On Frequency-Weighted l_2-Sensitivity Analysis and Minimization of 2-D State-Space Digital Filters Subject to l_2-Scaling Constraints

Takao Hinamoto, Toru Oumi, and Osemekhian I. Omoifo
Graduate School of Engineering
Hiroshima University
Higashi-Hiroshima 739-8527, Japan
Email: {hinamoto, oumi, osei}@hiroshima-u.ac.jp

Wu-Sheng Lu
Dept. of Elec. and Comp. Engineering
University of Victoria
Victoria, BC, Canada V8W 3P6
Email: wslu@ece.uvic.ca

***Abstract*— The minimization problem of a frequency-weighted l_2-sensitivity measure subject to l_2-scaling constraints for two-dimensional (2-D) state-space digital filters is formulated. First, an iterative method for solving the constrained optimization problem in question is developed by introducing a Lagrange function and applying some matrix-theoretic techniques as well as an efficient bisection method. The optimal filter structure is then synthesized so as to minimize the frequency-weighted l_2-sensitivity subject to l_2-scaling constraints. Finally, a numerical example is presented to illustrate the utility of the proposed technique.**

I. INTRODUCTION

When a transfer function with infinite accuracy coefficients that have met the design specifications is given, its state-space model with the transfer function is actually implemented using a finite binary representation. In such a case, the truncation or rounding of the coefficients in the state-space model is required to fit the finite word length (FWL) constraints. Consequently, the characteristics of a stable filter might be so altered that the filter may become unstable. This motivates the study of the coefficient sensitivity minimization problem. The problem of synthesizing 2-D state-space digital filter structures with minimum coefficient sensitivity has been investigated, i.e., the l_1/l_2-mixed sensitivity minimization problem [1]-[6] and l_2-sensitivity minimization problem [6]-[10]. It has been realized that solutions for *frequency-weighted* sensitivity minimization would be of practical use as these solutions allow to emphasize or de-emphasize the filter's sensitivity in certain frequency regions of interest. Synthesis procedures of the optimal FWL 2-D filtes that minimize the frequency weighted sensitivity measure have been considered [4]-[7]. However, the minimization methods proposed in the above work do not impose constraints on the scaling of the design variables. As a result, elimination of overflow oscillations can not be ensured. More recently, the minimization problem of l_2-sensitivity subject to l_2-scaling constraints has been explored for a class of 2-D state-space digital filters [11]. It is well known that the use of scaling constraints can be beneficial for suppressing overflow oscillations [12],[13]. However, frequency-weighted sensitivity measure has not yet been considered in [11].

In this paper, we investigate the problem of minimizing a frequency-weighted l_2-sensitivity measure subject to l_2-scaling constraints for 2-D state-space digital filters described by the Roesser local state-space (LSS) model [14]. First, the minimization problem of a frequency-weighted l_2-sensitivity measure subject to l_2-scaling constraints is formulated by introducing a Lagrange function. Next, an iterative method is developed by using some matrix-theoretic techniques and an efficient bisection method. The optimal filter structure with minimum frequency-weighted l_2-sensitivity and no overflow oscillations is then constructed by applying an appropriate coordinate transformation matrix. Finally, a numerical example is presented to demonstrate the validity and effectiveness of the proposed technique.

II. l_2-SENSITIVITY ANALYSIS

Consider a LSS model $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)_{m,n}$ for 2-D IIR digital filters which is stable, separately locally controllable and separately locally observable [14],[15]

$$\begin{aligned} \boldsymbol{x}_{11}(i,j) &= \boldsymbol{A}\boldsymbol{x}(i,j) + \boldsymbol{b}u(i,j) \\ y(i,j) &= \boldsymbol{c}\boldsymbol{x}(i,j) + du(i,j) \end{aligned} \tag{1}$$

where

$$\boldsymbol{x}_{11}(i,j) = \begin{bmatrix} \boldsymbol{x}^h(i+1,j) \\ \boldsymbol{x}^v(i,j+1) \end{bmatrix}, \qquad \boldsymbol{x}(i,j) = \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}, \qquad \boldsymbol{c} = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 \end{bmatrix}$$

with an $m \times 1$ horizontal state vector $\boldsymbol{x}^h(i,j)$, an $n \times 1$ vertical state vector $\boldsymbol{x}^v(i,j)$, a scalar input $u(i,j)$, a scalar output $y(i,j)$, and real constant matrices $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_3$, $\boldsymbol{A}_4$, $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{c}_1$, $\boldsymbol{c}_2$ and d of appropriate dimensions. The transfer function of the LSS model in (1) is given by

$$H(z_1, z_2) = \boldsymbol{c}(\boldsymbol{Z} - \boldsymbol{A})^{-1}\boldsymbol{b} + d \tag{2}$$

where $\boldsymbol{Z} = z_1\boldsymbol{I}_m \oplus z_2\boldsymbol{I}_n$.

Definition 1 : Let $\boldsymbol{X}$ be an $m \times n$ real matrix and let $f(\boldsymbol{X})$ be a scalar complex function of $\boldsymbol{X}$, differentiable with respect to all the entries of $\boldsymbol{X}$. The sensitivity function of $f(\boldsymbol{X})$ with respect to $\boldsymbol{X}$ is then defined as

$$\boldsymbol{S}_{\boldsymbol{X}} = \frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}}, \qquad (\boldsymbol{S}_{\boldsymbol{X}})_{ij} = \frac{\partial f(\boldsymbol{X})}{\partial x_{ij}} \tag{3}$$

where x_{ij} denotes the (i,j)th entry of matrix $\boldsymbol{X}$.

Definition 2: In order to take into account the sensitivity behavior of the transfer function in a specified frequency band, or even at some discrete frequency points, the weighted sensitivity functions are defined as

$$\begin{aligned}\frac{\delta H(z_1,z_2)}{\delta \boldsymbol{A}} &= W_A(z_1,z_2)\frac{\partial H(z_1,z_2)}{\partial \boldsymbol{A}}\\ \frac{\delta H(z_1,z_2)}{\delta \boldsymbol{b}} &= W_B(z_1,z_2)\frac{\partial H(z_1,z_2)}{\partial \boldsymbol{b}}\\ \frac{\delta H(z_1,z_2)}{\delta \boldsymbol{c}^T} &= W_C(z_1,z_2)\frac{\partial H(z_1,z_2)}{\partial \boldsymbol{c}^T}\end{aligned} \tag{4}$$

where $W_A(z_1,z_2)$, $W_B(z_1,z_2)$, and $W_C(z_1,z_2)$ are scalar, stable, causal functions of the complex variables z_1 and z_2.

Notice that δ in (4) is not meant to be a derivative operator, but rather a notation for defining the weighted parameter sensitivity.

Definition 3: Let $\boldsymbol{X}(z_1,z_2)$ be an $m\times n$ complex matrix valued function of the complex variables z_1 and z_2. The l_2 norm of $\boldsymbol{X}(z_1,z_2)$ is then defined by

$$\begin{aligned}&||\boldsymbol{X}(z_1,z_2)||_2\\ &=\left(\text{tr}\left[\frac{1}{(2\pi j)^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\boldsymbol{X}(z_1,z_2)\boldsymbol{X}^*(z_1,z_2)\frac{dz_1dz_2}{z_1z_2}\right]\right)^{\frac{1}{2}}\end{aligned} \tag{5}$$

where $\Gamma_i=\{z_i:|z_i|=1\}$ for $i=1,2$.

From Definitions 1-3, the overall weighted l_2-sensitivity measure for the LSS model in (1) can be evaluated by

$$\begin{aligned}S &= \left\|\frac{\delta H(z_1,z_2)}{\delta \boldsymbol{A}}\right\|_2^2+\left\|\frac{\delta H(z_1,z_2)}{\delta \boldsymbol{b}}\right\|_2^2+\left\|\frac{\delta H(z_1,z_2)}{\delta \boldsymbol{c}^T}\right\|_2^2\\ &= \left\|W_A(z_1,z_2)[\boldsymbol{F}(z_1,z_2)\boldsymbol{G}(z_1,z_2)]^T\right\|_2^2\\ &\quad+\left\|W_B(z_1,z_2)\boldsymbol{G}^T(z_1,z_2)\right\|_2^2+\left\|W_C(z_1,z_2)\boldsymbol{F}(z_1,z_2)\right\|_2^2\end{aligned} \tag{6}$$

where $\boldsymbol{F}(z_1,z_2)=(\boldsymbol{Z}-\boldsymbol{A})^{-1}\boldsymbol{b}$, $\boldsymbol{G}(z_1,z_2)=\boldsymbol{c}\,(\boldsymbol{Z}-\boldsymbol{A})^{-1}$. The weighted l_2-sensitivity measure in (6) is then written as

$$S=\text{tr}[\boldsymbol{M}_A]+\text{tr}[\boldsymbol{W}_B]+\text{tr}[\boldsymbol{K}_C] \tag{7}$$

where $\boldsymbol{M}_A$, $\boldsymbol{W}_B$, and $\boldsymbol{K}_C$ are obtained by the following general expression:

$$\boldsymbol{X}=\frac{1}{(2\pi j)^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\boldsymbol{Y}(z_1,z_2)\boldsymbol{Y}^*(z_1,z_2)\frac{dz_1dz_2}{z_1z_2}$$

with $\boldsymbol{Y}(z_1,z_2)=W_A(z_1,z_2)[\boldsymbol{F}(z_1,z_2)\boldsymbol{G}(z_1,z_2)]^T$ for $\boldsymbol{X}=\boldsymbol{M}_A$, $\boldsymbol{Y}(z_1,z_2)=W_B^*(z_1,z_2)\boldsymbol{G}^*(z_1,z_2)$ for $\boldsymbol{X}=\boldsymbol{W}_B$, and $\boldsymbol{Y}(z_1,z_2)=W_C(z_1,z_2)\boldsymbol{F}(z_1,z_2)$ for $\boldsymbol{X}=\boldsymbol{K}_C$.

III. l_2-SENSITIVITY MINIMIZATION

Let a 2-D coordinate transformation be defined by

$$\overline{\boldsymbol{x}}(i,j)=\boldsymbol{T}^{-1}\boldsymbol{x}(i,j) \tag{8}$$

where $\boldsymbol{T}=\boldsymbol{T}_1\oplus\boldsymbol{T}_4$ is a block-diagonal nonsingular matrix with an $m\times m$ submatrix $\boldsymbol{T}_1$ and an $n\times n$ submatrix $\boldsymbol{T}_4$. Then we obtain a new realization $(\overline{\boldsymbol{A}},\overline{\boldsymbol{b}},\overline{\boldsymbol{c}},d)_{m,n}$ characterized by

$$\overline{\boldsymbol{A}}=\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T},\quad \overline{\boldsymbol{b}}=\boldsymbol{T}^{-1}\boldsymbol{b},\quad \overline{\boldsymbol{c}}=\boldsymbol{c}\boldsymbol{T}. \tag{9}$$

Applying the coordinate transformation in (8) to the LSS model in (1), the weighted l_2-sensitivity measure in (7) is changed to

$$S(\boldsymbol{T})=\text{tr}[\boldsymbol{T}^T\boldsymbol{M}_A(\boldsymbol{T})\boldsymbol{T}]+\text{tr}[\boldsymbol{T}^T\boldsymbol{W}_B\boldsymbol{T}]+\text{tr}[\boldsymbol{T}^{-1}\boldsymbol{K}_C\boldsymbol{T}^{-T}] \tag{10}$$

where

$$\boldsymbol{M}_A(\boldsymbol{T})=\frac{1}{(2\pi j)^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\overline{\boldsymbol{Y}}(z_1,z_2)\overline{\boldsymbol{Y}}^*(z_1,z_2)\frac{dz_1dz_2}{z_1z_2}$$

with $\overline{\boldsymbol{Y}}(z_1,z_2)=W_A(z_1,z_2)[\boldsymbol{F}(z_1,z_2)\boldsymbol{G}(z_1,z_2)]^T\boldsymbol{T}^{-T}$.

It is noted that the local controllability Gramian $\boldsymbol{K}$ for the LSS model in (1) is defined by

$$\boldsymbol{K}=\frac{1}{(2\pi j)^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\boldsymbol{F}(z_1,z_2)\boldsymbol{F}^*(z_1,z_2)\frac{dz_1dz_2}{z_1z_2} \tag{11}$$

which is related to the local controllability Gramian $\overline{\boldsymbol{K}}$ for the new realization $(\overline{\boldsymbol{A}},\overline{\boldsymbol{b}},\overline{\boldsymbol{c}},d)_{m,n}$ in (9) by

$$\overline{\boldsymbol{K}}=\boldsymbol{T}^{-1}\boldsymbol{K}\boldsymbol{T}^{-T}. \tag{12}$$

If l_2-scaling constraints are imposed on the new local state vector $\overline{\boldsymbol{x}}(i,j)$, then it is required that

$$\begin{aligned}(\overline{\boldsymbol{K}}_1)_{ii} &= (\boldsymbol{T}_1^{-1}\boldsymbol{K}_1\boldsymbol{T}_1^{-T})_{ii}=1 \;\text{ for }\; i=1,2,\cdots,m\\ (\overline{\boldsymbol{K}}_4)_{jj} &= (\boldsymbol{T}_4^{-1}\boldsymbol{K}_4\boldsymbol{T}_4^{-T})_{jj}=1 \;\text{ for }\; j=1,2,\cdots,n\end{aligned} \tag{13}$$

where

$$\boldsymbol{K}=\begin{bmatrix}\boldsymbol{K}_1 & \boldsymbol{K}_2\\ \boldsymbol{K}_3 & \boldsymbol{K}_4\end{bmatrix}$$

with an $m\times m$ submatrix $\boldsymbol{K}_1$ and an $n\times n$ submatrix $\boldsymbol{K}_4$ along its diagonal.

The minimization problem of the weighted l_2-sensitivity subject to l_2-scaling constraints is now formulated as follows: *Given matrices $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, obtain an $(m+n)\times(m+n)$ block-diagonal nonsingular matrix $\boldsymbol{T}=\boldsymbol{T}_1\oplus\boldsymbol{T}_4$ which minimizes $S(\boldsymbol{T})$ in (10) subject to l_2-scaling constraints in (13).*

The l_2-sensitivity measure $S(\boldsymbol{T})$ in (10) is expressed as

$$\begin{aligned}S_o(\boldsymbol{P}) &= \text{tr}[\boldsymbol{M}(\boldsymbol{P})\boldsymbol{P}]+\text{tr}[\boldsymbol{W}_B\boldsymbol{P}]+\text{tr}[\boldsymbol{K}_C\boldsymbol{P}^{-1}]\\ &= \text{tr}[\boldsymbol{N}(\boldsymbol{P})\boldsymbol{P}^{-1}]+\text{tr}[\boldsymbol{W}_B\boldsymbol{P}]+\text{tr}[\boldsymbol{K}_C\boldsymbol{P}^{-1}]\end{aligned} \tag{14}$$

where $\boldsymbol{P}=\boldsymbol{P}_1\oplus\boldsymbol{P}_4$ with $\boldsymbol{P}_i=\boldsymbol{T}_i\boldsymbol{T}_i^T$ for $i=1,4$ and

$$\begin{aligned}\boldsymbol{M}(\boldsymbol{P}) &= \frac{1}{(2\pi j)^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\boldsymbol{Y}(z_1,z_2)\boldsymbol{P}^{-1}\boldsymbol{Y}^*(z_1,z_2)\frac{dz_1dz_2}{z_1z_2}\\ \boldsymbol{N}(\boldsymbol{P}) &= \frac{1}{(2\pi j)^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\boldsymbol{Y}^*(z_1,z_2)\boldsymbol{P}\boldsymbol{Y}(z_1,z_2)\frac{dz_1dz_2}{z_1z_2}\end{aligned}$$

with $\boldsymbol{Y}(z_1,z_2)=W_A(z_1,z_2)[\boldsymbol{F}(z_1,z_2)\boldsymbol{G}(z_1,z_2)]^T$. If we sum up the two sets of constraints in (13), respectively, then

$$\begin{aligned}\text{tr}[\boldsymbol{T}_1^{-1}\boldsymbol{K}_1\boldsymbol{T}_1^{-T}] &= \text{tr}[\boldsymbol{K}_1\boldsymbol{P}_1^{-1}]=m\\ \text{tr}[\boldsymbol{T}_4^{-1}\boldsymbol{K}_4\boldsymbol{T}_4^{-T}] &= \text{tr}[\boldsymbol{K}_4\boldsymbol{P}_4^{-1}]=n.\end{aligned} \tag{15}$$

Consequently, the problem of minimizing $S_o(\boldsymbol{P})$ in (14) subject to l_2-scaling constraints in (13) can be *relaxed* into the problem

$$\begin{array}{l} \text{minimize } S_o(\boldsymbol{P}) \text{ in (14)} \\ \text{subject to } \text{tr}[\boldsymbol{K}_1\boldsymbol{P}_1^{-1}] = m \text{ and } \text{tr}[\boldsymbol{K}_4\boldsymbol{P}_4^{-1}] = n. \end{array} \tag{16}$$

Under the constraints $\text{tr}[\boldsymbol{K}_1\boldsymbol{P}_1^{-1}] = m$ and $\text{tr}[\boldsymbol{K}_4\boldsymbol{P}_4^{-1}] = n$ in (15), there exist an $m \times m$ orthogonal matrix $\boldsymbol{U}_1$ and an $n \times n$ orthogonal matrix $\boldsymbol{U}_4$ such that matrix $\boldsymbol{T} = \boldsymbol{P}_1^{1/2}\boldsymbol{U}_1 \oplus \boldsymbol{P}_1^{1/2}\boldsymbol{U}_4$ satisfies the l_2-scaling constraints in (13).

To solve problem (16) for $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_4$, we define the Lagrange function of the problem as

$$\begin{aligned} J(\boldsymbol{P}, \lambda_1, \lambda_4) = & \text{tr}[\boldsymbol{M}_1(\boldsymbol{P})\boldsymbol{P}_1] + \text{tr}[\boldsymbol{M}_4(\boldsymbol{P})\boldsymbol{P}_4] + \text{tr}[\boldsymbol{W}_{1B}\boldsymbol{P}_1] \\ & + \text{tr}[\boldsymbol{W}_{4B}\boldsymbol{P}_4] + \text{tr}[\boldsymbol{K}_{1C}\boldsymbol{P}_1^{-1}] + \text{tr}[\boldsymbol{K}_{4C}\boldsymbol{P}_4^{-1}] \\ & + \lambda_1(\text{tr}[\boldsymbol{K}_1\boldsymbol{P}_1^{-1}] - m) + \lambda_4(\text{tr}[\boldsymbol{K}_4\boldsymbol{P}_4^{-1}] - n) \end{aligned} \tag{17}$$

where λ_1 and λ_4 are the Lagrange multipliers, and

$$\boldsymbol{M}(\boldsymbol{P}) = \begin{bmatrix} \boldsymbol{M}_1(\boldsymbol{P}) & \boldsymbol{M}_2(\boldsymbol{P}) \\ \boldsymbol{M}_3(\boldsymbol{P}) & \boldsymbol{M}_4(\boldsymbol{P}) \end{bmatrix}$$

$$\boldsymbol{W}_B = \begin{bmatrix} \boldsymbol{W}_{1B} & \boldsymbol{W}_{2B} \\ \boldsymbol{W}_{3B} & \boldsymbol{W}_{4B} \end{bmatrix}, \quad \boldsymbol{K}_C = \begin{bmatrix} \boldsymbol{K}_{1C} & \boldsymbol{K}_{2C} \\ \boldsymbol{K}_{3C} & \boldsymbol{K}_{4C} \end{bmatrix}$$

and set $\partial J(\boldsymbol{P}, \lambda_1, \lambda_4)/\partial \boldsymbol{P}_1 = \boldsymbol{0}$ and $\partial J(\boldsymbol{P}, \lambda_1, \lambda_4)/\partial \boldsymbol{P}_4 = \boldsymbol{0}$. The last two equations give

$$\begin{aligned} \boldsymbol{P}_1\boldsymbol{F}_1(\boldsymbol{P})\boldsymbol{P}_1 &= \boldsymbol{G}_1(\boldsymbol{P}, \lambda_1) \\ \boldsymbol{P}_4\boldsymbol{F}_4(\boldsymbol{P})\boldsymbol{P}_4 &= \boldsymbol{G}_4(\boldsymbol{P}, \lambda_4) \end{aligned} \tag{18}$$

where

$$\begin{aligned} \boldsymbol{F}_1(\boldsymbol{P}) &= \boldsymbol{M}_1(\boldsymbol{P}) + \boldsymbol{W}_{1B} \\ \boldsymbol{G}_1(\boldsymbol{P}, \lambda_1) &= \boldsymbol{N}_1(\boldsymbol{P}) + \boldsymbol{K}_{1C} + \lambda_1\boldsymbol{K}_1 \\ \boldsymbol{F}_4(\boldsymbol{P}) &= \boldsymbol{M}_4(\boldsymbol{P}) + \boldsymbol{W}_{4B} \\ \boldsymbol{G}_4(\boldsymbol{P}, \lambda_4) &= \boldsymbol{N}_4(\boldsymbol{P}) + \boldsymbol{K}_{4C} + \lambda_4\boldsymbol{K}_4 \\ \boldsymbol{N}(\boldsymbol{P}) &= \begin{bmatrix} \boldsymbol{N}_1(\boldsymbol{P}) & \boldsymbol{N}_2(\boldsymbol{P}) \\ \boldsymbol{N}_3(\boldsymbol{P}) & \boldsymbol{N}_4(\boldsymbol{P}) \end{bmatrix}. \end{aligned}$$

The equations in (18) are highly nonlinear with respect to $\boldsymbol{P}_1$ and $\boldsymbol{P}_4$. An effective approach for solving these equations is to *relax* them into the following recursive second-order matrix equations:

$$\begin{aligned} \boldsymbol{P}_1^{(k+1)}\boldsymbol{F}_1(\boldsymbol{P}^{(k)})\boldsymbol{P}_1^{(k+1)} &= \boldsymbol{G}_1(\boldsymbol{P}^{(k)}, \lambda_1^{(k+1)}) \\ \boldsymbol{P}_4^{(k+1)}\boldsymbol{F}_4(\boldsymbol{P}^{(k)})\boldsymbol{P}_4^{(k+1)} &= \boldsymbol{G}_4(\boldsymbol{P}^{(k)}, \lambda_4^{(k+1)}) \end{aligned} \tag{19}$$

with initial condition $\boldsymbol{P}^{(0)} = \boldsymbol{P}_1^{(0)} \oplus \boldsymbol{P}_4^{(0)} = \boldsymbol{I}_{m+n}$. The unique solutions $\boldsymbol{P}_1^{(k+1)}$ and $\boldsymbol{P}_4^{(k+1)}$ of (19) are found to be

$$\begin{aligned} \boldsymbol{P}_1^{(k+1)} = & \boldsymbol{F}_1(\boldsymbol{P}^{(k)})^{-\frac{1}{2}}[\boldsymbol{F}_1(\boldsymbol{P}^{(k)})^{\frac{1}{2}} \\ & \cdot \boldsymbol{G}_1(\boldsymbol{P}^{(k)}, \lambda_1^{(k+1)})\boldsymbol{F}_1(\boldsymbol{P}^{(k)})^{\frac{1}{2}}]^{\frac{1}{2}}\boldsymbol{F}_1(\boldsymbol{P}^{(k)})^{-\frac{1}{2}} \\ \boldsymbol{P}_4^{(k+1)} = & \boldsymbol{F}_4(\boldsymbol{P}^{(k)})^{-\frac{1}{2}}[\boldsymbol{F}_4(\boldsymbol{P}^{(k)})^{\frac{1}{2}} \\ & \cdot \boldsymbol{G}_4(\boldsymbol{P}^{(k)}, \lambda_4^{(k+1)})\boldsymbol{F}_4(\boldsymbol{P}^{(k)})^{\frac{1}{2}}]^{\frac{1}{2}}\boldsymbol{F}_4(\boldsymbol{P}^{(k)})^{-\frac{1}{2}}. \end{aligned} \tag{20}$$

The Lagrange multipliers $\lambda_1^{(k+1)}$ and $\lambda_4^{(k+1)}$ can be efficiently obtained using a bisection method so that

$$\begin{aligned} f_1(\lambda_1^{(k+1)}) &= m - \text{tr}[\tilde{\boldsymbol{K}}_1^{(k)}\,\tilde{\boldsymbol{G}}_1^{(k)}(\lambda_1^{(k+1)})] = 0 \\ f_4(\lambda_4^{(k+1)}) &= n - \text{tr}[\tilde{\boldsymbol{K}}_4^{(k)}\,\tilde{\boldsymbol{G}}_4^{(k)}(\lambda_4^{(k+1)})] = 0 \end{aligned} \tag{21}$$

are satisfied where

$$\begin{aligned} \tilde{\boldsymbol{K}}_1^{(k)} &= \boldsymbol{F}_1(\boldsymbol{P}^{(k)})^{\frac{1}{2}}\boldsymbol{K}_1\boldsymbol{F}_1(\boldsymbol{P}^{(k)})^{\frac{1}{2}} \\ \tilde{\boldsymbol{K}}_4^{(k)} &= \boldsymbol{F}_4(\boldsymbol{P}^{(k)})^{\frac{1}{2}}\boldsymbol{K}_4\boldsymbol{F}_4(\boldsymbol{P}^{(k)})^{\frac{1}{2}} \\ \tilde{\boldsymbol{G}}_1^{(k)}(\lambda_1^{(k+1)}) &= [\boldsymbol{F}_1(\boldsymbol{P}^{(k)})^{\frac{1}{2}}\boldsymbol{G}_1(\boldsymbol{P}^{(k)}, \lambda_1^{(k+1)})\boldsymbol{F}_1(\boldsymbol{P}^{(k)})^{\frac{1}{2}}]^{-\frac{1}{2}} \\ \tilde{\boldsymbol{G}}_4^{(k)}(\lambda_4^{(k+1)}) &= [\boldsymbol{F}_4(\boldsymbol{P}^{(k)})^{\frac{1}{2}}\boldsymbol{G}_4(\boldsymbol{P}^{(k)}, \lambda_4^{(k+1)})\boldsymbol{F}_4(\boldsymbol{P}^{(k)})^{\frac{1}{2}}]^{-\frac{1}{2}}. \end{aligned}$$

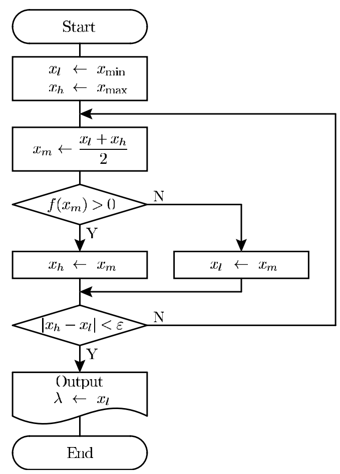

Fig. 1. A flow chart of the bisection method.

A flow chart of the bisection method used is shown in Fig. 1. The iteration process continues until

$$|J(\boldsymbol{P}^{(k+1)}, \lambda_1^{(k+1)}, \lambda_4^{(k+1)}) - J(\boldsymbol{P}^{(k)}, \lambda_1^{(k)}, \lambda_4^{(k)})| < \varepsilon \tag{22}$$

for a prescribed tolerance $\varepsilon > 0$. If the iteration is terminated at step k, then $\boldsymbol{x}_k$ is claimed to be a solution point.

IV. NUMERICAL EXAMPLE

Consider a 2-D stable recursive digital filter specified by $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)_{2,2}$ where

$$\boldsymbol{A} = \begin{bmatrix} 1.888990 & -0.912190 & -0.114079 & 0.000000 \\ 1.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.242902 & -0.226159 & 1.888990 & 0.926336 \\ -0.244144 & 0.230044 & -0.984729 & 0.000000 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} 0.023466 & 0.000000 & -0.027123 & 0.092453 \end{bmatrix}^T$$

$$\boldsymbol{c} = \begin{bmatrix} 0.269726 & -0.851677 & -0.233355 & 0.000000 \end{bmatrix}$$

$$d = 0.08900$$

The frequency weighted functions used in this example was a 2-D FIR low-pass filter with the unit-sample response

$$w_A(i,j) = w_B(i,j) = w_C(i,j)$$
$$= 0.256322 \exp[-0.103203\{(i-4)^2 + (j-4)^2\}]$$

for $(0,0) \le (i,j) \le (20,20)$, and zero elsewhere. Using (7), the frequency-weighted L_2-sensitivity of system $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)_{2,2}$ was found to be $S = 126.9935053243 \times 10^4$.

Choosing $\boldsymbol{P}^{(0)} = \boldsymbol{P}_1^{(0)} \oplus \boldsymbol{P}_4^{(0)} = \boldsymbol{I}_4$ in (20) as the initial estimate and tolerance $\varepsilon = 10^{-8}$ in (22), it took the algorithm proposed in Section III 9 iterations to converge to the solution

$$\boldsymbol{P}^{opt} = \begin{bmatrix} 1.635211 & 1.711154 \\ 1.711154 & 1.825293 \end{bmatrix} \oplus \begin{bmatrix} 0.901327 & -0.914948 \\ -0.914948 & 0.962358 \end{bmatrix}$$

or equivalently,

$$\boldsymbol{T}^{opt} = \begin{bmatrix} 1.141842 & 0.575681 \\ 1.111047 & 0.768680 \end{bmatrix} \oplus \begin{bmatrix} 0.266819 & -0.911117 \\ -0.094981 & 0.976390 \end{bmatrix}.$$

The minimized frequency-weighted l_2-sensitivity measure in (17) corresponding to the above solution was found to be $J(\mathbf{P}^{opt}, \lambda_1, \lambda_4) = 4.0943096873 \times 10^4$ with $\lambda_1 = -16960.331775$ and $\lambda_4 = 17042.623403$.

The profile of the frequency-weighted l_2-sensitivity measure $J(\boldsymbol{P}, \lambda_1, \lambda_4)$ and the profiles of the Lagrange multipliers λ_1 and λ_4 of the first 11 iterations are shown in Figs. 2 and 3, respectively, from which it is observed that with a tolerance $\varepsilon = 10^{-8}$ the algorithm converges within 11 iterations.

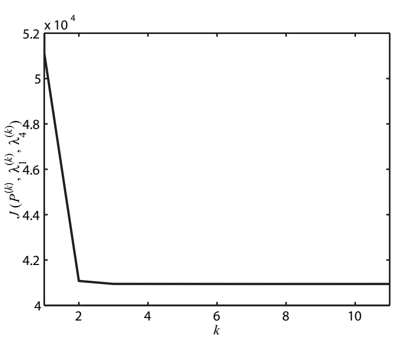

Fig. 2. Profile of $J(\boldsymbol{P}, \lambda_1, \lambda_4)$ during the first 11 iterations.

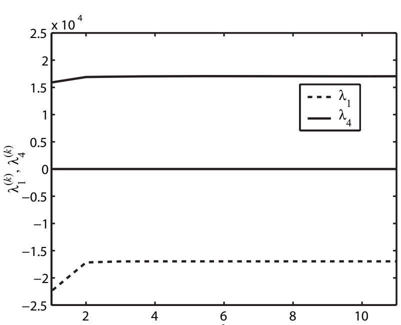

Fig. 3. The λ_1 and λ_4 profiles during the first 11 iterations.

V. CONCLUSION

The minimization problem of a frequency-weighted l_2-sensitivity measure subject to l_2-scaling constraints for 2-D state-space digital filters described by the Roesser LSS model have been investigated. An iterative method is proposed based on the introduction of the Lagrange function and by making use of some matrix-theoretic techniques as well as an efficient bisection method. The optimal state-space filter structure with minimum weighted l_2-sensitivity and no overflow oscillations has then been constructed by applying an appropriate coordinate-transformation matrix. Computer simulation results have demonstrated the validity and effectiveness of the proposed technique.

REFERENCES

[1] M. Kawamata, T. Lin and T. Higuchi, "Minimization of sensitivity of 2-D state-space digital filters and its relation to 2-D balanced realizations," in *Proc. 1987 IEEE Int. Symp. Circuits Syst.*, pp.710-713.

[2] T. Hinamoto, T. Hamanaka and S. Maekawa, "Synthesis of 2-D state-space digital filters with low sensitivity based on the Fornasini-Marchesini model," *IEEE Trans. Acoust., Speech, Signal Processing*, vol.ASSP-38, pp.1587-1594, Sept. 1990.

[3] T. Hinamoto, T. Takao and M. Muneyasu, "Synthesis of 2-D separable-denominator digital filters with low sensitivity," *J. Franklin Institute*, vol.329, pp.1063-1080, 1992.

[4] T. Hinamoto and T. Takao, "Synthesis of 2-D state-space filter structures with low frequency-weighted sensitivity," *IEEE Trans. Circuits Syst. II*, vol.39, pp.646-651, Sept. 1992.

[5] T. Hinamoto and T. Takao, "Minimization of frequency-weighting sensitivity in 2-D systems based on the Fornasini-Marchesini second model, " in *1992 IEEE Int. Conf. Acoust., Speech, Signal Processing*, pp.401-404.

[6] T. Hinamoto, Y. Zempo, Y. Nishino and W.-S. Lu, "An analytical approach for the synthesis of two-dimensional state-space filter structures with minimum weighted sensitivity," *IEEE Trans. Circuits Syst. I*, vol.46, pp.1172-1183, Oct. 1999.

[7] G. Li, "Two-dimensional system optimal realizations with L_2-sensitivity minimization," *IEEE Trans. Signal Processing*, vol.46, pp.809-813, Mar. 1998.

[8] G. Li, "On frequency weighted minimal L_2 sensitivity of 2-D systems using Fornasini-Marchesini LSS model", *IEEE Trans. Circuits Syst. I*, vol.44, pp.642-646, July 1997.

[9] T. Hinamoto, S. Yokoyama, T. Inoue, W. Zeng and W.-S. Lu, "Analysis and minimization of L_2-sensitivity for linear systems and two-dimensional state-space filters using general controllability and observability Gramians," *IEEE Trans. Circuits Syst. I*, vol.49, pp.1279-1289, Sept. 2002.

[10] T. Hinamoto and Y. Sugie, "L_2-sensitivity analysis and minimization of 2-D separable-denominator state-space digital filters," *IEEE Trans. Signal Processing*, vol.50, pp.3107-3114, Dec. 2002.

[11] T. Hinamoto, K. Iwata and W.-S. Lu, "L_2-sensitivity Minimization of one- and two-dimensional state-space digital filters subject to L_2-scaling constraints," in *IEEE Trans. Signal Processing*, vol.54, pp.1804-1812, May 2006.

[12] C. T. Mullis and R. A. Roberts, "Synthesis of minimum roundoff noise fixed-point digital filters," *IEEE Trans. Circuits Syst.*, vol. 23, pp. 551-562, Sept. 1976.

[13] S. Y. Hwang, "Minimum uncorrelated unit noise in state-space digital filtering," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 25, pp. 273-281, Aug. 1977.

[14] R. P. Roesser, "A discrete state-space model for linear image processing," *IEEE Trans. Automat. Contr.*, vol.AC-20, pp.1-10, Feb. 1975.

[15] S. Kung, B. C. Levy, M. Morf and T. Kailath, "New results in 2-D systems theory, Part II: 2-D state-space model—Realization and the notions of controllability, observability, and minimality," *Proc.* IEEE, vol.65, pp.945-961, June 1977.