Minimax Design of Complex-Coefficient FIR Filters with Low Group Delay

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Abstract — **This paper presents a least-pth approach for the design of complex-coefficient FIR digital filters with low group delay in the minimax sense. Features of the proposed approach include: it does not need to adapt the weighting function involved and no constrains are imposed during the course of optimization. More important, the algorithm enjoys global convergence to the minimax design regardless of the initial design used. This property is an immediate consequence of the fact that for each even power p, the weighted L_p objective function is convex in the entire parameter space. Two minimax designs of FIR filter with low passband group delay are included to illustrate the proposed method.**

I. Introduction

The Parks-McClellan algorithm and its variants have been the most efficient tools for the minimax design of FIR digital filters [1]–[3]. They however only apply to the class of linear-phase FIR filters. In many applications, nonlinear-phase FIR filters (e.g. those with low group-delay) are more desirable. Several methods for the minimax design of FIR filters with arbitrary magnitude *and* arbitrary phase responses are available in the literature. Among others, we mention the weighted least-squares approach [4] in which the weighting function is adapted until a near equiripple filter performance is achieved; the constrained optimization approach [5] in which the design is formulated as a linear or quadratic programming problem; the semidefinie programming approach [6] where the design is accomplished by minimizing an approximation-error bound subject to a set of linear and quadratic constraints that can be converted into linear matrix inequalities. The design of FIR filters with complex coefficients has also been studied by several authors using for example linear programming [7], multiple criterion optimization technique [8], and linear constrained approximation methods [9].

This paper presents a least-pth approach for the design of complex-coefficient FIR digital filters with low group delay in the minimax sense. Least-pth optimization as a design tool is not new. As a matter of fact, it was used quite successfully for the minimax design of IIR filters, see [3] and the references cited there. However, it appears that to date least-pth-based algorithms for minimax design of nonlinear-phase FIR filters have not been reported. In the proposed method, a (near) minimax design is obtained by minimizing a weighted L_p error function *without* constraints, where the weighting function is fixed during the course of optimization and power p is a sufficiently large even integer. We show that for any even power p, the L_p objective function is *convex* in the entire parameter space. This global convexity, in conjunction with the availability of closed-form gradient and Hessian of the objective function, provides a basis on that the proposed algorithm is shown to be globally convergent to the minimax design regardless of the initial design chosen. Compared with the existing design methods mentioned above, the proposed method does not need to update the weighting function, and it is a *unconstrained* convex minimization approach. In the rest of the paper, we describe the proposed design method by highlighting the following:

- Relation of L_p-minimization to minimax design
- Weighted L_p objective function and its gradient and Hessian
- Convexity of the L_p objective function
- Design algorithm and computational issues
- Examples

II. Design Formulation

A. The p-norm and infinity-norm

The p-norm and infinity-norm of an n-vector $\mathbf{v} = [v_1 \ \cdots \ v_n]^T$ are defined as

$$\|\mathbf{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}$$

and

$$\|\mathbf{v}\|_\infty = \max_i(|v_i|, \text{ for } 1 \le i \le n)$$

If p is even and the vector components are real numbers, then

$$\|\mathbf{v}\|_p = \left(\sum_{i=1}^{n} v_i^p\right)^{1/p} \tag{1}$$

It is well known [10] that the p-norm and infinity-norm are related by

$$\lim_{p\to\infty} \|\mathbf{v}\|_p = \|\mathbf{v}\|_\infty \tag{2}$$

To get a sense of how $\|\mathbf{v}\|_p$ approaches $\|\mathbf{v}\|_\infty$, we compute for $\mathbf{v} = [1 \ \ 2 \ \ \cdots \ \ 100]^T$ its p-norm $\|\mathbf{v}\|_2 = 581.68$,

$\|\mathbf{v}\|_{10} = 125.38$, $\|\mathbf{v}\|_{50} = 101.85$, $\|\mathbf{v}\|_{100} = 100.45$, $\|\mathbf{v}\|_{200} = 100.07$ and, of course, $\|\mathbf{v}\|_{\infty} = 100$. The point here is that for an even p, the p-norm of a vector is a *differentiable* function of its components but the infinite-norm is *not*. So when the infinity-norm is involved in a (design) problem, one can replace it by a p-norm (with p even) so that powerful calculus-based tools can be used to help solve the altered problem. Obviously, with respect to the "original" design problem the results obtained can only be *approximate*. However, as indicated by (2), the difference between the approximate and exact solutions becomes insignificant if power p is sufficiently large.

B. The objective function

Given a desired frequency response $H_d(\omega)$, we want to determine the complex-valued coefficients $\{h_i\}$ in the FIR transfer function

$$H(z) = \sum_{k=0}^{N} h_k z^{-k} \tag{3}$$

such that the weighted L_{2p} approximation error

$$f(\mathbf{h}) = \left[\int_{-\pi}^{\pi} W(\omega) |H(e^{j\omega}) - H_d(\omega)|^{2p} d\omega \right]^{1/2p} \tag{4}$$

is minimized, where $W(\omega) \geq 0$ is a weighting function, p is a positive integer, and $\mathbf{h}$ is defined below.

If we denote

$$\begin{aligned} h_k &= h_{rk} + jh_{ik} \\ H_d(\omega) &= H_{dr}(\omega) - jH_{di}(\omega) \\ \mathbf{c}(\omega) &= [1 \ \ \cos\omega \ \ \cdots \ \ \cos N\omega]^T \\ \mathbf{s}(\omega) &= [0 \ \ \sin\omega \ \ \cdots \ \ \sin N\omega]^T \\ \mathbf{h}_r &= [h_{r0} \ \ \cdots \ \ f_{rN}]^T \\ \mathbf{h}_i &= [h_{i0} \ \ \cdots \ \ h_{iN}]^T \\ \mathbf{h} &= [\mathbf{h}_r^T \ \ \mathbf{h}_i^T]^T \\ \mathbf{u}(\omega) &= [\mathbf{c}^T(\omega) \ \ \mathbf{s}^T(\omega)]^T \\ \mathbf{v}(\omega) &= [\mathbf{s}^T(\omega) \ \ -\mathbf{c}^T(\omega)]^T \end{aligned}$$

then $\mathbf{u}(\omega)$, $\mathbf{v}(\omega)$, and $\mathbf{h}$ are real-valued vectors of dimension $2N+2$ and (4) becomes

$$f(\mathbf{h}) = \left\{ \int_{-\pi}^{\pi} W[(\mathbf{h}^T\mathbf{u} - H_{dr})^2 + (\mathbf{h}^T\mathbf{v} - H_{di})^2]^p d\omega \right\}^{1/2p} \tag{5}$$

where for simplicity the frequency dependence of W, $\mathbf{u}$, $\mathbf{v}$, H_{dr}, and H_{di} has been omitted. Now if we definite

$$e_2(\omega) = (\mathbf{h}^T\mathbf{u} - H_{dr})^2 + (\mathbf{h}^T\mathbf{v} - H_{di})^2 \tag{6}$$

then the objective function can be expressed as

$$f(\mathbf{h}) = \left[\int_{-\pi}^{\pi} W(\omega) e_2^p(\omega) d\omega \right]^{1/2p} \tag{7}$$

C. Gradient and Hessian of $f(\mathbf{h})$

Using (7), it is straightforward to compute the gradient and Hessian of objective function $f(\mathbf{h})$ as

$$\nabla f(\mathbf{h}) = f^{1-2p}(\mathbf{h}) \int_{-\pi}^{\pi} W(\omega) e_2^{p-1}(\omega) \mathbf{q}(\omega) d\omega \tag{8a}$$

where

$$\mathbf{q}(\omega) = (\mathbf{h}^T\mathbf{u} - H_{dr})\mathbf{u} + (\mathbf{h}^T\mathbf{v} - H_{di})\mathbf{v} \tag{8b}$$

and

$$\nabla^2 f(\mathbf{h}) = \mathbf{H}_1 + \mathbf{H}_2 - \mathbf{H}_3 \tag{8c}$$

where

$$\mathbf{H}_1 = 2(p-1) f^{1-2p}(\mathbf{h}) \int_{-\pi}^{\pi} W e_2^{p-2} \mathbf{q}\mathbf{q}^T d\omega \tag{8d}$$

$$\mathbf{H}_2 = f^{1-2p}(\mathbf{h}) \int_{-\pi}^{\pi} W e_2^{p-1} (\mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T) d\omega \tag{8e}$$

$$\mathbf{H}_3 = (2p-1) f^{-1}(\mathbf{h}) \nabla f(\mathbf{h}) \nabla^T f(\mathbf{h}) \tag{8f}$$

Of central importance to the proposed design algorithm is the property that *for each and every positive integer p, the weighted L_{2p} objective function defined in* (4) *is convex in the entire parameter space* $(\mathbf{h} \in)\mathcal{R}^{2(N+1)}$. This property can be proved by showing that the Hessian $\nabla^2 f(\mathbf{h})$ is positive semidefinite for all $\mathbf{h} \in R^{2(N+1)}$ (see the Appendix).

We conclude this section with two remarks:

(a) It is obvious that a minimizer of $f(\mathbf{h})$ in (4) is also a minimizer of

$$\hat{f}(\mathbf{h}) = \int_{-\pi}^{\pi} W(\omega) |H(e^{j\omega}) - H_d(\omega)|^{2p} d\omega \tag{9}$$

and vice versa. Hence one would naturally consider minimizing $\hat{f}(\mathbf{h})$ instead because the global convexity of $\hat{f}(\mathbf{h})$ is not hard to prove. The problem is that minimizing $\hat{f}(\mathbf{h})$ with a large p encounters considerable numerical difficulties, but these difficulties do not present when one deals with function $f(\mathbf{h})$ even for a very large p.

(b) A fraction power of a convex function is not necessarily convex (e.g., $1 + x^2$ is convex, but $(1 + x^2)^{1/4}$ is not). So although the convexity of $\hat{f}(\mathbf{h})$ with an even p is well known, it turns out that proving the convexity of $f(\mathbf{h})$ is a nontrivial matter.

III. Design Algorithm

A. The L_{2p} minimization

It is now quite clear that up to a given tolerance, an FIR filter that approximates a rather arbitrary frequency response $H_d(\omega)$ in the minimax sense can be obtained by minimizing $f(\mathbf{h})$ in (4) with a sufficiently large p. It follows from the discussion in Sec. II that for a given p, $f(\mathbf{h})$ has a unique global minimizer. Therefore, in principle any descent minimization algorithm, e.g., the steepest descent method, modified Newton's method, and quasi-Newton methods [11] can be used to compute the minimax design regardless of the initial design chosen. On the other hand, however, the amount of computation required to accomplish the design is largely determined by the choice of optimization method as well as the initial point (design).

B. *Choice of initial design*

A reasonable initial design is the L_2-optimal design obtained by minimizing $f(\mathbf{h})$ in (4) with $p = 1$. In this case we have

$$f(\mathbf{h}) = (\mathbf{h}^T\mathbf{Q}\mathbf{h} - 2\mathbf{h}^T\mathbf{p} + \text{const})^{1/2} \tag{10a}$$

where

$$\mathbf{Q} = \int_{-\pi}^{\pi} W(\mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T)d\omega \tag{10b}$$

$$\mathbf{p} = \int_{-\pi}^{\pi} W(H_{dr}\mathbf{u} + H_{di}\mathbf{v})d\omega \tag{10c}$$

Since $\mathbf{Q}$ is positive definite, the global minimizer of $f(\mathbf{h})$ in (10) is given by

$$\mathbf{h} = \mathbf{Q}^{-1}\mathbf{p} \tag{11}$$

We note that $\mathbf{Q}$ in (10b) is a positive-definite, symmetric, block Toeplitz matrix for which fast algorithms to compute its inverse are available [12].

C. *Choice of optimization method*

Minimizing convex objective function $f(\mathbf{h})$ can be accomplished in a number of ways. Since the gradient and Hessian of $f(\mathbf{h})$ are available in closed-form, the Newton's method and the family of quasi-Newton methods are among the most appropriate.

From (8), we see that the evaluations of $f(\mathbf{h})$, $\nabla f(\mathbf{h})$, and $\nabla^2 f(\mathbf{h})$ all involve numerical integration. In computing $\nabla^2 f(\mathbf{h})$, the error introduced in the numerical integration slightly perturbs the Hessian so that the perturbed Hessian is no longer positive definite. The problem can be easily fixed by modifying $\nabla^2 f(\mathbf{h})$ to $\nabla^2 f(\mathbf{h}) + \varepsilon\mathbf{I}$ where $\varepsilon > 0$ is a small scalar. The Newton's method with above modification is called the *modified Newton's method* [11].

Quasi-Newton methods do not require $\nabla^2 f(\mathbf{h})$ yet provide efficiency comparable to that of the Newton's method. Among others, we choose the Broyden-Fletcher-Glodfarb-Shanno (BFGS) algorithm [11] which has been a preferred choice in DSP-related optimization problems [3].

D. *Direct and indirect implementations*

With power p, weighting function $W(\omega)$, and initial design $\mathbf{h}_0$ chosen, the design can be implemented directly or indirectly.

A direct implementation applies a selected unconstrained optimization method to minimize the L_{2p} objective function in (4). Based on rather extensive trials, it is found that to achieve a near minimax design the value of p should in any case be larger than 20, and for high-order FIR filters a power p comparable to filter order N should be used.

In an indirect implementation, the L_2-optimal design obtained by minimizing the L_{2p} function with $p = 1$ is taken to be the initial design $\mathbf{h}_0$ in a subsequent optimization step where the objective function is the L_{2p} function with p moderately increased to, say, $p = 2$. Evidently, it is an "easy" problem because the minimizer, $\mathbf{h}_1$, in this case cannot be far from the initial point. Next, $\mathbf{h}_1$ is used as the initial point to minimize the L_{2p} function with $p = 3$. Again, this is an "easy" problem. The sequential L_{2p} optimization continues until p reaches a prescribed value.

IV. EXAMPLES

We now present two design examples to illustrate the proposed method. The first is minimax design of a complex-coefficient lowpass FIR filter or order $N = 54$. The design parameters were as follows: normalized passband edge $\omega_p = 0.225$, stopband edge $\omega_a = 0.275$, passband group delay = 23, $W(\omega) \equiv 1$ in both passband and stopband and $W(\omega) \equiv 0$ elsewhere, and $p = 130$. Both direct and indirect implementations using modified Newton's method and BFGS algorithm were carried out. As was expected, all trials converge to the same near minimax design with the modified Newton's method in the direct implementation the most efficient: it took the algorithm 46 iterations with 4.15×10^7 Kflops to converge. The amplitude response, passband ripple, and passband group delay of the filter obtained are shown in Fig. 1. It is observed that equiripple stopband attenuation and passband gain have been achieved. Note that the passband group delay does not show equiripple variations. This is because the minimax optimization was carried out for the *complex-valued* frequency response $H_d(\omega)$, not the phase-response alone (see Eq. (4)).

The second example is minimax design of a complex-coefficient bandpass FIR filter of length $N = 160$. The design parameters were: normalized passband = $[0.2,\ 0.3]$; stopband = $[0,\ 0.1875] \bigcup [0.3125,\ 0.5]$; passband group delay = 65; $W(\omega) \equiv 1$ in passband, $W(\omega) \equiv 50$ in stopbands, and $W(\omega) \equiv 0$ elsewhere; and $p = 130$. When the modified Newton's method was directly implemented, it took the algorithm 82 iterations with 3.47×10^8 Kflops to converge. The amplitude response, passband ripple, and passband group delay are depicted in Fig. 2.

V. CONCLUDING REMARKS

We have described a weighted least-pth approach to designing near minimax nonlinear-phase FIR filters with complex-valued coefficients. The proposed algorithm is conceptually simple and user friendly as there is no need to ad-

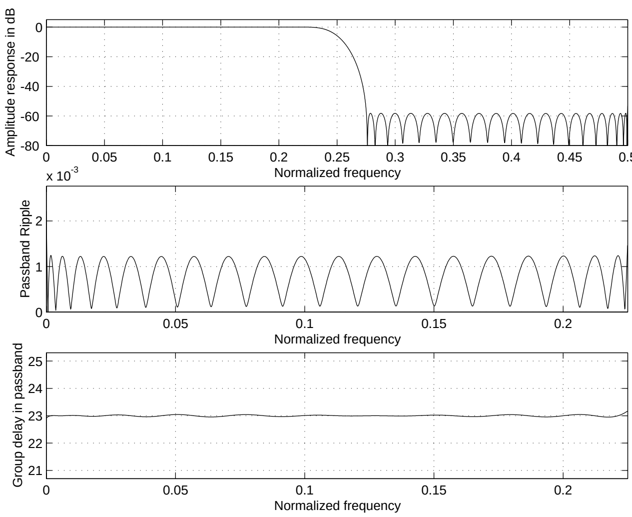

Fig. 1. Minimax design of a complex-coefficient lowpass filter with low passband group delay.

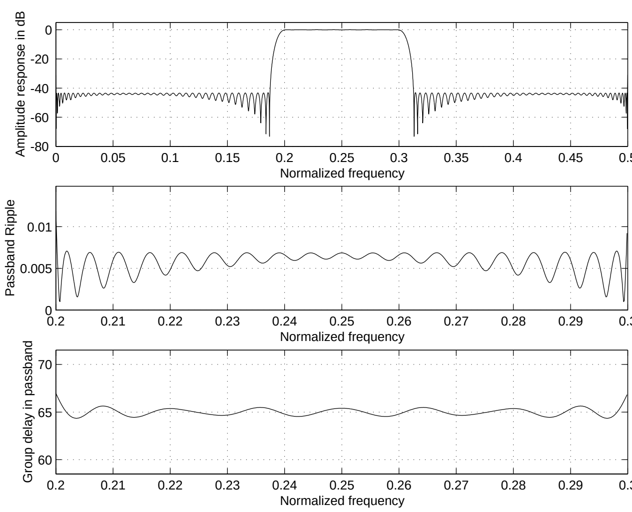

Fig. 2. Minimax design of a complex-coefficient bandpass filter with low passband group delay.

just the weight, no constraints are imposed, and the design can start "anywhere" in the parameter space.

The least-pth approach proposed in this paper can be extended to obtain minimax design of two-dimensional FIR filters with low passband group delay. Details of this development will be reported elsewhere.

APPENDIX

In what follows we show that $\mathbf{y}^T(\nabla^2 f(\mathbf{h}))\,\mathbf{y} \geq 0$ for any $\mathbf{y} \in R^{2(N+1)}$. We start by writing

$$\mathbf{y}^T \nabla^2 f \mathbf{y} = a_1 + a_2 - a_3$$

where

$$a_1 = \mathbf{y}^T \mathbf{H}_1 \mathbf{y} = 2(p-1) f^{1-2p}(\mathbf{h}) \int W e_2^{p-2} (\mathbf{q}^T \mathbf{y})^2$$

$$a_2 = \mathbf{y}^T \mathbf{H}_2 \mathbf{y} = f^{1-2p}(\mathbf{h}) \int W e_2^{p-1} [(\mathbf{u}^T \mathbf{y})^2 + (\mathbf{v}^T \mathbf{y})^2]$$

$$a_3 = \mathbf{y}^T \mathbf{H}_3 \mathbf{y} = (2p-1) f^{-1}(\mathbf{h}) (\mathbf{y}^T \nabla f)^2$$

For simplicity, in a_1 and a_2 the upper and lower limits as well as term $d\omega$ of the integrals have been omitted. Next we split a_1 as $a_1 = a_{11} - a_{12}$ where

$$a_{11} = (2p-1) f^{1-2p}(\mathbf{h}) \int W e_2^{p-2} (\mathbf{q}^T \mathbf{y})^2$$

$$a_{12} = f^{1-2p}(\mathbf{h}) \int W e_2^{p-2} (\mathbf{q}^T \mathbf{y})^2$$

Hence

$$\mathbf{y}^T \nabla^2 f \mathbf{y} = (a_{11} - a_3) + (a_2 - a_{12})$$

Below we show that $a_{11} - a_3 \geq 0$ and $a_2 - a_{12} \geq 0$.

• *Proof of* $a_{11} - a_3 \geq 0$

By (8a), a_3 can be expressed as

$$a_3 = (2p-1) f^{1-4p}(\mathbf{h}) \left[\int W e_2^{p-1} (\mathbf{q}^T \mathbf{y}) \right]^2$$

thus

$$\begin{aligned} &\frac{a_{11} - a_3}{(2p-1) f^{1-4p}(\mathbf{h})} \\ &= f^{2p}(\mathbf{h}) \int W e_2^{p-1} (\mathbf{q}^T \mathbf{y})^2 - \left[\int W e_2^{p-1} (\mathbf{q}^T \mathbf{y}) \right]^2 \\ &= \int W e_2^{p} \int W e_2^{p-1} (\mathbf{q}^T \mathbf{y})^2 - \left[\int W e_2^{p-1} (\mathbf{q}^T \mathbf{y}) \right]^2 \end{aligned}$$

Writing the integrand in the second term as

$$W e_2^{p-1} (\mathbf{q}^T \mathbf{y}) = W^{\frac{1}{2}} e_2^{\frac{p}{2}} \cdot W^{\frac{1}{2}} e_2^{\frac{p-2}{2}} (\mathbf{q}^T \mathbf{y})$$

and applying the Canchy-Schwarz inequality, we obtain

$$\left[\int W e_2^{p-1} (\mathbf{q}^T \mathbf{y}) \right]^2 \leq \int W e_2^{p} \cdot \int W e_2^{p-2} (\mathbf{q}^T \mathbf{y})^2$$

which implies that

$$\frac{a_{11} - a_3}{(2p-1) f^{1-4p}(\mathbf{h})} \geq 0$$

Since $(2p-1) f^{1-4p}(\mathbf{h}) > 0$ we conclude that $a_{11} - a_3 \geq 0$.

• *Proof of* $a_2 - a_{12} \geq 0$

$$\begin{aligned} &\frac{a_2 - a_{12}}{f^{1-2p}(\mathbf{h})} \\ &= \int W e_2^{p-1} [(\mathbf{u}^T \mathbf{y})^2 + (\mathbf{v}^T \mathbf{y})^2] - \int W e_2^{p-2} (\mathbf{q}^T \mathbf{y})^2 \end{aligned}$$

Using (8b), (6), and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&(\mathbf{q}^T\mathbf{y})^2 \\
&= [(\mathbf{h}^T\mathbf{u} - H_{dr})(\mathbf{u}^T\mathbf{y}) + (\mathbf{h}^T\mathbf{v} - H_{di})(\mathbf{v}^T\mathbf{y})]^2 \\
&\le [(\mathbf{h}^T\mathbf{u} - H_{dr})^2 + (\mathbf{h}^T\mathbf{v} - H_{di})^2][(\mathbf{u}^T\mathbf{y})^2 + (\mathbf{v}^T\mathbf{y})^2] \\
&= e_2[(\mathbf{u}^T\mathbf{y})^2 + (\mathbf{v}^T\mathbf{y})^2]
\end{aligned}$$

Hence

$$\int We_2^{p-2}(\mathbf{q}^T\mathbf{y})^2 \le \int We_2^{p-1}[(\mathbf{u}^T\mathbf{y})^2 + (\mathbf{v}^T\mathbf{y})^2]$$

which implies that

$$\frac{a_2 - a_{12}}{f^{1-2p}(\mathbf{h})} \ge 0$$

Since $f^{1-2p}(\mathbf{h}) > 0$, we conclude $a_2 - a_{12} \ge 0$ that completes the proof. ■

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