A Second-Order Cone Programming Approach for Minimax Design of 2-D FIR Filters with Low Group Delay

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***Abstract*— A design algorithm based on second-order cone programming (SOCP) for minimax design of 2-D FIR filters with low group delay is proposed. SOCP is a special class of convex programming problems that can be carried out considerably more efficiently than the popular semidefinite programming. The simulation studies presented in this paper also confirm this in a filter design context. The proposed algorithm is compared favorably with a recently proposed design method based on sequential quadratic programming. The proposed design method is expected to be a useful utility for 2-D digital filter designers whose interest is not limited to linear phase responses and low order filters.**

I. Introduction

There have been several algorithms for the design of two-dimensional (2-D) digital filters, see for example [1]–[7] and the reference cited therein. Concerning the minimax designs, a successful early attempt is documented in [1]. More recent efforts include [6] and [7] where methods based on semidefinite programming (SDP) and sequential quadratic programming (SQP) have been proposed. These methods work quite well except that the design complexity becomes rather high even for filters of moderate order. The high computational complexity is attributed to two facts: the large number of design variables in the order of $O(N1 \cdot N_2)$ for filters of size $N_1 \times N_2$ and the large number of constraints imposed on a dense set of frequency grid points over the baseband $[-\pi,\ \pi] \times [-\pi,\ \pi]$.

In this paper, the minimax design of 2-D FIR filters approximating arbitrary frequency responses (including those with low group delays), is addressed in a convex programming framework known as second-order cone programming (SOCP) [8]. The paper provides details in design formulation and algorithmic description of the method, and presents simulation results to demonstrate that the SOCP-based algorithm can achieve the same design results with much reduced computational complexity relative to the algorithms in the literature.

II. Problem Formulation

Consider a 2-D FIR transfer function

$$H(z_1, z_2) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} h_{ij} z_1^{-i} z_2^{-j} \tag{1}$$

We seek to find a transfer function $H(z_1, z_2)$ in (1) whose frequency response best approximates a desired frequency response $H_d(\omega_1, \omega_2)$ in the minimax sense. Namely, we are interested in solving the optimization problem

$$\underset{\boldsymbol{h}}{\text{minimize}} \underset{(\omega_1,\omega_2)\in\Omega}{\text{maximize}}\ W(\omega_1, \omega_2)|H(\omega_1, \omega_2) - H_d(\omega_1, \omega_2)| \tag{2}$$

where $\Omega = \{(\omega_1, \omega_2),\ -\pi \le \omega_1,\ \omega_2 \le \pi\}$ and $W(\omega_1, \omega_2) \ge 0$ is a known weighting function defined on Ω.

For notation simplicity, we assume $N_1 = N_2 = N$ and denote

$$\begin{aligned} H(\omega_1, \omega_2) &= H(e^{j\omega_1}, e^{j\omega_2}) \\ \boldsymbol{c}(\omega) &= [1 \quad \cos\omega \ \cdots \ \cos(N-1)\omega]^T \\ \boldsymbol{s}(\omega) &= [0 \quad \sin\omega \ \cdots \ \sin(N-1)\omega]^T \end{aligned}$$

and

$$\boldsymbol{H} = \{h_{ij},\ 0 \le i,\ j \le N-1\}$$

The frequency response of the 2-D FIR filter characterized in (1) can be expressed as

$$\begin{aligned} H(\omega_1, \omega_2) = &\ [\boldsymbol{c}^T(\omega_1)\boldsymbol{H}\boldsymbol{c}(\omega_2) - \boldsymbol{s}^T(\omega_1)\boldsymbol{H}\boldsymbol{s}(\omega_2)] \\ &- j[\boldsymbol{s}^T(\omega_1)\boldsymbol{H}\boldsymbol{c}(\omega_2) + \boldsymbol{c}^T(\omega_1)\boldsymbol{H}\boldsymbol{s}(\omega_2)] \end{aligned}$$

If we let $\boldsymbol{c}_i = \boldsymbol{c}(\omega_i)$ and $\boldsymbol{s}_i = \boldsymbol{s}(\omega_i)$ for $i = 1,\ 2$, then we can write

$$\begin{aligned} H(\omega_1, \omega_2) &= \text{tr}[\boldsymbol{P}(\omega_1, \omega_2)\boldsymbol{H}] - j\text{tr}[\boldsymbol{Q}(\omega_1, \omega_2)\boldsymbol{H}] \\ \boldsymbol{P}(\omega_1, \omega_2) &= \boldsymbol{c}_2\boldsymbol{c}_1^T - \boldsymbol{s}_2\boldsymbol{s}_1^T \\ \boldsymbol{Q}(\omega_1, \omega_2) &= \boldsymbol{c}_2\boldsymbol{s}_1^T + \boldsymbol{s}_2\boldsymbol{c}_1^T \end{aligned}$$

where tr$[\cdot]$ denotes matrix trace. Further, if we denote the column vectors generated by stacking the transposed rows of $\boldsymbol{P}(\omega_1, \omega_2)$, $\boldsymbol{Q}(\omega_1, \omega_2)$ and $\boldsymbol{H}$ by $\boldsymbol{p}(\omega_1, \omega_2)$, $\boldsymbol{q}(\omega_1, \omega_2)$ and

$\boldsymbol{h}$, respectively, then the frequency response can simply be expressed as

$$H(\omega_1, \omega_2) = \boldsymbol{p}^T(\omega_1, \omega_2)\boldsymbol{h} - j\boldsymbol{q}^T(\omega_1, \omega_2)\boldsymbol{h} \tag{3}$$

By placing an upper bound η on the objective function in (2), the solution of (2) shall minimize the bound, thus the problem at hand can be converted to

$$\begin{aligned} &\text{minimize} \quad \eta && \text{(4a)} \\ &\text{subject to:} \quad W(\omega_1, \omega_2)\{[\boldsymbol{p}^T(\omega_1, \omega_2)\boldsymbol{h} - H_{dr}(\omega_1, \omega_2)]^2 \\ &\qquad + [\boldsymbol{q}^T(\omega_1, \omega_2)\boldsymbol{h} - H_{di}(\omega_1, \omega_2)]^2\}^{1/2} \le \eta && \text{(4b)} \\ &\qquad \text{for } (\omega_1, \omega_2) \in \Omega \end{aligned}$$

where $H_{dr}(\omega_1, \omega_2)$ and $-H_{di}(\omega_1, \omega_2)$ are the real and imaginary parts of $H_d(\omega_1, \omega_2)$, i.e.,

$$H_d(\omega_1, \omega_2) = H_{dr}(\omega_1, \omega_2) - jH_{di}(\omega_1, \omega_2)$$

For the design of low group delay filters, the desired frequency response assumes the form

$$H_d(\omega_1, \omega_2) = A_d(\omega_1, \omega_2)e^{-jd(\omega_1 + \omega_2)}$$

where $A_d(\omega_1, \omega_2)$ is a nonnegative real-valued function on Ω, and d is a constant group delay between 0 and $(N-1)/2$. In this case we have

$$H_{dr}(\omega_1, \omega_2) = A_d(\omega_1, \omega_2)\cos[d(\omega_1 + \omega_2)]$$

and

$$H_{di}(\omega_1, \omega_2) = A_d(\omega_1, \omega_2)\sin[d(\omega_1 + \omega_2)]$$

It should be stressed, however, that both the problem formulation in (4) and the solution process to be described below are applicable to the design of 2-D FIR filters approximating a given *arbitrary* frequency response.

For a feasible exercise of SOCP, the constraints in (4b) are discretized to a dense grid of frequencies $\Omega_d = \{(\omega_{1k}, \omega_{2k}), k = 1, 2, \ldots, K\} \subseteq \Omega$ and, in doing so, the problem in (4) becomes

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{e}^T\boldsymbol{x} && \text{(5a)} \\ &\text{subject to:} \quad a_k(\boldsymbol{h}) \le \boldsymbol{e}^T\boldsymbol{x} \quad \text{for } k = 1, 2, \ldots, K && \text{(5b)} \end{aligned}$$

where $\boldsymbol{x} = [\eta \;\; \boldsymbol{h}^T]^T$, $\boldsymbol{e} = [1 \;\; 0 \;\; \cdots \;\; 0]^T$, and

$$a_k(\boldsymbol{h}) = W_k[(\boldsymbol{p}_k^T\boldsymbol{h} - H_{drk})^2 + [\boldsymbol{q}_k^T\boldsymbol{h} - H_{dik})^2]^{1/2} \tag{5c}$$

with

$$\begin{aligned} W_k &= W(\omega_{1k}, \omega_{2k}) \\ \boldsymbol{p}_k &= \boldsymbol{p}(\omega_{1k}, \omega_{2k}) \\ \boldsymbol{q}_k &= \boldsymbol{q}(\omega_{1k}, \omega_{2k}) \\ H_{drk} &= H_{dr}(\omega_{1k}, \omega_{2k}) \\ H_{dik} &= H_{di}(\omega_{1k}, \omega_{2k}) \end{aligned}$$

III. An SOCP-Based Solution

A. SOCP

SOCP is a class of convex programming problems, in which a linear objective function is minimized subject to second-order cone constraints [8]. Also known as quadratic or Lorentz cone, a second-order cone of dimension n is defined as

$$\mathcal{K} = \left\{ \begin{bmatrix} t \\ \boldsymbol{u} \end{bmatrix}, \; t \in R, \; \boldsymbol{u} \in R^{n-1}, \; \|\boldsymbol{u}\|_2 \le t \right\}$$

For $n = 1$, the second-order cone is degenerated to a ray on the t-axis starting from $t = 0$ and for $n = 3$ the second-order cone is depicted in Fig. 1.

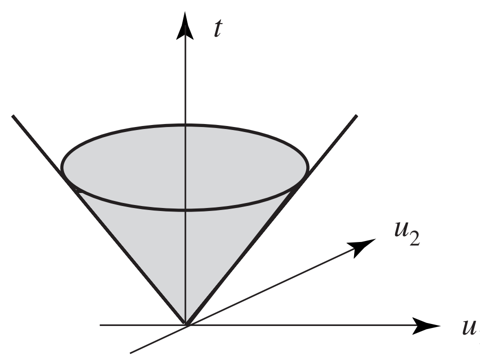

Fig. 1. Second-order cone of dimension $n = 3$.

A standard SOCP formulation favored in many engineering applications can be expressed as

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{b}^T\boldsymbol{x} && \text{(6a)} \\ &\text{subject to:} \quad \|\boldsymbol{A}_k^T\boldsymbol{x} + \boldsymbol{c}_k\|_2 \le \boldsymbol{b}_k^T\boldsymbol{x} + d_k && \text{(6b)} \\ &\qquad \text{for } k = 1, 2, \ldots, K \end{aligned}$$

The reason we call (6) an SOCP problem is because of the fact that if we let

$$\hat{\boldsymbol{A}}_k^T = \begin{bmatrix} \boldsymbol{b}_k^T \\ \boldsymbol{A}_k^T \end{bmatrix}, \; \hat{\boldsymbol{c}}_k = \begin{bmatrix} d_k \\ \boldsymbol{c}_k \end{bmatrix}$$

then the constraints in (6b) are equivalent to that each $\hat{\boldsymbol{A}}_k^T\boldsymbol{x} + \hat{\boldsymbol{c}}_k$ belongs to a second-order cone $\mathcal{K}_k$.

B. An SOCP formulation of problem (5)

The function $a_k(\boldsymbol{h})$ in (5c) can be written as

$$\begin{aligned} a_k(\boldsymbol{h}) &= [(\hat{\boldsymbol{p}}_k^T\boldsymbol{h} - \hat{H}_{drk})^2 + (\hat{\boldsymbol{q}}_k^T\boldsymbol{h} - \hat{H}_{dik})^2]^{1/2} \\ &= \left\| \begin{bmatrix} \hat{\boldsymbol{p}}_k^T \\ \hat{\boldsymbol{q}}_k^T \end{bmatrix} \boldsymbol{h} - \begin{bmatrix} \hat{H}_{drk} \\ \hat{H}_{dik} \end{bmatrix} \right\|_2 \\ &= \left\| \begin{bmatrix} 0 & \hat{\boldsymbol{p}}_k^T \\ 0 & \hat{\boldsymbol{q}}_k^T \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \hat{H}_{drk} \\ \hat{H}_{dik} \end{bmatrix} \right\|_2 \end{aligned}$$

where $\hat{\boldsymbol{p}}_k = W_k^{1/2}\boldsymbol{p}_k,\ \hat{\boldsymbol{q}}_k = W_k^{1/2}\boldsymbol{q}_k,\ \hat{H}_{drk} = W_k^{1/2}H_{drk}$, and $\hat{H}_{dik} = W_k^{1/2}H_{dik}$. Hence the constraints in (5b) become the constraints in (6b) with

$$\boldsymbol{A}_k = \begin{bmatrix} 0 & 0 \\ \hat{\boldsymbol{p}}_k & \hat{\boldsymbol{q}}_k \end{bmatrix},\ \boldsymbol{b}_k = \boldsymbol{e},\ \boldsymbol{c}_k = -\begin{bmatrix} \hat{H}_{drk} \\ \hat{H}_{dik} \end{bmatrix},\ d_k = 0 \quad (7)$$

and the objective function in (5a) becomes the objective functin in (6a) with $\boldsymbol{b} = \boldsymbol{e}$.

C. MATLAB implementation

MATLAB toolbox SeDuMi version 1.1R2 was used to solve the SOCP problem in (6) where the data required are specified in (7). The name of the toolbox stands for self-dual minimization as it implements a self-dual embedding technique for optimization over self-dual homogeneous cones [9].

For implementation purposes, define

$$\begin{aligned} \boldsymbol{A}_t &= [\tilde{\boldsymbol{A}}_1 \ \ \tilde{\boldsymbol{A}}_2 \ \ \cdots \ \ \tilde{\boldsymbol{A}}_K] \\ \tilde{\boldsymbol{A}}_k &= -[\boldsymbol{b}_k \ \ \boldsymbol{A}_k] \\ \boldsymbol{b}_t &= = -\boldsymbol{b} \\ \boldsymbol{c}_t &= \begin{bmatrix} \tilde{\boldsymbol{c}}_1 \\ \vdots \\ \tilde{\boldsymbol{c}}_K \end{bmatrix},\ \tilde{\boldsymbol{c}}_k = \begin{bmatrix} d_k \\ \boldsymbol{c}_k \end{bmatrix} \end{aligned}$$

In addition, define a K-dimensional vector $\boldsymbol{q} = [n_1 \quad n_2 \quad \ldots \quad n_K]$ where n_k denotes the number of columns of matrix $\tilde{\boldsymbol{A}}_k$. The key commands for SeDuMi to solve problem (6) are as follows:

```
K.q = q;
pars.fid = 0;
[xs, ys, info] = sedumi(At, bt, ct, K, pars);
x = ys;
info
```

The solution of the problem is then given as vector $\boldsymbol{x}$.

IV. Design Examples

The proposed algorithm was applied to design 2-D FIR filters of various types and sizes. As a representative, Fig. 2 depicts the amplitude response and passband group delay of a circularly symmetric lowpass filter of size 43×43 with normalized passband edge $\omega_p = 0.5$ and stopband edge $\omega_1 = 0.65$ that approximates a desired lowpass frequency response with passband group delay = 19. A total of $K = 1986$ grid points. It took the proposed algorithm 25 iterations and 652.69 seconds CPU time on a 3.4 GHz Pentium 4 PC to converge to the solution as shown in Fig. 2 whose maximum passband error and minimum stopband attenuation were found to be 0.0021 and 54.1529 dB, respectively. The maximum relative deviation of group delay in passband was 0.0279.

For comparison purposes, the proposed algorithm was also applied to design five linear-phase circularly symmetric lowpass filters with $f_p = 0.425$, $f_a = 0.575$, and $N = 7$, 11, 15, 19, and 23. The filters with the same design specifications were also carried out using the algorithms of [3] and [6] and the

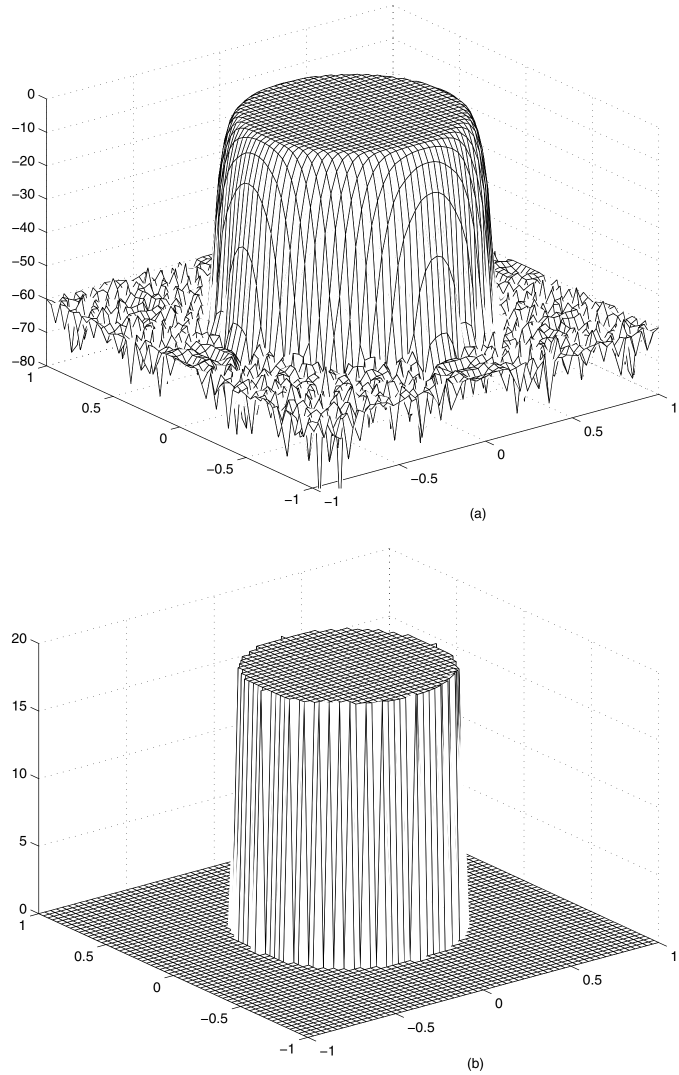

Fig. 2. (a) Amplitude response and (b) passband group delay of the 43×43 circularly symmetric lowpass filter.

results are summarized in Table 1. The amplitude responses of these filters are shown in Fig. 3. From these design examples, it is quite obvious that the SOCP-based algorithm can be used to design filters of fairly high order and offers considerably reduced computational complexity relative to that of several existing design methods.

V. Conclusions

The minimax design of 2-D filters with low group delay can be formulated as an SOCP problem which is a convex constrained optimization that admits a unique, global solution. Thanks to the public-domain toolbox SeDuMi which has proven to be a reliable and efficient LP-SDP-SOCP solver, the proposed design method is expected to be a useful utility for 2-D digital filter designers whose interest is not limited to linear phase responses and low order filters.

TABLE I

COMPARISON OF THE PROPOSED METHOD WITH THE METHODS IN [3], [6]

N	Grid points K	maximum ripple		CPU time in seconds		
		passband	stopband	proposed method	[3]	[6]
7	345	0.1769	0.1787	0.36	8.10	1.72
11	345	0.0931	0.1093	0.83	15.74	3.36
15	351	0.0629	0.0643	1.88	33.67	6.65
19	629	0.0317	0.0451	8.71	117.99	23.42
23	761	0.0193	0.0227	20.02	282.18	55.44

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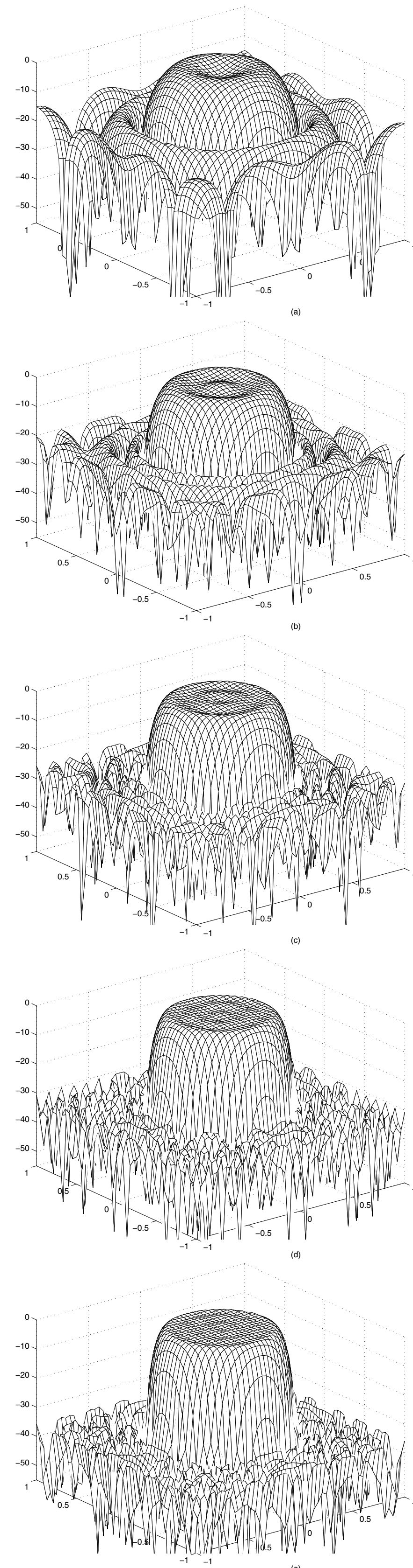

Fig. 3. Amplitudes of circularly symmetric linear-phase lowpass 2-D FIR filters of size (a) 7×7, (b) 11×11, (c) 15×15, (d) 19×19, and (e) 23×23.