Realization of MIMO Linear Discrete-Time Systems with Minimum L_2-Sensitivity and No Overflow Oscillations

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Abstract— **The minimization problem of an L_2-sensitivity measure subject to L_2-scaling constraints on the dynamic range for multi-input/multi-output (MIMO) linear discrete-time systems is formulated. An iterative technique is developed to solve the constrained optimization problem directly. The proposed solution method largely relies on the use of a Lagrange function and some matrix-theoretic techniques. A numerical example is presented to illustrate the utility of the proposed technique.**

I. INTRODUCTION

The state-space realization of a multi-input/multi-output (MIMO) linear discrete-time system is known as the problem of obtaining a suitable set of state-space equations that realize a desired MIMO transfer function $\boldsymbol{H}(z)$. However, the state-space equations corresponding to a transfer function $\boldsymbol{H}(z)$ are not unique. Naturally, among the infinite number of realizations of $\boldsymbol{H}(z)$, one wants to identify a state-space realization that minimizes a suitable sensitivity measure. When realizing a fixed-point state-space description with finite word length (FWL) from a transfer function with infinite accuracy coefficients, the coefficients in the state-space description must be truncated or rounded to fit the FWL constraints. This coefficient quantization usually alters the characteristics of the system. For instance, a stable system may be turned to an unstable one. This motivates the study of the coefficient sensitivity minimization problem. In [1]-[10], two main classes of techniques have been proposed for constructing state-space descriptions that minimize the coefficient sensitivity: L_1/L_2-sensitivity minimization [1]-[5] and L_2-sensitivity minimization [6]-[10]. It has been argued in [6]-[10] that the sensitivity measure based on the L_2 norm is more natural and reasonable relative to that based on the L_1/L_2-sensitivity minimization. Alternatively, it is well known that the use of scaling constraints can be beneficial for suppressing overflow oscillations [11],[12]. The L_2-sensitivity minimization problem subject to L_2-scaling constraints for state-space digital filters has been solved iteratively by converting it into an unconstrained optimization problem with an appropriate linear transformation [13]. However, to our best knowledge, there is no study on the minimization of the L_2-sensitivity subject to the L_2-scaling constraints for MIMO linear discrete-time systems.

In this paper, we investigate the problem of minimizing an L_2-sensitivity measure subject to L_2-scaling constraints for MIMO linear discrete-time systems. An expression for evaluating the L_2-sensitivity is explored, and the L_2-sensitivity minimization problem subject to the L_2-scaling constraints is formulated. Next, an iterative procedure is developed for minimizing the L_2-sensitivity measure subject to L_2-scaling constraints. This is largely based on the use of a Lagrange function and some matrix-theoretic techniques. Computer simulation results demonstrate the validity of the proposed technique.

II. L_2-SENSITIVITY ANALYSIS

Consider a stable, controllable and observable MIMO linear discrete-time system $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D})_n$ described by

$$\begin{aligned}\boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{B}\boldsymbol{u}(k)\\ \boldsymbol{y}(k) &= \boldsymbol{C}\boldsymbol{x}(k)+\boldsymbol{D}\boldsymbol{u}(k)\end{aligned} \tag{1}$$

where $\boldsymbol{x}(k)$ is an $n\times 1$ state-variable vector, $\boldsymbol{u}(k)$ is a $q\times 1$ input vector, $\boldsymbol{y}(k)$ is a $p\times 1$ output vector, and $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ and $\boldsymbol{D}$ are real constant matrices of appropriate dimensions. The transfer function of the linear system in (1) is given by

$$\boldsymbol{H}(z)=\boldsymbol{C}(z\boldsymbol{I}_n-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D} \tag{2}$$

whose (i,j)th element is described by

$$H_{ij}(z)=\boldsymbol{c}_i(z\boldsymbol{I}_n-\boldsymbol{A})^{-1}\boldsymbol{b}_j+d_{ij} \tag{3}$$

where

$$\boldsymbol{B}=\begin{bmatrix}\boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_q\end{bmatrix}$$

$$\boldsymbol{C}=\begin{bmatrix}\boldsymbol{c}_1\\ \boldsymbol{c}_2\\ \vdots\\ \boldsymbol{c}_p\end{bmatrix},\qquad \boldsymbol{D}=\begin{bmatrix}d_{11} & d_{12} & \cdots & d_{1q}\\ d_{21} & d_{22} & \cdots & d_{2q}\\ \vdots & \vdots & \ddots & \vdots\\ d_{p1} & d_{p2} & \cdots & d_{pq}\end{bmatrix}.$$

The L_2-sensitivity of the linear system in (1) is defined as follows.

Definition 1: Let $\boldsymbol{X}$ be an $m\times n$ real matrix and let $f(\boldsymbol{X})$ be a scalar complex function of $\boldsymbol{X}$, differentiable with respect to all the entries of $\boldsymbol{X}$. The sensitivity function of f with respect to $\boldsymbol{X}$ is then defined as

$$\boldsymbol{S}_{\boldsymbol{X}}=\frac{\partial f}{\partial \boldsymbol{X}},\qquad (\boldsymbol{S}_{\boldsymbol{X}})_{ij}=\frac{\partial f}{\partial x_{ij}} \tag{4}$$

where x_{ij} denotes the (i,j)th entry of matrix $\boldsymbol{X}$.

Definition 2: Let $\boldsymbol{X}(z)$ be an $m \times n$ complex matrix-valued function of a complex variable z and let $x_{pq}(z)$ be the (p,q)th entry of $\boldsymbol{X}(z)$. The L_2-norm of $\boldsymbol{X}(z)$ is then defined as

$$\begin{aligned} \|\boldsymbol{X}(z)\|_2 &= \left[\frac{1}{2\pi}\int_0^{2\pi}\sum_{p=1}^{m}\sum_{q=1}^{n}\left|x_{pq}(e^{j\omega})\right|^2 d\omega\right]^{\frac{1}{2}} \\ &= \left(\mathrm{tr}\left[\frac{1}{2\pi j}\oint_{|z|=1}\boldsymbol{X}(z)\boldsymbol{X}^*(z)\frac{dz}{z}\right]\right)^{\frac{1}{2}}. \quad (5) \end{aligned}$$

From (3) and Definitions 1 and 2, the overall L_2-sensitivity measure for the linear system in (1) is defined by

$$\begin{aligned} S &= \sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\frac{\partial H_{ij}(z)}{\partial \boldsymbol{A}}\right\|_2^2 + \sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\frac{\partial H_{ij}(z)}{\partial \boldsymbol{b}_j}\right\|_2^2 \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\frac{\partial H_{ij}(z)}{\partial \boldsymbol{c}_i^T}\right\|_2^2 \\ &= \sum_{i=1}^{p}\sum_{j=1}^{q}\left\|[\boldsymbol{f}_j(z)\boldsymbol{g}_i(z)]^T\right\|_2^2 + q\sum_{i=1}^{p}\left\|\boldsymbol{g}_i^T(z)\right\|_2^2 \\ &\quad + p\sum_{j=1}^{q}\left\|\boldsymbol{f}_j(z)\right\|_2^2 \end{aligned} \quad (6)$$

where $\boldsymbol{f}_j(z) = (z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b}_j$ and $\boldsymbol{g}_i(z) = \boldsymbol{c}_i(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}$. Since the term $\boldsymbol{D}$ in (2) and the sensitivities with respect to its elements are independent of the state-space coordinate, they are neglected in (6).

Using simple algebraic manipulations, the L_2-sensitivity measure in (6) can be expressed as

$$S = \sum_{i=1}^{p}\sum_{j=1}^{q}\mathrm{tr}[\boldsymbol{M}_{ij}(\boldsymbol{I}_n)] + q\,\mathrm{tr}[\boldsymbol{W}_o] + p\,\mathrm{tr}[\boldsymbol{K}_c] \quad (7)$$

with

$$\begin{aligned} \boldsymbol{K}_c &= \frac{1}{2\pi j}\oint_{|z|=1}\boldsymbol{F}(z)\boldsymbol{F}^T(z^{-1})\frac{dz}{z} \\ \boldsymbol{W}_o &= \frac{1}{2\pi j}\oint_{|z|=1}\boldsymbol{G}^T(z)\boldsymbol{G}(z^{-1})\frac{dz}{z} \\ \boldsymbol{M}_{ij}(\boldsymbol{I}_n) &= \frac{1}{2\pi j}\oint_{|z|=1}[\boldsymbol{f}_j(z)\boldsymbol{g}_i(z)]^T\boldsymbol{f}_j(z^{-1})\boldsymbol{g}_i(z^{-1})\frac{dz}{z} \end{aligned}$$

where $\boldsymbol{F}(z) = (z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B}$ and $\boldsymbol{G}(z) = \boldsymbol{C}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}$. The matrices $\boldsymbol{K}_c$ and $\boldsymbol{W}_o$ in (7) are called the controllability and observability Gramians, respectively, and can be obtained by solving the following Lyapunov equations [14]:

$$\begin{aligned} \boldsymbol{K}_c &= \boldsymbol{A}\boldsymbol{K}_c\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T \\ \boldsymbol{W}_o &= \boldsymbol{A}^T\boldsymbol{W}_o\boldsymbol{A} + \boldsymbol{C}^T\boldsymbol{C}. \end{aligned} \quad (8)$$

If a coordinate transformation defined by

$$\overline{\boldsymbol{x}}(k) = \boldsymbol{T}^{-1}\boldsymbol{x}(k) \quad (9)$$

is applied to the linear system in (1), then the new realization $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \overline{\boldsymbol{C}}, \boldsymbol{D})_n$ can be characterized by

$$\begin{aligned} \overline{\boldsymbol{A}} &= \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}, \quad \overline{\boldsymbol{B}} = \boldsymbol{T}^{-1}\boldsymbol{B}, \quad \overline{\boldsymbol{C}} = \boldsymbol{C}\boldsymbol{T} \\ \overline{\boldsymbol{W}}_o &= \boldsymbol{T}^T\boldsymbol{W}_o\boldsymbol{T}, \quad \overline{\boldsymbol{K}}_c = \boldsymbol{T}^{-1}\boldsymbol{K}_c\boldsymbol{T}^{-T} \end{aligned} \quad (10)$$

and

$$\begin{aligned} \overline{\boldsymbol{f}}_j(z) &= (z\boldsymbol{I}_n - \overline{\boldsymbol{A}})^{-1}\overline{\boldsymbol{b}}_j = \boldsymbol{T}^{-1}\boldsymbol{f}_j(z) \\ \overline{\boldsymbol{g}}_i(z) &= \overline{\boldsymbol{c}}_i(z\boldsymbol{I}_n - \overline{\boldsymbol{A}})^{-1} = \boldsymbol{g}_i(z)\boldsymbol{T} \end{aligned} \quad (11)$$

where

$$\begin{aligned} \overline{\boldsymbol{B}} &= \begin{bmatrix} \overline{\boldsymbol{b}}_1 & \overline{\boldsymbol{b}}_2 & \cdots & \overline{\boldsymbol{b}}_q \end{bmatrix} \\ \overline{\boldsymbol{C}} &= \begin{bmatrix} \overline{\boldsymbol{c}}_1^T & \overline{\boldsymbol{c}}_2^T & \cdots & \overline{\boldsymbol{c}}_p^T \end{bmatrix}^T. \end{aligned}$$

Moreover, $\boldsymbol{M}_{ij}(\boldsymbol{I}_n)$ is transformed to $\overline{\boldsymbol{M}}_{ij}(\boldsymbol{I}_n)$ as follows:

$$\begin{aligned} \overline{\boldsymbol{M}}_{ij}(\boldsymbol{I}_n) &= \frac{1}{2\pi j}\oint_{|z|=1}[\overline{\boldsymbol{f}}_j(z)\overline{\boldsymbol{g}}_i(z)]^T\overline{\boldsymbol{f}}_j(z^{-1})\overline{\boldsymbol{g}}_i(z^{-1})\frac{dz}{z} \\ &= \boldsymbol{T}^T\boldsymbol{M}_{ij}(\boldsymbol{P})\boldsymbol{T} \end{aligned} \quad (12)$$

with $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{T}^T$ where

$$\boldsymbol{M}_{ij}(\boldsymbol{P}) = \frac{1}{2\pi j}\oint_{|z|=1}[\boldsymbol{f}_j(z)\boldsymbol{g}_i(z)]^T\boldsymbol{P}^{-1}\boldsymbol{f}_j(z^{-1})\boldsymbol{g}_i(z^{-1})\frac{dz}{z}.$$

Noting that

$$\begin{aligned} &\overline{\boldsymbol{f}}_j(z)\overline{\boldsymbol{g}}_i(z) \\ &= \boldsymbol{T}^{-1}\boldsymbol{f}_j(z)\boldsymbol{g}_i(z)\boldsymbol{T} \\ &= \begin{bmatrix} \boldsymbol{T}^{-1} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} z\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{b}_j\boldsymbol{c}_i \\ \boldsymbol{0} & z\boldsymbol{I}_n - \boldsymbol{A} \end{bmatrix}^{-1}\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{T} \end{bmatrix} \end{aligned} \quad (13)$$

and denoting the observability Gramian of a composite system in (13) by $\boldsymbol{Y}_{ij}(\boldsymbol{P})$, it can be shown that for an arbitrary $\boldsymbol{P}$, matrix $\boldsymbol{M}_{ij}(\boldsymbol{P})$ can be obtained by solving the Lyapunov equation

$$\begin{aligned} \boldsymbol{Y}_{ij}(\boldsymbol{P}) &= \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b}_j\boldsymbol{c}_i \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}^T\boldsymbol{Y}_{ij}(\boldsymbol{P})\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b}_j\boldsymbol{c}_i \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix} \\ &\quad + \begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \end{aligned} \quad (14)$$

and then taking the lower-right $n \times n$ block of $\boldsymbol{Y}_{ij}(\boldsymbol{P})$ as $\boldsymbol{M}_{ij}(\boldsymbol{P})$, i.e.,

$$\boldsymbol{M}_{ij}(\boldsymbol{P}) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_n \end{bmatrix}\boldsymbol{Y}_{ij}(\boldsymbol{P})\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_n \end{bmatrix}. \quad (15)$$

Thus, the L_2-sensitivity measure in (7) is changed to

$$\begin{aligned} S(\boldsymbol{P}) = &\sum_{i=1}^{p}\sum_{j=1}^{q}\mathrm{tr}[\boldsymbol{M}_{ij}(\boldsymbol{P})\,\boldsymbol{P}] + q\,\mathrm{tr}[\boldsymbol{W}_o\boldsymbol{P}] \\ &+ p\,\mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}]. \end{aligned} \quad (16)$$

From (2) and (10), it is clear that the transfer function $\boldsymbol{H}(z)$ is invariant under the coordinate transformation in (9).

Moreover, if the L_2-norm dynamic-range scaling constraints are imposed on the new state-variable vector $\overline{\boldsymbol{x}}(k)$, then it is required that for $i = 1, 2, \cdots, n$

$$(\overline{\boldsymbol{K}}_c)_{ii} = (\boldsymbol{T}^{-1}\boldsymbol{K}_c\boldsymbol{T}^{-T})_{ii} = 1. \quad (17)$$

The problem of minimizing an L_2-sensitivity measure subject to L_2-scaling constraints is now formulated as follows: *Given $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, obtain an $n \times n$ nonsingular matrix $\boldsymbol{T}$ which minimizes (16) subject to the scaling constraints in (17).*

III. L_2-SENSITIVITY MINIMIZATION

The problem of minimizing $S(\boldsymbol{P})$ in (16) subject to the constraints in (17) is a constrained nonlinear optimization problem where the variable is matrix $\boldsymbol{P}$. If we sum up the n constraints in (17), then we have

$$\mathrm{tr}[\boldsymbol{T}^{-1}\boldsymbol{K}_c\boldsymbol{T}^{-T}] = \mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] = n. \tag{18}$$

Consequently, the problem of minimizing (16) subject to the constraints in (17) can be *relaxed* into the following problem:

$$\begin{aligned} &\text{minimize } S(\boldsymbol{P}) \text{ in (16)} \\ &\text{subject to } \mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] = n. \end{aligned} \tag{19}$$

We now address problem (19) as the first step of our solution strategy. In order to solve (19), we define the Lagrange function of the problem as

$$\begin{aligned} J(\boldsymbol{P},\lambda) = &\sum_{i=1}^{p}\sum_{j=1}^{q}\mathrm{tr}[\boldsymbol{M}_{ij}(\boldsymbol{P})\boldsymbol{P}] + q\,\mathrm{tr}[\boldsymbol{W}_o\boldsymbol{P}] \\ &+ p\,\mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] + \lambda(\mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] - n) \end{aligned} \tag{20}$$

where λ is a Lagrange multiplier. It is well known that the solution of problem (19) must satisfy the Karush-Kuhn-Tucker (KKT) conditions $\partial J(\boldsymbol{P},\lambda)/\partial \boldsymbol{P} = \boldsymbol{0}$ and $\partial J(\boldsymbol{P},\lambda)/\partial\lambda = 0$ where the gradients are found to be

$$\begin{aligned} \frac{\partial J(\boldsymbol{P},\lambda)}{\partial \boldsymbol{P}} &= \sum_{i=1}^{p}\sum_{j=1}^{q}\boldsymbol{M}_{ij}(\boldsymbol{P}) + q\,\boldsymbol{W}_o \\ &\quad - \boldsymbol{P}^{-1}\sum_{i=1}^{p}\sum_{j=1}^{q}\boldsymbol{N}_{ij}(\boldsymbol{P})\boldsymbol{P}^{-1} \\ &\quad - (\lambda+p)\boldsymbol{P}^{-1}\boldsymbol{K}_c\boldsymbol{P}^{-1} \\ \frac{\partial J(\boldsymbol{P},\lambda)}{\partial\lambda} &= \mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] - n \end{aligned} \tag{21}$$

where $\boldsymbol{N}_{ij}(\boldsymbol{P})$ is obtained by solving the Lyapunov equation

$$\begin{aligned} \boldsymbol{Z}_{ij}(\boldsymbol{P}) = &\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b}_j\boldsymbol{c}_i \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix} \boldsymbol{Z}_{ij}(\boldsymbol{P}) \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b}_j\boldsymbol{c}_i \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}^T \\ &+ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{P} & \boldsymbol{0} \end{bmatrix} \end{aligned}$$

and then taking the upper-left $n \times n$ block of $\boldsymbol{Z}_{ij}(\boldsymbol{P})$, i.e.,

$$\boldsymbol{N}_{ij}(\boldsymbol{P}) = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \end{bmatrix} \boldsymbol{Z}_{ij}(\boldsymbol{P}) \begin{bmatrix} \boldsymbol{I}_n \\ \boldsymbol{0} \end{bmatrix}.$$

Hence the KKT conditions in (21) become

$$\boldsymbol{P}\,\boldsymbol{F}(\boldsymbol{P})\boldsymbol{P} = \boldsymbol{G}(\boldsymbol{P},\lambda), \qquad \mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] = n \tag{22}$$

where

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{P}) &= \sum_{i=1}^{p}\sum_{j=1}^{q}\boldsymbol{M}_{ij}(\boldsymbol{P}) + q\,\boldsymbol{W}_o \\ \boldsymbol{G}(\boldsymbol{P},\lambda) &= \sum_{i=1}^{p}\sum_{j=1}^{q}\boldsymbol{N}_{ij}(\boldsymbol{P}) + (\lambda+p)\boldsymbol{K}_c. \end{aligned}$$

The first equation in (22) is highly nonlinear with respect to $\boldsymbol{P}$. An effective approach to solving the first equation in (22) is to *relax* it into the following recursive second-order matrix equation:

$$\boldsymbol{P}_{k+1}\boldsymbol{F}(\boldsymbol{P}_k)\boldsymbol{P}_{k+1} = \boldsymbol{G}(\boldsymbol{P}_k,\lambda_k) \tag{23}$$

where $\boldsymbol{P}_k$ is assumed to be known from the previous recursion and the solution $\boldsymbol{P}_{k+1}$ is given by [10]

$$\boldsymbol{P}_{k+1} = \boldsymbol{F}(\boldsymbol{P}_k)^{-\frac{1}{2}}[\boldsymbol{F}(\boldsymbol{P}_k)^{\frac{1}{2}}\boldsymbol{G}(\boldsymbol{P}_k,\lambda_k)\boldsymbol{F}(\boldsymbol{P}_k)^{\frac{1}{2}}]^{\frac{1}{2}}\boldsymbol{F}(\boldsymbol{P}_k)^{-\frac{1}{2}}. \tag{24}$$

To derive a recursive formula for the Lagrange multiplier λ, we use (22) to write

$$\mathrm{tr}[\boldsymbol{P}\boldsymbol{F}(\boldsymbol{P})] = \sum_{i=1}^{p}\sum_{j=1}^{q}\mathrm{tr}[\boldsymbol{N}_{ij}(\boldsymbol{P})\boldsymbol{P}^{-1}] + n(\lambda+p) \tag{25}$$

which naturally suggests the following recursion for λ:

$$\lambda_{k+1} = \frac{\mathrm{tr}[\boldsymbol{P}_k\boldsymbol{F}(\boldsymbol{P}_k)] - \displaystyle\sum_{i=1}^{p}\sum_{j=1}^{q}\mathrm{tr}[\boldsymbol{N}_{ij}(\boldsymbol{P}_k)\boldsymbol{P}_k^{-1}]}{n} - p. \tag{26}$$

The initial estimates are given by $\boldsymbol{P}_0 = \boldsymbol{I}_n$ and any value of $\lambda_0 > 0$. This iteration process continues until (22) is satisfied within a prescribed numerical tolerance.

As the second step of the solution strategy, we now turn our attention to the construction of the optimal coordinate transformation matrix $\boldsymbol{T}$ that solves the problem of minimizing (16) subject to the constraints in (17). Since $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{T}^T$, the optimal $\boldsymbol{T}$ assumes the form

$$\boldsymbol{T} = \boldsymbol{P}^{\frac{1}{2}}\boldsymbol{U} \tag{27}$$

where $\boldsymbol{P}^{1/2}$ is the square root of the matrix $\boldsymbol{P}$ obtained above, and $\boldsymbol{U}$ is an $n \times n$ orthogonal matrix to be determined as follows. From (10) and (27) it follows that

$$\overline{\boldsymbol{K}}_c = \boldsymbol{T}^{-1}\boldsymbol{K}_c\boldsymbol{T}^{-T} = \boldsymbol{U}^T\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{K}_c\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{U}. \tag{28}$$

In order to find an $n \times n$ orthogonal matrix $\boldsymbol{U}$ such that the matrix $\overline{\boldsymbol{K}}_c$ in (28) satisfies the scaling constraints in (17), we perform the eigenvalue-eigenvector decomposition for the positive definite matrix $\boldsymbol{P}^{-1/2}\boldsymbol{K}_c\boldsymbol{P}^{-1/2}$ as

$$\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{K}_c\boldsymbol{P}^{-\frac{1}{2}} = \boldsymbol{R}\boldsymbol{\Theta}\boldsymbol{R}^T \tag{29}$$

where $\boldsymbol{\Theta} = \mathrm{diag}\{\theta_1, \theta_2, \cdots, \theta_n\}$ with $\theta_i > 0$ and $\boldsymbol{R}$ is an orthogonal matrix. Next, an orthogonal matrix $\boldsymbol{S}$ such that

$$S\Theta S^T = \begin{bmatrix} 1 & * & \cdots & * \\ * & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ * & \cdots & * & 1 \end{bmatrix} \tag{30}$$

can be obtained by numerical manipulations [12, p.278]. Using (28), (29) and (30), it can be readily verified that the orthogonal matrix $\boldsymbol{U} = \boldsymbol{R}\boldsymbol{S}^T$ leads to a $\overline{\boldsymbol{K}}_c$ in (28) whose diagonal elements are equal to unity, hence the constraints in (17) are now satisfied. This matrix $\boldsymbol{T}$ together with (27) gives the solution of the problem of minimizing (16) subject to the constraints in (17) as

$$\boldsymbol{T} = \boldsymbol{P}^{\frac{1}{2}}\boldsymbol{R}\boldsymbol{S}^T. \tag{31}$$

IV. NUMERICAL EXAMPLE

Consider a two-input/three-output linear discrete-time system $(\boldsymbol{A}_o, \boldsymbol{B}_o, \boldsymbol{C}_o, \boldsymbol{D})_n$ specified by

$$\boldsymbol{A}_o = \begin{bmatrix} 0 & 0 & 0.072 & 0 & 1.50 \\ 1 & 0 & 0.300 & 0 & 0.20 \\ 0 & 1 & -0.100 & 0 & 0.90 \\ 0 & 0 & 0 & 0 & 0.05 \\ 0 & 0 & 0 & 1 & 0.40 \end{bmatrix}, \quad \boldsymbol{B}_o = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{C}_o = \begin{bmatrix} 1.1 & 2.7 & 0.9 & 0.4 & 1.5 \\ 2.1 & 3.1 & 0.3 & 0.2 & 0.1 \\ 5.4 & 1.6 & -1.7 & -6.6 & 3.0 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 1.0 & 0.8 \\ 0.3 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}.$$

After carrying out the L_2-scaling for the above system with a diagonal coordinate transformation matrix, the L_2-sensitivity of the scaled system was computed as $S = 3.130822 \times 10^4$. When applying the iterative algorithm in (24) and (26) to the scaled system, the profiles of the L_2-sensitivity, parameter λ, as well as $\mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}]$ during the first 200 iterations of the algorithm are shown in Figs. 1 and 2, respectively, where $S(\boldsymbol{P}) = 2.065454 \times 10^4$ at $k = 200$.

V. CONCLUSION

The minimization problem of an L_2-sensitivity measure subject to L_2-scaling constraints has been investigated for MIMO linear discrete-time systems. An efficient iterative technique has been developed by using a Lagrange function and some matrix-theoretic techniques. This makes it possible to solve the constraint optimization problem directly. Our computer simulation results have demonstrated the effectiveness of the proposed technique.

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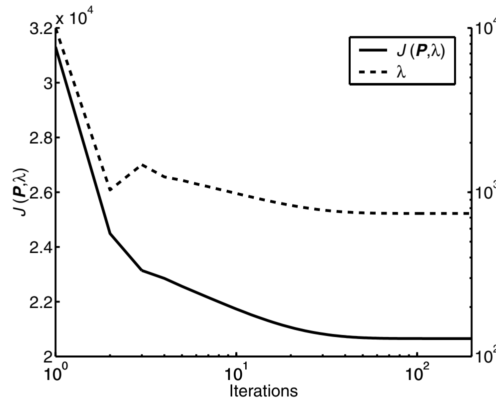

Fig. 1. L_2-Sensitivity and λ Performances.

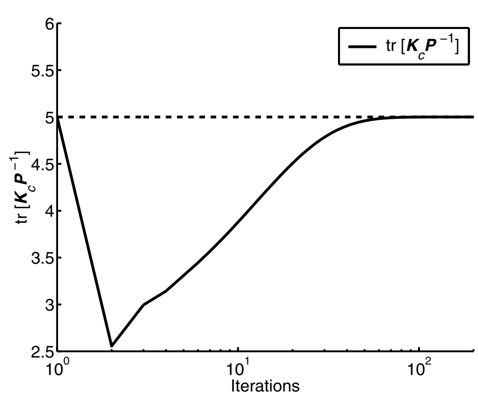

Fig. 2. $\mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}]$ Performance.