STATE-SPACE DIGITAL FILTERS WITH MINIMUM L_2-SENSITIVITY SUBJECT TO L_2-SCALING CONSTRAINTS

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ABSTRACT

The problem of minimizing an L_2-sensitivity measure subject to L_2-norm dynamic-range scaling constraints in state-space digital filters is considered. A novel iterative technique is developed to solve the constraint optimization problem directly. This relies on a Lagrange function and some matrix-theoretic techniques. Computer simulation results are also given to demonstrate the effectiveness of the proposed technique.

I. INTRODUCTION

In the implementation of fixed-point state-space digital filters with finite word length (FWL), the efficiency and performance of the filter are directly affected by the choice of its state-space filter structure. If a transfer function satisfying specification requirements is designed with infinite accuracy coefficients and realized by a state-space model, the coefficients in the state-space model must be truncated or rounded to fit the FWL constraints. The characteristics of the filter is then altered due to the coefficient quantization, which may turn a stable filter into an unstable one. Therefore, the problem of minimizing the coefficient sensitivity of a digital filter is a significant research topic. Several techniques have been proposed for synthesizing state-space digital filter structures that minimize the coefficient sensitivity. These can be divided into two main classes: the L_1/L_2-sensitivity minimization [1]-[5] and the L_2-sensitivity minimization [6]-[11]. It is noted that the sensitivity measure based on the L_2 norm is more natural and reasonable relative to the L_1/L_2-sensitivity measure. It is well known that applying the L_2-scaling constraints to a state-space digital filter is beneficial for suppressing overflow oscillation [12],[13]. However, not enough research has been done on the minimization of the L_2-sensitivity subject to the L_2-norm dynamic-range scaling constraints [11].

In this paper, the problem of minimizing the L_2-sensitivity measure subject to L_2-norm dynamic-range scaling constraints is investigated for state-space digital filters. To this end, an expression for evaluating the L_2-sensitivity is introduced. An L_2-sensitivity minimization problem subject to the scaling constraints is formulated. An iterative algorithm is then developed to solve the constraint optimization problem directly. Unlike the work reported in [11], the proposed iterative technique relies on neither converting the problem into an unconstrained optimization one nor using a quasi-Newton algorithm. From computer simulation results, it has turned out that the proposed iterative technique requires less than half amount of computations to attain almost the same convergence accuracy as compared to the technique reported in [11].

II. L_2-SENSITIVITY ANALYSIS

Consider a state-space digital filter $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)_n$ which is stable, controllable and observable

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}u(k) \\ y(k) &= \boldsymbol{c}\boldsymbol{x}(k) + du(k) \end{aligned} \tag{1}$$

where $\boldsymbol{x}(k)$ is an $n \times 1$ state-variable vector, $u(k)$ is a scalar input, $y(k)$ is a scalar output, and $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$ and d are real constant matrices of appropriate dimensions. The transfer function of (1) is given by

$$H(z) = \boldsymbol{c}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b} + d. \tag{2}$$

Definition 1: Let $\boldsymbol{X}$ be an $m \times n$ real matrix and let $f(\boldsymbol{X})$ be a scalar complex function of $\boldsymbol{X}$, differentiable with respect to all the entries of $\boldsymbol{X}$. The sensitivity function of f with respect to $\boldsymbol{X}$ is then defined as

$$\boldsymbol{S}_{\boldsymbol{X}} = \frac{\partial f}{\partial \boldsymbol{X}}, \qquad (\boldsymbol{S}_{\boldsymbol{X}})_{ij} = \frac{\partial f}{\partial x_{ij}} \tag{3}$$

where x_{ij} denotes the (i, j)th entry of matrix $\boldsymbol{X}$.

Definition 2: Let $\boldsymbol{X}(z)$ be an $m \times n$ complex matrix-valued function of a complex variable z and let $x_{pq}(z)$ be the (p,q)th entry of $\boldsymbol{X}(z)$. The L_2-norm of $\boldsymbol{X}(z)$ is then defined as

$$\|\boldsymbol{X}(z)\|_2 = \left(\mathrm{tr}\left[\frac{1}{2\pi j} \oint_{|z|=1} \boldsymbol{X}(z)\boldsymbol{X}^*(z)\frac{dz}{z} \right] \right)^{\frac{1}{2}}. \quad (4)$$

From (2) and *Definitions 1* and *2*, the overall L_2-sensitivity measure for the state-space digital filter in (1) is defined as

$$\begin{aligned} S &= \left\| \frac{\partial H(z)}{\partial \boldsymbol{A}} \right\|_2^2 + \left\| \frac{\partial H(z)}{\partial \boldsymbol{b}} \right\|_2^2 + \left\| \frac{\partial H(z)}{\partial \boldsymbol{c}^T} \right\|_2^2 \\ &= \left\| [\boldsymbol{F}(z)\boldsymbol{G}(z)]^T \right\|_2^2 + \left\| \boldsymbol{G}^T(z) \right\|_2^2 + \|\boldsymbol{F}(z)\|_2^2 \end{aligned} \quad (5)$$

where

$$\boldsymbol{F}(z) = (z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b}, \qquad \boldsymbol{G}(z) = \boldsymbol{c}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}.$$

The term d in (2) and the sensitivity with respect to it are coordinate-independent and therefore they are neglected here.

It is easy to show that the L_2-sensitivity measure in (5) can be expressed as

$$S = \mathrm{tr}[\boldsymbol{M}(\boldsymbol{I}_n)] + \mathrm{tr}[\boldsymbol{W}_o] + \mathrm{tr}[\boldsymbol{K}_c] \quad (6)$$

where

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{P}) &= \frac{1}{2\pi j} \oint_{|z|=1} [\boldsymbol{F}(z)\boldsymbol{G}(z)]^T \boldsymbol{P}^{-1}\boldsymbol{F}(z^{-1})\boldsymbol{G}(z^{-1})\frac{dz}{z} \\ \boldsymbol{K}_c &= \frac{1}{2\pi j} \oint_{|z|=1} \boldsymbol{F}(z)\boldsymbol{F}^T(z^{-1})\frac{dz}{z} \\ \boldsymbol{W}_o &= \frac{1}{2\pi j} \oint_{|z|=1} \boldsymbol{G}^T(z)\boldsymbol{G}(z^{-1})\frac{dz}{z}. \end{aligned}$$

The matrices $\boldsymbol{K}_c$ and $\boldsymbol{W}_o$ are called the controllability and observability Gramians, respectively. The Gramians $\boldsymbol{M}(\boldsymbol{P})$ with $\boldsymbol{P} = \boldsymbol{I}_n$, $\boldsymbol{K}_c$ and $\boldsymbol{W}_o$ can be obtained by solving the Lyapunov equations [14]

$$\begin{aligned} \begin{bmatrix} * & * \\ * & \boldsymbol{M}(\boldsymbol{P}) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A} & \boldsymbol{bc} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}^T \begin{bmatrix} * & * \\ * & \boldsymbol{M}(\boldsymbol{P}) \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \boldsymbol{A} & \boldsymbol{bc} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix} + \begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \\ \boldsymbol{K}_c &= \boldsymbol{A}\boldsymbol{K}_c\boldsymbol{A}^T + \boldsymbol{b}\boldsymbol{b}^T \\ \boldsymbol{W}_o &= \boldsymbol{A}^T\boldsymbol{W}_o\boldsymbol{A} + \boldsymbol{c}^T\boldsymbol{c}. \end{aligned} \quad (7)$$

If a coordinate transformation defined by

$$\overline{\boldsymbol{x}}(k) = \boldsymbol{T}^{-1}\boldsymbol{x}(k) \quad (8)$$

is applied to the state-space model (1), then the new realization $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{c}}, d)_n$ can be characterized by

$$\overline{\boldsymbol{A}} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}, \qquad \overline{\boldsymbol{b}} = \boldsymbol{T}^{-1}\boldsymbol{b}, \qquad \overline{\boldsymbol{c}} = \boldsymbol{c}\boldsymbol{T}. \quad (9)$$

From (2) and (9), it is clear that the transfer function $H(z)$ is invariant under the coordinate transformation in (8). The coordinate transformation defined by (8) changes (6) to

$$S(\boldsymbol{P}) = \mathrm{tr}[\boldsymbol{M}(\boldsymbol{P})\boldsymbol{P}] + \mathrm{tr}[\boldsymbol{W}_o\boldsymbol{P}] + \mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] \quad (10)$$

where $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{T}^T$.

Moreover, if the L_2-norm dynamic-range scaling constraints are imposed on the new state-variable vector $\overline{\boldsymbol{x}}(k)$, it is required that for $i = 1, 2, \cdots, n$

$$(\overline{\boldsymbol{K}}_c)_{ii} = (\boldsymbol{T}^{-1}\boldsymbol{K}_c\boldsymbol{T}^{-T})_{ii} = 1. \quad (11)$$

The problem of L_2-sensitivity minimization subject to L_2-norm dynamic-range scaling constraints is now formulated as follows: For given $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, obtain an $n \times n$ nonsingular matrix $\boldsymbol{T}$ which minimizes (10) subject to the constraints in (11).

III. L_2-SENSITIVITY MINIMIZATION

In order to minimize (10) over an $n \times n$ symmetric positive-definite matrix $\boldsymbol{P}$ subject to the constraints shown in (11), we define the Lagrange function

$$\begin{aligned} J(\boldsymbol{P}, \lambda) &= \mathrm{tr}[\boldsymbol{M}(\boldsymbol{P})\boldsymbol{P}] + \mathrm{tr}[\boldsymbol{W}_o\boldsymbol{P}] \\ &\quad + \mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] + \lambda(\mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] - n) \end{aligned} \quad (12)$$

where λ is a Lagrange multiplier. We compute

$$\begin{aligned} \frac{\partial J(\boldsymbol{P}, \lambda)}{\partial \boldsymbol{P}} &= \boldsymbol{M}(\boldsymbol{P}) - \boldsymbol{P}^{-1}\boldsymbol{N}(\boldsymbol{P})\boldsymbol{P}^{-1} + \boldsymbol{W}_o \\ &\quad - (\lambda + 1)\boldsymbol{P}^{-1}\boldsymbol{K}_c\boldsymbol{P}^{-1} \\ \frac{\partial J(\boldsymbol{P}, \lambda)}{\partial \lambda} &= \mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] - n \end{aligned} \quad (13)$$

where $\boldsymbol{N}(\boldsymbol{P})$ can be obtained by solving the Lyapunov equation [14]

$$\begin{aligned} \begin{bmatrix} \boldsymbol{N}(\boldsymbol{P}) & * \\ * & * \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A} & \boldsymbol{bc} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{N}(\boldsymbol{P}) & * \\ * & * \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \boldsymbol{A} & \boldsymbol{bc} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}^T + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{P} & \boldsymbol{0} \end{bmatrix}. \end{aligned}$$

From $\partial J(\boldsymbol{P}, \lambda)/\partial \boldsymbol{P} = \boldsymbol{0}$ and $\partial J(\boldsymbol{P}, \lambda)/\partial \lambda = 0$, we get

$$\boldsymbol{P}\boldsymbol{F}(\boldsymbol{P})\boldsymbol{P} = \boldsymbol{G}(\boldsymbol{P}, \lambda), \qquad \mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] = n \quad (14)$$

where

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{P}) &= \boldsymbol{M}(\boldsymbol{P}) + \boldsymbol{W}_o \\ \boldsymbol{G}(\boldsymbol{P}, \lambda) &= \boldsymbol{N}(\boldsymbol{P}) + (\lambda + 1)\boldsymbol{K}_c. \end{aligned}$$

It follows that the value $\boldsymbol{P}_{i+1}$ satisfying

$$\boldsymbol{P}_{i+1}\boldsymbol{F}(\boldsymbol{P}_i)\boldsymbol{P}_{i+1} = \boldsymbol{G}(\boldsymbol{P}_i, \lambda_i) \tag{15}$$

is given by

$$\boldsymbol{P}_{i+1} = \boldsymbol{F}(\boldsymbol{P}_i)^{-\frac{1}{2}}[\boldsymbol{F}(\boldsymbol{P}_i)^{\frac{1}{2}}\boldsymbol{G}(\boldsymbol{P}_i, \lambda_i)\boldsymbol{F}(\boldsymbol{P}_i)^{\frac{1}{2}}]^{\frac{1}{2}}\boldsymbol{F}(\boldsymbol{P}_i)^{-\frac{1}{2}} \tag{16}$$

and that the value λ_{i+1} satisfying

$$\boldsymbol{P}_i\boldsymbol{F}(\boldsymbol{P}_i) = \boldsymbol{G}(\boldsymbol{P}_i, \lambda_{i+1})\boldsymbol{P}_i^{-1}, \quad \mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}_i^{-1}] = n \tag{17}$$

is obtained as

$$\lambda_{i+1} = \frac{\mathrm{tr}[\boldsymbol{P}_i\boldsymbol{F}(\boldsymbol{P}_i)] - \mathrm{tr}[\boldsymbol{N}(\boldsymbol{P}_i)\boldsymbol{P}_i^{-1}]}{n} - 1. \tag{18}$$

In the above algorithm, $\boldsymbol{P}_i$ and λ_i are the solutions of the previous iteration. The initial estimates are given by $\boldsymbol{P}_0 = \boldsymbol{I}_n$ and any value of $\lambda_0 > 0$. This iteration process continues until (14) is satisfied within a prescribed numerical tolerance.

Next, the coordinate transformation matrix $\boldsymbol{T}$ will be constructed so that (11) is satisfied. From $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{T}^T$, the optimal coordinate transformation matrix $\boldsymbol{T}$ that minimizes (12) can be obtained in closed form as

$$\boldsymbol{T} = \boldsymbol{P}^{\frac{1}{2}}\boldsymbol{U} \tag{19}$$

where $\boldsymbol{U}$ is an arbitrary $n\times n$ orthogonal matrix. From (19) it follows that

$$\begin{aligned} \overline{\boldsymbol{K}}_c &= \boldsymbol{T}^{-1}\boldsymbol{K}_c\boldsymbol{T}^{-T} \\ &= \boldsymbol{U}^T\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{K}_c\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{U}. \end{aligned} \tag{20}$$

Let us choose the $n\times n$ orthogonal matrix $\boldsymbol{U}$ such that the matrix $\overline{\boldsymbol{K}}_c$ in (20) satisfies the l_2-norm dynamic-range scaling constraints, (11), on the state-variables. To this end, we perform the eigenvalue-eigenvector decomposition

$$\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{K}_c\boldsymbol{P}^{-\frac{1}{2}} = \boldsymbol{R}\boldsymbol{\Theta}\boldsymbol{R}^T \tag{21}$$

where $\boldsymbol{\Theta} = \mathrm{diag}\{\theta_1, \theta_2, \cdots, \theta_n\}$ and $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}_n$. Now an $n\times n$ orthogonal matrix $\boldsymbol{S}$ such that

$$\boldsymbol{S}\boldsymbol{\Theta}\boldsymbol{S}^T = \begin{bmatrix} 1 & * & \cdots & * \\ * & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ * & \cdots & * & 1 \end{bmatrix} \tag{22}$$

can be obtained by numerical manipulations [13, p.278]. By choosing $\boldsymbol{U} = \boldsymbol{R}\boldsymbol{S}^T$ in (19), the optimal coordinate transformation matrix $\boldsymbol{T}$ both satisfying (11) and minimizing (10) can now be constructed as

$$\boldsymbol{T} = \boldsymbol{P}^{\frac{1}{2}}\boldsymbol{R}\boldsymbol{S}^T. \tag{23}$$

IV. NUMERICAL EXAMPLE

Let a state-space digital filter in (1) be specified by

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.453770 & -1.556160 & 1.974860 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 0.242096 \end{bmatrix}^T$$

$$\boldsymbol{c} = \begin{bmatrix} 0.095706 & 0.095086 & 0.327556 \end{bmatrix}$$

$$d = 0.015940.$$

Performing the computation of (7) and the L_2-scaling, the Grammians $\boldsymbol{K}_c$, $\boldsymbol{W}_o$ and $\boldsymbol{M}(\boldsymbol{I}_3)$ are calculated as

$$\boldsymbol{K}_c = \begin{bmatrix} 1.000000 & 0.872501 & 0.562821 \\ 0.872501 & 1.000000 & 0.872501 \\ 0.562821 & 0.872501 & 1.000000 \end{bmatrix}$$

$$\boldsymbol{W}_o = \begin{bmatrix} 0.820741 & -2.035328 & 1.628161 \\ -2.035328 & 5.307273 & -4.264903 \\ 1.628161 & -4.264903 & 3.941491 \end{bmatrix}$$

$$\boldsymbol{M}(\boldsymbol{I}_3) = \begin{bmatrix} 8.921380 & -22.046457 & 17.916285 \\ -22.046457 & 55.671710 & -46.052011 \\ 17.916285 & -46.052011 & 42.522082 \end{bmatrix}.$$

The L_2-sensitivity measure in (6) is computed as

$$S = 120.184677.$$

Choosing $\boldsymbol{P}_0 = \boldsymbol{I}_3$ and $\lambda_0 = 100$ as the initial estimates, it took the proposed iterative algorithm 500 iterations to converge to

$$\boldsymbol{P}^{opt} = \begin{bmatrix} 2.307529 & 1.375667 & 0.514400 \\ 1.375667 & 1.103115 & 0.678193 \\ 0.514400 & 0.678193 & 0.666912 \end{bmatrix}$$

which yields

$$\boldsymbol{T}^{opt} = \begin{bmatrix} 0.906372 & 0.756223 & 0.956110 \\ 0.196978 & 0.857123 & 0.574155 \\ -0.369823 & 0.597630 & 0.415910 \end{bmatrix}.$$

In this case, the L_2-sensitivity measure in (10) is minimized subject to the scaling constraints in (11) to

$$S(\boldsymbol{P}^{opt}) = 8.672129.$$

The L_2-sensitivity and λ performances of 500 iterations are shown in Fig.1, from which it is seen that the proposed iterative algorithm sufficiently converges with 500 iterations.

For comparison purposes, the existing method reported in [10] is applied to minimize the L_2-sensivivity

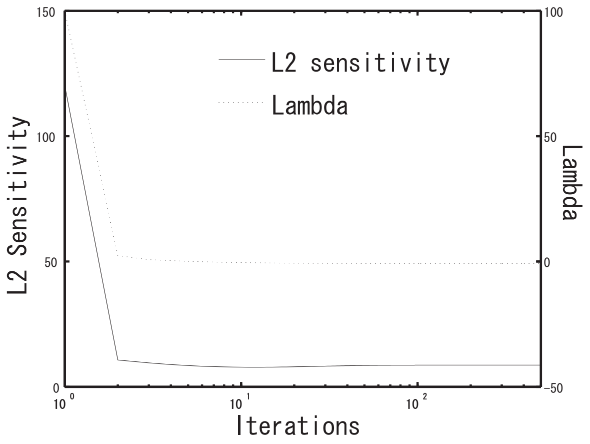

Figure 1: L_2-Sensitivity and λ Performances

measure in (10) (without considering the scaling constraints in (11)) and then the resulting optimal coordinate transformation matrix is scaled by an appropriate nonsingular diagonal matrix, so that the scaling constraints in (11) are satisfied. Then the result is $S(\boldsymbol{T}) = 9.817579$. Applying the technique reported in [11] yields $S(\boldsymbol{T}^{opt}) = 8.683279$. Moreover, by applying the method in [13], the optimal filter structure which minimizes the roundoff noise at the filter output subject to the scaling constraints in (11) has the L_2-sensitivity, $S(\boldsymbol{T}) = 8.797931$.

VI. CONCLUSION

This paper has considered the problem of minimizing an L_2-sensitivity measure subject to L_2-norm dynamic range scaling constraints in state-space digital filters. An efficient iterative technique has been developed by using a Lagrange function and some matrix-theoretic techniques in order to solve the constraint optimization problem directly. Our computer simulation results have demonstrated the effectiveness of the proposed technique compared with several existing methods.

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