Design of FIR Filters with Discrete Coefficients via Polynomial Programming: Towards the Global Solution

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***Abstract*— Polynomial programming (PP) deals with a class of optimization problems where both the objective function and constraint functions are multivariable polynomials. PP covers several popular classes of convex optimization problems such as linear, convex quadratic, semidefinite, and second-order cone programming problems, it also includes a good many *non-convex* problems that are encountered in engineering analysis and design. This paper describes a preliminary attempt to apply a recently developed PP algorithm to the design of FIR digital filters with discrete coefficients. Computer simulations are presented to demonstrate the efficiency of the PP-based algorithm and its ability to provide globally or near-globally optimal designs.**

I. Introduction

We in this paper are concerned with the design of FIR digital filters with discrete coefficients. Filters with sum-of-power-of-2 coefficients admit fast implementations which require no conventional multiplications but superposition of shifted versions of the input. On the other hand, the design of a general optimum filter with discrete coefficients is known to be NP-hard as it is essentially an integer programming problem [1]. It is the importance of the design problem and the technical challenge encountered that have attracted researchers' attention for more than two decades [2]–[17].

In this paper, we describe a preliminary attempt to apply a polynomial programming (PP) algorithm to derive a solution to the design problem in question. PP deals with a class of constrained optimization problems where both the objective function and constraint functions are multivariable polynomials. It covers several popular classes of convex optimization problems such as linear, convex quadratic, semidefinite, and second-order cone programming problems, it also includes a good many nonconvex problems that are encountered in engineering analysis and design. A brief introduction of PP formulation and its potential application to several filter design problems are presented recently [17], while this paper is focused on filters where each coefficient is a sum of signed power of 2 (SP2) terms. The main components of the paper include a formulation of the design problem as a quadratic optimization with discrete constraints; a review of the PP formulation in connection with the problem at hand; software implementation and computer simulations to demonstrate the efficiency of the algorithm and its ability to provide globally or near-globally optimal designs.

II. Polynomial Programming

A. Polynomial Optimization Problems

Polynomial programming deals with the constrained optimization problem

$$\text{minimize} \quad p_0(\boldsymbol{x}) \tag{1a}$$

$$\text{subject to:} \quad p_k(\boldsymbol{x}) \geq 0 \quad \text{for } k = 1,\ 2,\ \ldots,\ K \tag{1b}$$

where $p_k(\boldsymbol{x})$ for $k = 0,\ 1,\ \ldots,\ K$ are real-valued polynomials, and $\boldsymbol{x} = [x_1\ \ x_2\ \ \cdots\ \ x_n]^T$ is a vector of n real variables. Each polynomial in (1) assumes the form

$$p(\boldsymbol{x}) = \sum_{d \in \mathcal{F}} c(\alpha)\boldsymbol{x}^{\boldsymbol{\alpha}} \tag{2}$$

where $\boldsymbol{\alpha} = [\alpha_1\ \ \alpha_2\ \ \cdots\ \ \alpha_n] \in \mathcal{F} \subset \mathcal{Z}_+^n$ — the set of vectors with nonnegative integer components, $c(\boldsymbol{\alpha}) \in R$ and $\boldsymbol{x}_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. The degree (order) of $p(\boldsymbol{x})$ is defined to be the largest $\Sigma_i \alpha_i$ amongst all $\boldsymbol{x}^{\boldsymbol{\alpha}}$ appeared in (2). Because linear programming, quadratic programming and second-order cone programming problems only involve polynomials in their formulations, they are subclasses of PP. A semidefinite programming (SDP) problem involves positive semidefiniteness of the matrix variable, since the positive definiteness can be characterized as all principal minors of the matrix being nonnegative, and minors of a matrix are polynomials of its entries, SDP is also a subclass of PP. Since a polynomial can be nonconvex and the region defined by a set of polynomials can be nonconvex as well, PP also covers a great many nonconvex optimization problems. In [17] it is pointed out that several filter design problems of current interest, including the design of stable IIR filters, frequency-response-masking filters, and FIR and IIR filters with SP2 coefficients, are nonconvex in nature, and they fit nicely into the PP framework.

B. Global Solution of Problem (1)

Studies on solution methods for problem (1) are relatively recent. Shor [18] is among the first, along the way important progress has been made by several authors [19]–[24]. In particular, Lasserre [21] shows that when the feasible region defined by (1b) is compact (not necessarily convex), the global solution of (1) can be asymptotically approached by the solutions of a sequence of SDP problems which are formulated by relaxing problem (1). Noticeable features of this solution methodology are: (i) In theory the global solution can only be approached asymptotically, in practice however the exact minimizer can often be obtained by solving a *finite* sequence of SDP problems. This is especially the case for polynomial minimization subject to discrete constraints, e.g., the quadratic 0-1 programs [22]; (ii) The size of the SDP problems involved grows very quickly that may lead to numerical difficulties even for PP problems of moderate scales [21], [22]. This is obviously an issue relating to software implementation of PP-based algorithms. We shall come back to this in Sec. IV where simulation results are presented.

C. SDP Relaxation of Polynomial Optimization Problems: An Example

The basic idea behind the SDP relaxation of a general polynomial optimization problem is linearization of the polynomials involved by introducing new variables and imposing additional linear and/or SDP constraints. These techniques, especially the later one, can be quite tricky and involved. The example given below illustrates these steps, and we refer the reader to references [21], [22] for the technical details.

Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & x_1 - 2x_2 && \text{(3a)}\\ \text{subject to:} \quad & x_1 \geq 0,\ x_2 \geq 0 && \text{(3b)}\\ & (x_1-1)^2 + x_2^2 \leq 1 && \text{(3c)}\\ & (x_1-1)^2 + (x_2-1)^2 \geq 1 && \text{(3d)} \end{aligned}$$

As can be seen from Fig.1, the feasible region defined by (3b)–(3d) is not convex, thus (3) is a nonconvex PP problem.

By introducing new variables $y_{20} = x_1^2$ and $y_{02} = x_2^2$, the constraints in (3c) and (3d) can be written as

$$-y_{20} + 2x_1 - y_{02} \geq 0 \tag{4a}$$

and

$$y_{20} - 2x_1 + y_{02} - 2x_2 + 1 \geq 0 \tag{4b}$$

respectively. In this way, all constraints are linearized. The linear programming problem so obtained offers an approximate solution for problem (3) as $(x_1,\ x_2) = (0,\ 0.5)$ which is reasonably close to the global minimizer (see Fig. 1), but this approximate solution is not feasible as it violates constraint (3c). This means that additional variables and adequate constraints are needed to construct a more useful yet convex problem. To this end, we define $y_{11} = x_1x_2$ and note that $y_{20} \geq 0$, $y_{11} \geq 0$ (because $x_1 \geq 0,\ x_2 \geq 0$), and $y_{02} \geq 0$. Moreover, note that matrix $[1 \quad x_1 \quad x_2]^T[1 \quad x_1 \quad x_2]$ is always positive semidefinite. In terms of the new variable set $[x_1 \quad x_2 \quad y_{20} \quad y_{11} \quad y_{02}]$, the above matrix becomes linear as

$$\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & y_{20} & y_{11} \\ x_2 & y_{11} & y_{02} \end{bmatrix}$$

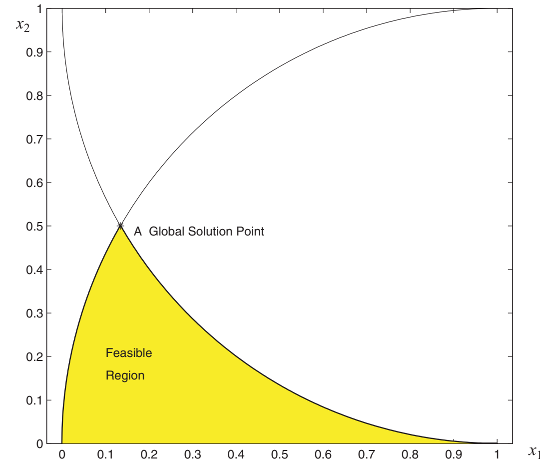

Fig. 1. Feasible region for problem (3). It can be easily verified that point A with coordinates (0.1340, 0.5) is the global minimizer.

This leads to an SDP relaxation of problem (3) as

$$\begin{aligned} \text{minimize} \quad & x_1 - 2x_2 && \text{(5a)}\\ \text{subject to:} \quad & x_1 \geq 0,\ x_2 \geq 0,\ y_{20} \geq 0,\ y_{11} \geq 0 && \text{(5b)}\\ & y_{02} \geq 0,\ -y_{20} + 2x_1 - y_{02} \geq 0 && \text{(5c)}\\ & y_{20} - 2x_1 + y_{02} - 2x_2 + 1 \geq 0 && \text{(5d)}\\ & \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & y_{20} & y_{11} \\ x_2 & y_{11} & y_{02} \end{bmatrix} \succeq \mathbf{0} && \text{(5e)} \end{aligned}$$

where "$\succeq \mathbf{0}$" means positive semidefinite. The relaxed problem in (5) has five variables and eight constrains. The $(x_1,\ x_2)$ components of the solution of (5) was found to be $(x_1,\ x_2) = (0.1340,\ 0.5)$ which is the exact global solution of problem (3), see Fig.1. We see that by introducing additional variables and constraints, a nonconvex problem can be relaxed into an SDP problem of increased scale, through which the global solution of the original nonconvex problem may be identified.

III. Design of FIR Filters with SP2 Coefficients

A. A Least-Squares $\{-1,\ 1\}$-Optimization Formulation

We seek to find a linear-phase FIR filter of length N whose transfer function assumes the form

$$H(z) = \sum_{k=0}^{N-1} d_k z^{-k}$$

For notation simplicity, let the filter length be an odd integer, thus the corresponding zero-phase frequency response can be written as

$$A(\omega) = \sum_{k=0}^{n} a_k \cos k\omega, \quad n = (N-1)/2 \tag{6}$$

The design problem at hand is to find PS2 coefficients $\{a_k\}$ so that $A(\omega)$ best approximates a desired frequency response $A_d(\omega)$ subject to a bit budget for the representation of $\{a_k\}$.

Following the usual notation in [16], let

$$\hat{A}(\omega) = \sum_{k=0}^{n} \hat{a}_k \cos k\omega \tag{7}$$

be the optimal zero-phase frequency response with infinite-precision coefficients $\{\hat{a}_k\}$, which can readily be obtained using a well-established least-squares design algorithm. For each $\hat{a}_k$ and the number of SP2 terms allocated to the kth discrete coefficient, let $\underline{a}_k$ and $\bar{a}_k$ be the largest SP2 lower bound and smallest SP2 upper bound of $\hat{a}_k$, respectively. The candidate SP2 coefficients $\{a_k\}$ can then be expressed as

$$\begin{aligned} a_k &= a_{mk} + x_k\delta_k \quad \text{for } k = 0,\ 1,\ \ldots, n, \\ a_{mk} &= (\bar{a}_k + \underline{a}_k)/2 \\ \delta_k &= (\bar{a}_k - \underline{a}_k)/2 \\ x_k &\in \{-1,\ 1\} \end{aligned}$$

It follows that the filter's zero-phase frequency response can be written in terms of $\boldsymbol{x} = [x_0 \quad x_1 \quad \cdots \quad x_n]^T$ as

$$\begin{aligned} A(\omega) &= A_m(\omega) + \boldsymbol{x}^T\boldsymbol{c}(\omega) \\ A_m(\omega) &= \sum_{k=0}^{n} a_{mk} \cos k\omega \\ \boldsymbol{c}(\omega) &= [\delta_0 \quad \delta_1 \cos\omega \quad \cdots \quad \delta_n \cos n\omega]^T \end{aligned}$$

and the least-squares design problem can be formulated as

$$\text{minimize} \quad \int_{-\pi}^{\pi} W(\omega)[A_m(\omega) + \boldsymbol{x}^T\boldsymbol{c}(\omega) - A_d(\omega)]^2 \tag{8a}$$

$$\text{subject to:} \quad x_k \in \{-1,\ 1\} \quad \text{for } k = 0,\ \ldots,\ n \tag{8b}$$

B. A Polynomial Programming Formulation

The objective function in (8a) is a convex quadratic function, thus we can write (8) more compactly as

$$\text{minimize} \quad p(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}^T\boldsymbol{x} \tag{9a}$$

$$\text{subject to:} \quad x_k \in \{-1,\ 1\} \tag{9b}$$

where

$$\boldsymbol{Q} = \int_{-\pi}^{\pi} W(\omega)\boldsymbol{c}(\omega)\boldsymbol{c}^T(\omega)d\omega$$

$$\boldsymbol{q} = 2\int_{-\pi}^{\pi} W(\omega)[A_m(\omega) - A_d(\omega)]\boldsymbol{c}(\omega)d\omega$$

and the constant term in (8a) has been neglected in (9a). Concerning the discrete constraints in (9b), we note that $x_k \in \{-1,\ 1\}$ can be characterized as $x_k^2 = 1$ which is in turn equivalent to $x_k^2 - 1 \geq 0$ *and* $-x_k^2 + 1 \geq 0$. Consequently, problem (9) becomes

$$\text{minimize} \quad p(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}^T\boldsymbol{x} \tag{10a}$$

$$\text{subject to:} \quad p_k(\boldsymbol{x}) = x_k^2 - 1 \geq 0 \tag{10b}$$

$$p_{n+1+k}(\boldsymbol{x}) = -x_k^2 + 1 \geq 0 \tag{10c}$$

$$\text{for } k = 0,\ 1,\ \ldots,\ n$$

Problem (10) is to minimize a quadratic (and convex) polynomial objective function subject to a total of $2(n+1)$ 2nd-order polynomial constraints that fits nicely into the formulation in (1), hence it is a PP problem.

IV. Software and Simulation Results

A. Software for Polynomial Programming

Two easy-to-use, MATLAB-compatible software implementations of the SDP relaxation methods reported in [21], [22] and [24] are available as public-domain shareware [25], [26]. Glopti Poly [26] is in principle applicable to problem (10). However, since the number of variables is limited to 19 in its current version, the simulation results reported below were obtained using SparsePOP [25].

B. Design Exmaples

The SDP-relaxation algorithm for PP [21], [22] was applied to design a total of 25 linear-phase lowpass FIR filters with SP2 coefficients and odd lengths from $N = 5$ to $N = 53$. The normalized passband and stopband edges were set to $\omega_p = 0.4\pi$ and $\omega_a = 0.5\pi$, respectively. The weighting function was set to $W(\omega) \equiv 1$ for both passband and stopband, and $W(\omega) \equiv 0$ elsewhere. A critical parameter in the algorithm is the *relaxation order* which starts with the smallest value being an half of the highest order of all the polynomials involved. In the present case this relaxation order starts with one. In case the global minimizer cannot be identified, the relaxation order should start with a larger value, but this would in turn cause a considerable increase in the size of the subsequent SDP sequence, leading to increase in CPU time as well as computer memory problems.

For these reasons, in our simulations the relaxation order was set to one for all the designs conducted. As a result, global solutions may not always be reached. It should be stressed that with the support of the rigorous theory [21], [22], this is merely a technical difficulty that shall be overcome as the algorithm gets more efficient and numerically more robust. For the first nine designs with $N = 5,\ 7,\ \ldots,\ 21$, the SDP-relaxation algorithm implemented by SparsePOP all converged to the global designs. This was verified by comparing them to the designs obtained by exhaustive global search. For the remaining sixteen designs with length $N = 23,\ 25,\ \ldots,\ 53$, five of them were globally optimal and the rest of eleven designs were only suboptimal. The numerical results in terms of L_2 approximation error in (8a) and CPU time (in seconds) for a given filter length N and a bit budget M for the last 16 designs are given in Table 1. For comparison purposes, the L_2-error and CPU time consumed by global search to obtain the global solutions are also given in the table. It is observed that the SDP relaxation method is able to provide excellent designs that are either exactly globally optimal or satisfactory suboptimal with a very low design complexity. As a sample plot, the amplitude responses of the FIR filter of length $N = 53$ and $M = 68$ bits of the SDP relaxation design versus globally optimal design are shown in Fig. 2.

TABLE I

PERFORMANCE OF THE SDP-RELAXATION DESIGN METHOD

Length N	Bit # M	SDP Relaxation		Global Search	
		L_2-error	CPU	L_2-error	CPU
23	17	0.0025	1.55	0.0025	6.97
25	18	0.0021	1.60	0.0021	10.22
27	20	0.0012	1.63	0.0012	10.58
29	24	0.8186e−3	1.67	0.7683e-3	10.92
31	25	0.7210e−3	1.80	0.7177e-3	11.19
33	28	0.3732e−3	1.92	0.3637e-3	11.98
35	29	0.3626e−3	2.10	0.3242e-3	13.89
37	31	0.3519e−3	2.22	0.3048e-3	17.31
39	33	0.2896e−3	2.48	0.2590e-3	24.31
41	38	0.2368e−3	2.86	0.2265e-3	38.22
43	41	0.1622e−3	3.25	0.1622e-3	67.06
45	45	0.7125e−4	3.52	0.6870e-4	123.56
47	48	0.3239e−4	3.94	0.3052e-4	237.41
49	55	0.1686e−4	4.05	0.1421e-4	470.64
51	65	0.1148e−4	5.45	0.1148e-4	928.94
53	68	0.9940e−5	6.34	0.9685e-5	1.91e3

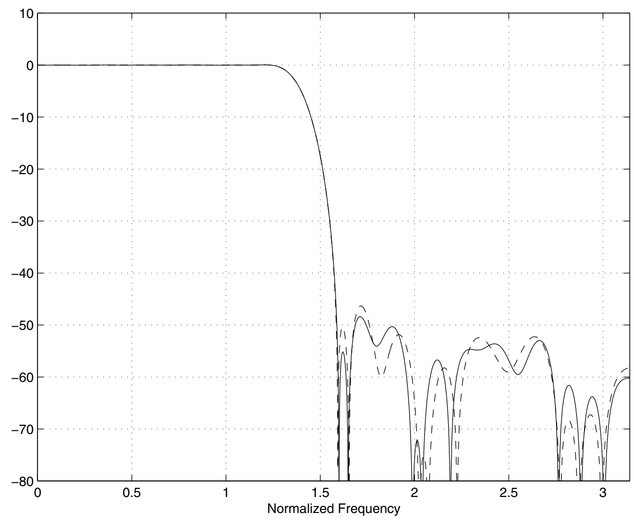

Fig. 2. Amplitude responses of the FIR filter with $N = 53$ and $M = 68$: SDP-relaxation design (solid line) versus globally optimal design (dashed line).

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