AN IMPROVED WEIGHTED LEAST-SQUARES DESIGN OF FIR DIGITAL FILTERS WITH VARIABLE FRACTIONAL DELAY

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ABSTRACT

Digital filters capable of changing their frequency response characteristics are often referred to as variable digital filters (VDFs) and have found useful in a number of digital signal processing applications. An important class of VDFs is the digital filters with variable fractional delay. This paper describes an improved WLS design for variable-fractional-delay FIR filters, which offers improved performance of the filters obtained with considerably reduced computational complexity compared to a recently proposed WLS design method. The design improvement is achieved by deriving a closed-form formula for evaluating the WLS objective function. The formula facilitates accurate and efficient function evaluations as compared to summing up a large number of discrete terms, which would be time consuming and inevitably introduce additional errors into the design.

1. INTRODUCTION

Digital filters capable of changing their frequency response characteristics, such as group delay, magnitude response, and resonance frequency etc., are often referred to as *variable digital filters* (VDFs). Typically the transfer function of a VDF contains a number of *parameters* that can be used to tune the frequency response of the VDF. Thus the main objective in the design of a VDF is to find a *parameterized* transfer function which in a certain sense best approximates a given set of frequency response characteristics that vary with the parameters in a desired manner. Applications of VDFs in image processing, digital telecommunications, and music instrument modeling have been reported in, for example, [1]-[5]. A detailed account of the basic theory, design, and implementation of various VDFs can be found in the survey papers [6, 7].

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In this paper, we focus our attention on the design of FIR VDFs with variable fractional delay. Digital filters with fractional delay represent an important class of digital filters as they find many applications, and several algorithms for the design of such filters have recently been proposed [7, 8]. In [8], a weighted least-squares (WLS) method is proposed to design a single-parameter FIR VDF, and the method was applied to design a variable fractional delay filter. This paper describes an improved WLS design for single-parameter FIR VDFs. The proposed algorithm is applied to design a variable fractional delay filter that demonstrates improved performance with reduced computational complexity compared to that of [8]. Essentially, the improved performance and design efficiency is achieved by deriving a closed-form formula for evaluating the WLS objective function in which the weighting function is assumed to be separable and piecewise constant. It avoids using large number of frequency and parameter grids and allows one to carry out the needed function evaluations accurately and quickly. A design example is included to illustrate the proposed method.

2. PROBLEM FORMULATION

We adopt the notation used in [8] to denote the transfer function of the fractional delay FIR filter by

$$H(z,p) = \sum_{n=0}^{N} a_n(p) z^{-n} \tag{1}$$

where p is the parameter representing the fractional delay, and $a_n(p)$ $(n = 0, 1, \cdots, N)$ are polynomials of degree K, i.e.,

$$a_n(p) = \sum_{k=0}^{K} a_{nk} p^k. \tag{2}$$

In matrix notation the frequency response of the filter can be expressed as

$$H(\omega, p) = \boldsymbol{\omega}^T \boldsymbol{A} \boldsymbol{p} \tag{3}$$

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where

$$\boldsymbol{\omega} = [\ 1 \quad e^{-j\omega} \quad e^{-j2\omega} \quad \cdots \quad e^{-jN\omega}\ ]^T$$
$$\boldsymbol{p} = [\ 1 \quad p \quad p^2 \quad \cdots \quad p^K\ ]^T$$

and

$$\boldsymbol{A} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0K} \\ a_{10} & a_{11} & \cdots & a_{1K} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N0} & a_{N1} & \cdots & a_{NK} \end{bmatrix}.$$

For the design of fractional delay filters, the desired (variable) frequency response is given by

$$H_d(\omega, p) = e^{-j\omega(D+p)} \tag{4}$$

where $p \in [0, 1]$ is the parameter encountered in (1)-(3), and D is an integer usually chosen to be $(N-1)/2$ if N is odd or $N/2$ if N is even. For a WLS design, the objective function is set to be

$$J(\boldsymbol{A}) = \int_0^\pi \int_0^1 W(\omega, p)|H(\omega, p) - H_d(\omega, p)|^2 dp d\omega \tag{5}$$

and one seeks to find coefficient matrix $\boldsymbol{A}$ that minimizes $J(\boldsymbol{A})$ in (5).

3. A CLOSED-FORM FORMULA FOR $J(\boldsymbol{A})$

Throughout we assume a separable and piecewise constant weighting function $W(\omega, p)$, namely,

$$W(\omega, p) = W_1(\omega) W_2(p). \tag{6}$$

In (6), $W_1(\omega)$ is assumed to be a constant w_{1l} on interval $[\omega_{l-1}, \omega_l)$, and $W_2(p)$ is assumed to be a constant w_{2m} on interval $[p_{m-1}, p_m)$, where $\{[\omega_{l-1}, \omega_l), l = 1, \cdots, L\}$ and $\{[p_{m-1}, p_m), m = 1, \cdots, M\}$ are partitions of frequency interval $[0, \pi]$ and parameter interval $[0, 1]$, respectively. The objective function $J(\boldsymbol{A})$ in (5) can be evaluated as follows.

First, we write $J(\boldsymbol{A})$ as

$$\begin{aligned} J(\boldsymbol{A}) &= \int_0^\pi \int_0^1 W(\omega, p) I(\omega, p) dp d\omega \\ &= J_1 + J_2 + \text{const} \end{aligned} \tag{7}$$

where

$$I(\omega, p) = \boldsymbol{\omega}^T \boldsymbol{A} \boldsymbol{p} \boldsymbol{p}^T \boldsymbol{A}^T \overline{\boldsymbol{\omega}} - 2\text{Re}[\boldsymbol{\omega}^T \boldsymbol{A} \boldsymbol{p} e^{j\omega(D+p)}] + 1.$$

By using the property that $\text{tr}(\boldsymbol{AB}) = \text{tr}(\boldsymbol{BA})$ where $\text{tr}(\cdot)$ denotes matrix trace, term J_1 in (7) can be calculated as

$$\begin{aligned} J_1 &= \int_0^\pi \int_0^1 W_1(\omega) W_2(p) \boldsymbol{\omega}^T \boldsymbol{A} \boldsymbol{p} \boldsymbol{p}^T \boldsymbol{A}^T \overline{\boldsymbol{\omega}} dp d\omega \\ &= \text{tr}(\boldsymbol{P} \boldsymbol{A}^T \boldsymbol{\Omega} \boldsymbol{A}) \end{aligned} \tag{8}$$

where $\overline{\boldsymbol{\omega}}$ is the complex-conjugate of $\boldsymbol{\omega}$,

$$\boldsymbol{P} = \int_0^1 W_2(p) \boldsymbol{p} \boldsymbol{p}^T dp = \sum_{m=1}^M w_{2m} \boldsymbol{P}_m \tag{9}$$

with the (i, j)th entry of $\boldsymbol{P}_m$ given by

$$P_m(i, j) = \frac{p_m^{i+j-1} - p_{m-1}^{i+j-1}}{i+j-1}, \quad 1 \le i, j \le K+1 \tag{10}$$

and

$$\boldsymbol{\Omega} = \text{Re}\left[\int_0^\pi W_1(\omega) \overline{\boldsymbol{\omega}} \boldsymbol{\omega}^T d\omega\right] = \sum_{l=1}^L w_{1l} \boldsymbol{\Omega}_l \tag{11}$$

with the (i, j)th entry of $\boldsymbol{\Omega}_l$ given by

$$\begin{aligned} &\boldsymbol{\Omega}_l(i, j) = \omega_l \text{sinc}[(i-j)\omega_l] - \omega_{l-1} \text{sinc}[(i-j)\omega_{l-1}] \\ &1 \le i, j \le N+1 \end{aligned} \tag{12}$$

and

$$\text{sinc}(x) = \begin{cases} 1 & \text{for } x = 0 \\ \dfrac{\sin x}{x} & \text{elsewhere.} \end{cases}$$

From (11) and (12) we see that $\boldsymbol{\Omega}_l$ and hence $\boldsymbol{\Omega}$ are Toeplitz matrices.

By using the trace property, the second term in (7) can be written as

$$\begin{aligned} J_2 &= -2\text{Re}\left[\int_0^\pi \int_0^1 W_1(\omega) W_2(p) \boldsymbol{\omega}^T \boldsymbol{A} \boldsymbol{p} e^{j\omega(D+p)} dp d\omega\right] \\ &= -2\text{tr}\left[\int_0^1 W_2(p) \boldsymbol{p} \boldsymbol{\omega}_p^T dp \cdot \boldsymbol{A}\right] \end{aligned} \tag{13}$$

where

$$\boldsymbol{\omega}_p^T = \text{Re}\left[\int_0^\pi W_1(\omega) \boldsymbol{\omega}^T e^{j\omega(D+p)} d\omega\right] = \sum_{l=1}^L w_{1l} \boldsymbol{\omega}_{lp}^T$$

with

$$\boldsymbol{\omega}_{lp}^T = [\ q_{l,1}(p) \quad q_{l,2}(p) \quad \cdots \quad q_{l,N+1}(p)\ ]$$

and

$$q_{l,j}(p) = \omega_l \text{sinc}[(D+p-j+1)\omega_l] - \omega_{l-1} \text{sinc}[(D+p-j+1)\omega_{l-1}].$$

This leads (13) to

$$J_2 = -2\text{tr}(\boldsymbol{SA}) \tag{14}$$

where

$$\boldsymbol{S} = \int_0^1 W_2(p) \boldsymbol{p} \boldsymbol{\omega}_p^T dp = \sum_{l=1}^L \sum_{m=1}^M w_{1l} w_{2m} \boldsymbol{S}_{lm} \tag{15}$$

with

$$S_{lm} = \int_{p_{m-1}}^{p_m} \boldsymbol{p}\boldsymbol{\omega}_{lp}^T dp.$$

It follows that the (i,j)th entry of $\boldsymbol{S}_{lm}$ is given by

$$S_{lm}(i,j) = \int_{p_{m-1}}^{p_m} p^{i-1} q_{l,j}(p) dp. \tag{16}$$

Using (7), (8), and (14), we obtain

$$J(\boldsymbol{A}) = \text{tr}(\boldsymbol{P}\boldsymbol{A}^T\boldsymbol{\Omega}\boldsymbol{A}) - 2\text{tr}(\boldsymbol{S}\boldsymbol{A}) + \text{const.} \tag{17}$$

The gradient of $J(\boldsymbol{A})$ with respect to $\boldsymbol{A}$ is given by

$$\frac{\partial J(\boldsymbol{A})}{\partial \boldsymbol{A}} = 2\boldsymbol{\Omega}\boldsymbol{A}\boldsymbol{P} - 2\boldsymbol{S}^T. \tag{18}$$

By setting $\partial J(\boldsymbol{A})/\partial \boldsymbol{A} = 0$, the optimal coefficient matrix $\boldsymbol{A}$ is obtained as

$$\boldsymbol{A} = \boldsymbol{\Omega}^{-1}\boldsymbol{S}^T\boldsymbol{P}^{-1}. \tag{19}$$

We now conclude this section with several remarks on the derivation. First, the integral in (16) can be evaluated using fast and reliable numerical integration methods such as adaptive Simpson's rule or adaptive Newton-Cotes rule [9]. Second, (19) indicates that the design is determined by three matrices, i.e., $\boldsymbol{\Omega}$, $\boldsymbol{S}$, and $\boldsymbol{P}$. It follows from (9)-(12) that $\boldsymbol{P}$ and $\boldsymbol{\Omega}$ are symmetric and positive definite, and are entirely determined by N, K, and the weighting function $W(\omega, p)$, and matrix $\boldsymbol{S}$ is the only entity that depends on the desired $H_d(\omega, p)$. This suggests that the same $\boldsymbol{P}$ and $\boldsymbol{\Omega}$ can be used in the different variable filter design as long as the same N, K, and $W(\omega, p)$ are employed. Third, although $\boldsymbol{P}$ and $\boldsymbol{\Omega}$ are positive definite hence their inverses do exist, computer simulations have indicated that these matrices (especially matrix $\boldsymbol{P}$) may become ill-conditioned even for moderate filter order N and polynomial degree K. This means that numerical difficulties may be encountered when (19) is used to compute the solution. We shall address this issue in the next section when presenting a design example using the proposed method.

4. AN ILLUSTRATIVE EXAMPLE

In this section, we illustrate the proposed algorithm by applying it to design a variable fractional delay filter with the same specifications as adopted in [8]: $N = 67$, $K = 7$, and the cutoff frequency $\omega_c = 0.9\pi$. Many existing methods for designing variable fractional delay filters have been considered in [7]. It appears that the best result achieved so far was that reported in [8]: by using a set carefully selected weights, the design obtained in [8] was able to keep the frequency-domain error

$$e(\omega,p) = 20\log_{10}|H(\omega,p) - H_d(\omega,p)| \tag{20}$$

below -100 dB in the entire region $0 \le \omega \le 0.9\pi$ and $0 \le p \le 1$.

With $N = 67$, $K = 7$, $W_2(p) \equiv 1$ for $p \in [0,1]$, and

$$W_1(\omega) = \begin{cases} 1 & \text{for } \omega \in [0,\omega_1) \\ 3 & \text{for } \omega \in [\omega_1,\omega_2) \\ 0 & \text{for } \omega \in [\omega_2,\pi] \end{cases}$$

where $\omega_1 = 0.88\pi$ and $\omega_2 = 0.8994\pi$, the new algorithm yielded a design with improved performance, see Table 1 for the comparisons of the proposed method with that of [8] in terms of the maximum error defined by

$$e_{max} = \max\{e(\omega,p), 0 \le \omega \le 0.9\pi, 0 \le p \le 1\}$$

the L_2-error defined by

$$e_2 = \left[\int_0^{0.9\pi}\int_0^1 |H(\omega,p) - H_d(\omega,p)|^2 dp d\omega\right]^{1/2}$$

the number of floating point operations (flops), and the CPU time used. The simulations are conducted using MATLAB 5 on a SUN Ultrasparc I. Fig. 1 shows the error function $e(\omega,p)$ defined in (20). The group delays of the filter in frequency range $[0, 0.9\pi]$ for different p are depicted in Fig. 2.

It is observed that the proposed design leads to reduced e_{max} and e_2, and needs only a small fraction of computations required by the method of [8]. This is mainly attributed to the closed-form evaluation of the matrices that are involved in (19). The integrals in (16) were evaluated numerically using MATLAB command `quad8` which makes use of an adaptive Newton-Cotes 8 panel rule [9]. As mentioned in the preceding section, the condition numbers of matrices $\boldsymbol{\Omega}$ and $\boldsymbol{P}$ are fairly large ($\sim 10^8$ for $\boldsymbol{\Omega}$ and $\sim 10^{10}$ for $\boldsymbol{P}$ in the present design). An effective remedy for dealing with ill-conditioned matrices is to perform the Cholesky decomposition of matrices $\boldsymbol{\Omega}$ and $\boldsymbol{P}$, i.e.,

$$\boldsymbol{P} = \boldsymbol{P}_1^T\boldsymbol{P}_1, \quad \boldsymbol{\Omega} = \boldsymbol{\Omega}_1^T\boldsymbol{\Omega}_1 \tag{21}$$

where $\boldsymbol{P}_1$ and $\boldsymbol{\Omega}_1$ are lower triangular matrices, and then multiplying the matrices involved in (19) in a right order:

$$\boldsymbol{A} = \boldsymbol{\Omega}_1^{-1}\left[\boldsymbol{\Omega}_1^{-T}(\boldsymbol{S}^T\boldsymbol{P}_1^{-1})\boldsymbol{P}_1^{-T}\right] \tag{22}$$

where grouping $\boldsymbol{S}^T$ with $\boldsymbol{P}_1^{-1}$ turns out critical in obtaining a numerically stable solution. The role of the Cholesky decomposition in (21) is to obtain matrices $\boldsymbol{P}_1$ and $\boldsymbol{\Omega}_1$ whose condition numbers are significantly reduced to $\sim 10^5$ and $\sim 10^4$, respectively. This in conjunction with the fact that most entries in $\boldsymbol{S}$ are small in magnitude ($\sim 10^{-2}$) suggests that the multiplication of $\boldsymbol{S}^T$ with $\boldsymbol{P}_1^{-1}$ would largely "cancel out" the large-magnitude entries in $\boldsymbol{P}_1^{-1}$, which in turn considerably eases off the numerical instability. Alternatively, other matrix decompositions such as QR and SVD

Table 1: Performance Comparisons

	Method of [8]	Proposed Method
e_{max} (dB)	-100.0088	-100.7215
e_2	1.9375×10^{-4}	1.7975×10^{-4}
Flops ($\times 10^6$)	55.65	1.70
CPU time (Sec.)	22.69	15.91

have also been tried with similar design results. But in the present case the Cholesky decomposition offers a slightly reduced computational complexity compared to other matrix decompositions.

5. CONCLUSION

An algorithm for the weighted-least-squares design of variable fractional delay FIR filters has been proposed. The design is accomplished by developing a closed-form formula that can be used to evaluate the WLS error function accurately and quickly, which leads to improved filter performance with reduced computational complexity.

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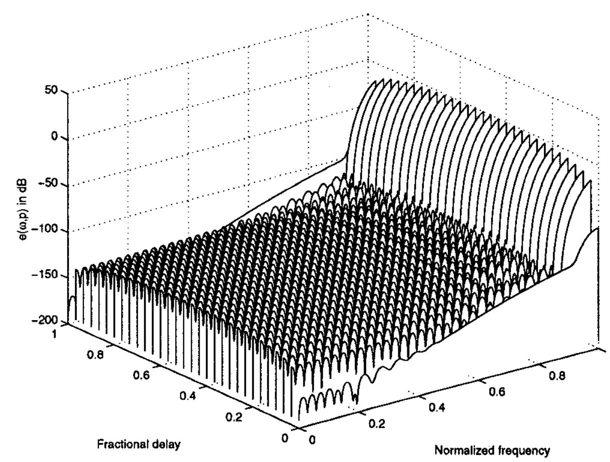

Fig. 1. Error function $e(\omega, p)$.

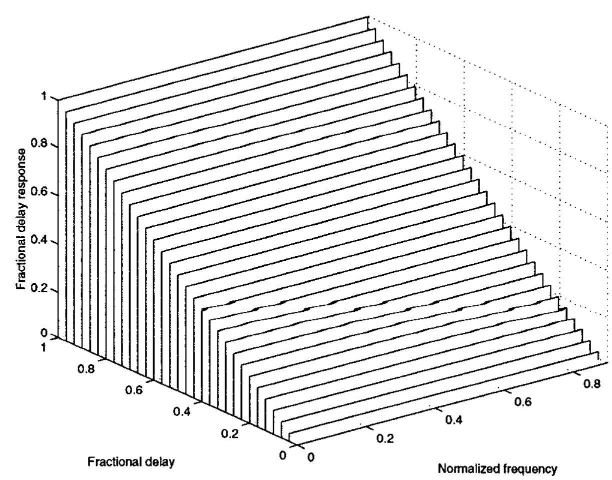

Fig. 2. Fractional delay response of $H(\omega, p)$.