b) Symmetric partition of 16-QAM constellation.

C. PAPR Reduction by Choosing Optimal Signs

The modulated symbol X_k is referred to as the *subsymbol* at the kth subcarrier, and the sign to be assigned to X_k is denoted as s_k where $s_k \in \{1, -1\}$. Vector $\boldsymbol{X} = [s_0 X_0, \ldots, s_{N-1} X_{N-1}]$ is referred to as the $OFDM\ symbol$, and $\boldsymbol{s} = [s_0 \ \cdots \ s_{N-1}]$ is referred to as the $sign\ vector$. The discrete complex baseband representation of the OFDM symbol can be described as

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} s_k X_k e^{\frac{j2\pi kn}{N}} \quad \text{for } n = 0, \ldots, N-1 \qquad (1)$$

where x_n represents the nth sample of the time-domain OFDM signal. If we let $\mathbf{x} = [x_0 \ \cdots \ x_{N-1}]^T$, the PAPR of signal $\mathbf{x}$ can be defined as

$$\mathrm{PAPR} = \frac{\|\mathbf{x}\|_\infty^2}{\mathcal{E}[\|\mathbf{x}\|_2^2]/N} \qquad (2)$$

where $\mathcal{E}[\cdot]$ denotes expectation, and $\|\mathbf{x}\|_\infty$ and $\|\mathbf{x}\|_2$ represent the infinity- and 2-norm of vector $\mathbf{x}$, respectively. Our objective is to obtain an optimum sign pattern, $\{s_k : s_k \in \{1, -1\}, k = 0, \ldots, N-1\}$, such that the PAPR of the OFDM symbol $\boldsymbol{X}$ is minimized. Note that the denominator in (2) is constant regardless of sign patterns. Using (1) and (2), the PAPR-minimization problem can be formulated as

$$\underset{\boldsymbol{s}}{\text{minimize}} \quad \max_{1\leq n\leq N} \left| \sum_{k=0}^{N-1} s_k X_k e^{\frac{j2\pi kn}{N}} \right| \tag{3a}$$

$$\text{subject to: } s_k \in \{1, -1\} \quad \text{for } k = 0, \ldots, N-1. \tag{3b}$$

In (3), variable X_k is complex valued. If we define

$$d_{nk} = \begin{cases} \text{Re}\left[X_k e^{\frac{j2\pi kn}{N}}\right] & 0 \leq n \leq N-1 \\ \text{Im}\left[X_k e^{\frac{j2\pi k(n-N)}{N}}\right] & N \leq n \leq 2N-1 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The problem in (3) can be relaxed to

$$\underset{\boldsymbol{s}}{\text{minimize}} \quad \max_{0\leq n\leq 2N-1} \left| \sum_{k=0}^{N-1} d_{nk} s_k \right| \tag{5a}$$

$$\text{subject to: } \; s_k \in \{1, -1\} \quad \text{for } k = 0, \ldots, N-1 \tag{5b}$$

where the variables involved are real valued. If $\tilde{\boldsymbol{s}}$ is the solution of the problem in (3) and $\tilde{n}$ is the index n at which the maximum of $\left|\sum_{k=0}^{N-1} e^{j2\pi kn/N} X_k \tilde{s}_k\right|$ is achieved, and $\boldsymbol{s}^\star$ is the solution of the problem in (5) and $n^\star$ is the index n at which the maximum of $\left|\sum_{k=0}^{N-1} d_{nkl} s_k^\star\right|$ is achieved, then it can be shown that

$$\left|\sum_{k=0}^{N-1} d_{n^\star k} s_k^\star\right| \leq \left|\sum_{k=0}^{N-1} e^{\frac{j2\pi kn}{N}} X_k \tilde{s}_k\right| \leq \sqrt{2} \left|\sum_{k=0}^{N-1} d_{n^\star k} s_k^\star\right|. \tag{6}$$

It follows from (6) that the solution of the problem in (5) can be regarded as a good approximation of that of the problem in (3).

III. New PAPR-Reduction Algorithm

A. Review of a Design Method for Sign Vector

The optimization problem in (5) is an integer programming problem which can be solved by using a derandomization method, known as the *method of conditional probabilities* (MCP) [5]-[7]. In this method, the components of the sign vectors $\boldsymbol{s}$ are treated as random variables which can assume values of 1 or -1 with equal probability. Let A_n^λ be the event that $\left|\sum_{k=0}^{N-1} d_{nk} s_k\right| \geq \lambda$ and $\Pr\left(A_n^\lambda\right)$ be the probability that event A_p^λ occurs and assume that λ is chosen such that $\sum_{n=0}^{2N-1} \Pr\left(A_n^\lambda\right) < 1$. By assuming that $s_0^\star = 1$, a suboptimal sign vector $\boldsymbol{s}^\star$ can be obtained sequentially as

$$s_j^\star = \arg\left[\min_{s_j\in\{1,-1\}} \sum_{n=0}^{2N-1} \Pr\left(A_n^\lambda | s_0^\star, \ldots, s_{j-1}^\star, s_j\right)\right] \tag{7}$$

for $j = 1, \ldots, N-1$. Consequently, we have

$$\sum_{n=0}^{2N-1} \Pr\left(A_n^\lambda | s_0^\star, \ldots, s_{N-1}^\star\right) \leq \sum_{n=0}^{2N-1} \Pr\left(A_n^\lambda\right) < 1. \tag{8}$$

With the entire vector $\boldsymbol{s}^\star$ known, the probability $\Pr\left(A_n^\lambda | s_0^\star, \ldots, s_{N-1}^\star\right)$ for each n is either zero or one. Hence, it can be inferred that the conditional probabilities in (8) are all zero. In other words, the sign vector $\boldsymbol{s}^\star = [s_0^\star \cdots s_{N-1}^\star]$ obtained using (7) can be regarded as a suboptimal solution at which the objective function in the problem in (5) is guaranteed to be smaller than λ. Because numerical evaluation of the conditional probabilities is often difficult, it is of use to derive an easy-to-evaluate upper bound of the conditional probability so that suboptimal solutions can be developed by working with the upper bound. For the problem in (5), there is a type of upper bound $U_p^\lambda\left(s_0, \ldots, s_j\right)$ known as *pessimistic estimator* [6] for the conditional probability, which can be characterized by

$$\Pr\left(A_n^\lambda | s_0, \ldots, s_j\right) \quad \leq \quad U_n^\lambda\left(s_0, \ldots, s_j\right) \tag{9a}$$

for $j = 0, \ldots, N-1$, where the following conditions are satisfied

$$\min_{s_j\in\{1,-1\}} \sum_{n=0}^{2N-1} U_n^\lambda\left(s_0, \ldots, s_j\right) \leq \sum_{n=0}^{2N-1} U_n^\lambda\left(s_0, \ldots, s_{j-1}\right) \tag{9b}$$

and

$$\sum_{p=0}^{2N-1} U_n^\lambda < 1. \tag{9c}$$

In (9c), U_p^λ denotes the upper bound of $\Pr\left(A_n^\lambda\right)$ with all the components of $\mathbf{s}$ treated as random variables. Replace the conditional probability by the pessimistic estimator $U_n^\lambda\left(s_0^\star, \ldots, s_{j-1}^\star, s_j\right)$ in (7), a suboptimal sign vector $\boldsymbol{s}^\star$ can be determined sequentially as

$$s_j^\star = \arg\left[\min_{s_j\in\{1,-1\}} \sum_{n=0}^{2N-1} U_n^\lambda\left(s_0^\star, \ldots, s_{j-1}^\star, s_j\right)\right] \tag{10}$$

for $j = 1, \ldots, N-1$. In [5], the Chernoff bound [8] on the conditional probability is applied to derive a pessimistic estimator as

$$\begin{aligned} &U_n^{\lambda^\star}\left(s_0, \ldots, s_j\right) \\ &= 2e^{-\gamma^\star\lambda^\star} \cosh\left(\gamma^\star \sum_{k=0}^{j} d_{nk} s_k\right) \prod_{k=j+1}^{N-1} \cosh\left(\gamma^\star d_{nk}\right) \end{aligned} \tag{11}$$

for $j = 0, \ldots, N-1$, where $\lambda^\star = \sqrt{2\varepsilon \log(4N)}$, $\gamma^\star = \lambda^\star/\varepsilon$, and

$$\varepsilon = \max_{0\leq p\leq 2N-1} \sum_{k=0}^{N-1} d_{nk}^2.$$

Eq. (11) leads to a suboptimal solution $\boldsymbol{s}^\star$ for the problem in (5) as

$$\begin{aligned} s_j^\star \quad = \quad -\text{sign}&\left[\sum_{n=0}^{2N-1} \sinh\left(\gamma^\star \sum_{k=0}^{j-1} s_l^\star d_{nk}\right) \sinh\left(\gamma^\star d_{nj}\right)\right. \\ &\left.\cdot \prod_{k=j+1}^{N-1} \cosh\left(\gamma^\star d_{nk}\right)\right] \quad \text{for } j = 1, \ldots, N-1. \end{aligned} \tag{12}$$

B. Generalization of the De-Randomization Algorithm

For the algorithm in [5], the constellation extension is applied on all subcarriers. Assume that for each subcarrier the size of modulation constellation is 2^q. By using the constellation extension technique, the effective data transmission rate is reduced by a factor of $1/q$. For example, if 16-QAM is adopted for each subcarrier, then the effective data transmission rate is reduced by 25%. On the other hand, by switching the sign of subsymbol X_k, we actually insert a sine signal with frequency proportional to k/N. Obviously, a sine signal with higher frequency has more peaks within an OFDM symbol duration and thus will be more helpful for PAPR reduction. Therefore, it is of interest to examine the performance of the system for the case where the constellation extension is applied to only some of the subcarriers in the high frequency range. For example, if the constellation extension is applied to the subcarriers with indices from N_h to $N-1$, then the problem at hand can be formulated as

$$\underset{\boldsymbol{s}}{\text{minimize}} \quad \max_{0\leq n\leq 2N-1} \left| \sum_{k=0}^{N_h-1} d_{nk} + \sum_{k=N_h}^{N-1} d_{nk} s_k \right| \tag{13a}$$

$$\text{subject to: } \; s_k \in \{1, -1\} \quad \text{for } k = N_h, \ldots, N-1. \tag{13b}$$

Let B_n^λ be the event that $\left|\sum_{k=0}^{N_h-1} d_{nk} + \sum_{k=N_h}^{N-1} d_{nk}s_k\right| \geq \lambda$ and $\Pr\left(B_n^\lambda\right)$ be the probability that event B_n^λ occurs. A pessimistic estimator of the probability $\Pr\left(B_n^\lambda | s_{N_h}, \ldots, s_j\right)$ can be derived as

$$U_n^{\lambda^\star}\left(s_{N_h}, \ldots, s_j\right) = 2e^{-\gamma^\star\lambda^\star}\cosh\left(\gamma^\star\sum_{k=0}^{N_h-1} d_{nk} + \gamma^\star\sum_{k=N_h}^{j} d_{nk}s_k\right)\cdot\prod_{k=j+1}^{N-1}\cosh\left(\gamma^\star d_{nk}\right) \quad (14)$$

for $j = N_h, \ldots, N-1$, where $\lambda^\star = \sqrt{2\varepsilon\log(4N)}$, $\gamma^\star = \lambda^\star/\varepsilon$, and

$$\varepsilon = \max_{0\leq n\leq 2N-1}\left[\sum_{k=N_h}^{N-1} d_{nk}^2 + \left(\sum_{k=0}^{N_h-1} d_{nk}\right)^2\right].$$

In this case, a suboptimal sign vector $\boldsymbol{s}^\star$ can be determined sequentially as

$$s_j^\star = \arg\left[\min_{s_j\in\{1,-1\}}\sum_{n=0}^{2N-1} U_n^{\lambda^\star}\left(s_{N_h}^\star, \ldots, s_{j-1}^\star, s_j\right)\right] \quad (15)$$

for $j = N_h, \ldots, N-1$. Note that the algorithm in [5] can be viewed as a special case of the proposed algorithm with $N_h = 0$.

C. Performance Improvement by Selective Rotations

The objective function in (14a) remains unchanged if all complex-valued terms are rotated by an angle θ because

$$\begin{aligned}&\left|\sum_{k=0}^{N_h-1} X_k e^{\frac{j2\pi kn}{N}} + \sum_{k=N_h}^{N-1} s_k X_k e^{\frac{j2\pi kn}{N}}\right| \\ =\ &\left|e^{j\theta}\left(\sum_{k=0}^{N_h-1} X_k e^{\frac{j2\pi kn}{N}} + \sum_{k=N_h}^{N-1} s_k X_k e^{\frac{j2\pi kn}{N}}\right)\right|. \quad (16)\end{aligned}$$

On the other hand, this rotation leads to a different set of d_{kl} which can be obtained using (4) with X_k replaced by $e^{j\theta}X_k$. If we use the parameters d_{nk} generated by K different rotation angles $\theta_0, \theta_1, \ldots, \theta_{K-1}$ with $\theta_0 = 0$, then we can obtain K suboptimal sign vectors $\boldsymbol{s}_0^\star, \boldsymbol{s}_1^\star, \ldots, \boldsymbol{s}_{K-1}^\star$ from which the best sign vector can be identified by comparing the performance of the corresponding suboptimal solutions. Since the set of rotation angles includes $\theta = 0$, the suboptimal solution obtained using this technique is always superior to the suboptimal solution described in Sec. III.B.

D. Performance Improvement by Subset-by-Subset Optimization of Sign Vector

For the proposed algorithm in Sec. III.B, the sign vector is optimized *sequentially*. This algorithm is fast because only one sign is optimized at any given time and, as indicated by Eq. (15), the computation involved is not heavy. If additional computational resources are available, improved performance can be achieved by replacing the sequential optimization by a *subset-by-subset* (SBS) optimization, where the components of the sign vector are partitioned into r subsets with each subset containing K_s sign components. The components of each subset are optimized *simultaneously* and the entire sign vector is optimized sequentially in a subset by subset manner. More specifically, we have

$$\begin{aligned}&\{s_j^\star \ \cdots\ s_{j+K_s-1}^\star\} \\ &= \arg\left[\min_{s_j,\ldots,s_{j+K_s-1}\in\{1,-1\}}\sum_{n=0}^{2N-1}\cosh\left(\gamma^\star\sum_{k=0}^{N_h} d_{nk} + \gamma^\star\right.\right. \\ &\left.\left.\cdot\sum_{k=N_h}^{j-1} d_{nk}s_k^\star + \gamma^\star\sum_{k=j}^{j+K_s-1} d_{nk}s_k\right)\prod_{k=j+K_s}^{N}\cosh\left(\gamma^\star d_{nk}\right)\right] \quad (17)\end{aligned}$$

for $j = N_h, \ldots, N-1$, where $\lambda^\star = \sqrt{2\varepsilon\log(4N)}$, $\gamma^\star = \lambda^\star/\varepsilon$, and

$$\varepsilon = \max_{0\leq n\leq 2N-1}\left[\sum_{k=N_h}^{N-1} d_{nk}^2 + \left(\sum_{k=0}^{N_h-1} d_{nk}\right)^2\right].$$

Evidently, the performance of the algorithm with SBS optimization is always better than that of the algorithm with sequential-optimization. The simulations presented in the next section will demonstrate that the improvement can be significant even with a moderate K_s, say $K_s = 3$.

IV. Simulations

The proposed derandomization algorithm was applied to an OFDM system with $N = 64$ subcarriers and its PAPR-reduction performance was evaluated and compared with that of the algorithms proposed in [4][5]. A commonly used performance measure for PAPR-reduction algorithms is the clipping probability (CP) which is defined as the probability that the PAPR of the multicarrier signal exceeds a given PAPR threshold PAPR_0. The average transmission power used to calculate the PAPR was obtained based on the 16-QAM constellation with each constellation point assumed to have the same probability. Since large peaks of the analog signal may occur after the D/A conversion, oversampling was applied to approximate the analog signal. In our simulations, all algorithms were applied using signals that were oversampled by a factor of 2, the sampling rate was then increased to 8 times the Nyquist sampling rate by a root-raised cosine filter with a rolloff factor of 0.12. For the SLM algorithm [4], the number of candidate sequences is denoted as U. For the proposed derandomization algorithm, N_h was set to 32, and the algorithm in [5] was considered as a special case of the proposed algorithm with $N_h = 0$. In the case where the derandomization algorithm incorporates the selective rotation scheme, the number of rotations is denoted as K_r and the rotation angles θ assume the values $\theta = 0, \pi/K_r, \ldots, (K_r-2)\pi/K_r, (K_r-1)\pi/K_r$. In the case where the derandomization algorithm incorporates the SBS optimization scheme, the number of components of the sign vector which were simultaneously optimized is denoted as K_s.

Example: A 16-QAM constellation was adopted for each subcarrier. For the derandomization algorithm with $N_h = 32$ and $N_h = 0$, the CPs versus various PAPR threshold values are plotted as the solid and dashed curves in Fig. 3, respectively. For the sake of comparison, the CPs obtained using the SLM algorithm and for the original OFDM signal are plotted in the same figure as dot-dashed curves. It can be observed that, by incorporating the selective rotation, a significant performance improvement can be achieved by the derandomization algorithm over the SLM algorithm. For example, for a CP of 10^{-3}, a 0.75-dB improvement can be obtained by the derandomization algorithm with $N_h = 0$ and $K_r = 4$ compared with that of the SLM algorithm with $U = 16$. On the other hand, it can be observed from Fig. 4 that for the derandomization algorithm with $N_h = 32$, significant performance improvement can be achieved by using the

TABLE I: Performance and Computational Complexity of various PAPR-Reduction Algorithms									
Algorithms	SLM Algorithm	Derandomization Algorithm ($N_h = 32$)				Derandomization Algorithm ($N_h = 0$)			
		$K_r = 1$		$K_r = 4$		$K_r = 1$		$K_r = 4$	
	$U = 16$	$K_s = 1$	$K_s = 3$	$K_s = 1$	$K_s = 3$	$K_s = 1$	$K_s = 3$	$K_s = 1$	$K_s = 3$
Performance Gain (dB)	3.45	2.8	3.1	3.45	3.8	3	3.1	4.2	4.25
Normalized CPU Time	1	0.028	0.056	0.12	0.23	0.085	0.0175	0.33	0.7
Effective Data Rate	100%	87.5%				75%			

SBS optimization over sequential optimization. For example, for the derandomization algorithm, for a CP of 10^{-3} a 0.3-dB improvement can be obtained by using SBS optimization with $K_s = 3$ compared with sequential optimization.

The computational complexity of the proposed algorithm is compared with those of the algorithms in [4][5] and the results obtained are given in Tables I where the performance of each algorithm is quantified in terms of its PAPR-reduction improvement in dB over the original data for a clipping probability of 10^{-3}. The computational complexity of the algorithms is measured in terms of the ratio of the CPU time required for each algorithm to that of the SLM algorithm with $U = 16$, for which the CPU time was normalized to unity. The effective data transmission rate for each algorithm is also included.

V. Conclusions

A new PAPR-reduction algorithm has been proposed based on a derandomization method. The performance of the proposed algorithm can be siginificantly improved by incorporating selective rotation and SBS optimization schemes. Simulations have demonstrated that the proposed algorithm outperforms the algorithms in [4] and [5]. It has also been shown that a tradeoff is inherent in the proposed algorithm between effective data rate, computational complexity, and PAPR-reduction performance.

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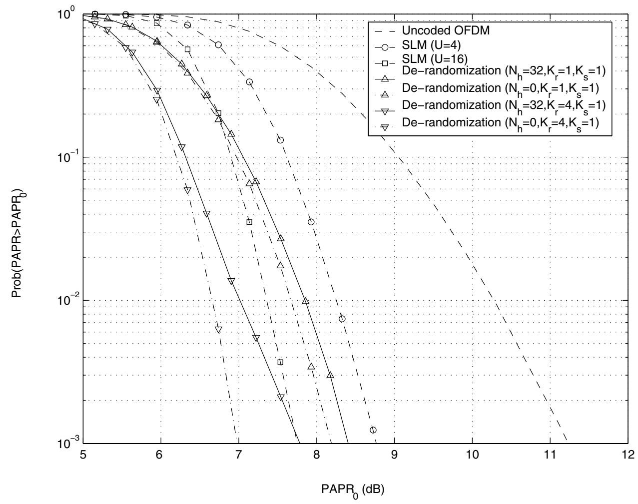

Fig. 3. Performance comparison of various PAPR-reduction algorithms.

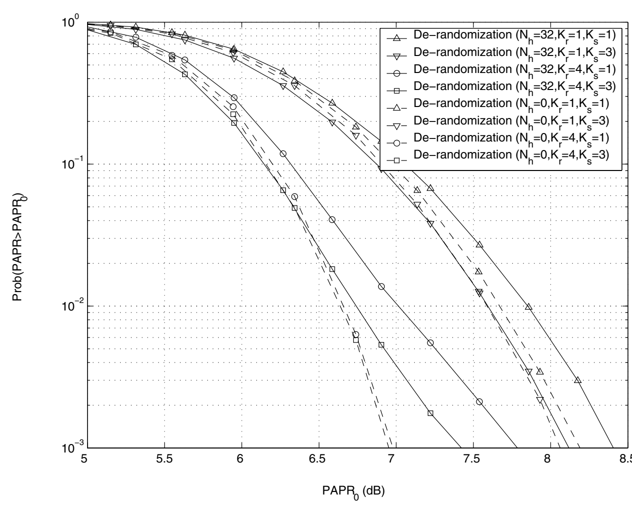

Fig. 4. Performance comparison of PAPR-reduction algorithms using SBS and sequential optimization.