An Iterative Method for the Design of FIR Partial Filter Banks

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Abstract— An iterative algorithm for the design of FIR partial filter banks (PFBs) is described. The design problem is formulated as a 4th-order nonlinear optimization problem in which the objective function is a weighted sum of a reconstruction-error term and an aliasing-error term. The optimization problem is solved by iteratively minimizing a pair of quadratic functions. Explicit formulas for evaluating the minimums of these quadratic functions are derived, which lead to an efficient and fast algorithm. Two design examples are presented to illustrate the proposed algorithm. The PFBs obtained are then used to compress bathymetric signals and electrocardiograms.

I. INTRODUCTION

Multirate digital signal processing systems have found many applications in areas such as subband coding for speech, image, and video, spectrum analysis, frequency-domain speech scramblers, time-varying system identification, and subband adaptive filtering [1]-[4]. A typical M-band filter bank consists of an M-channel analysis section that decomposes the input signal into subband signals and an M-channel synthesis section that combines the subband signals to reconstruct the output signal as shown in Fig. 1. $H_k(z)$ and $G_k(z)$ ($k = 0, \cdots, K$) are the transfer functions of the analysis and synthesis filters, respectively. The downsampler, which decreases the sampling rate of a subband signal by a factor M, and the upsampler, which increases the sampling rate of a downsampled signal by the same factor M, are represented by the blocks with the down arrow and up arrow, respectively. If the filter bank is a perfect reconstruction system, the reconstructed signal is an exact copy of the input signal although a delay is introduced.

In this paper, we investigate the problem of approximating discrete signals by using the modified multirate system shown in Fig. 2. The system has only K channels with $K < M$ and the filters used are of the FIR type. Such a system will be referred to as an M/K FIR *partial filter bank* (PFB). PFBs are often used instead of conventional filter banks in order to achieve reduced implementation cost and improved processing efficiency.

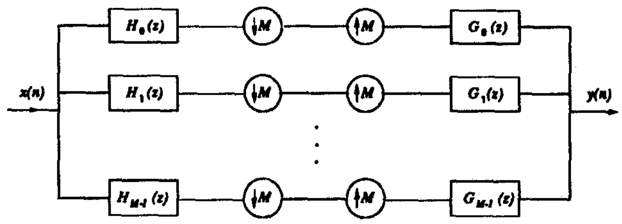

Fig. 1. An M-channel filter bank.

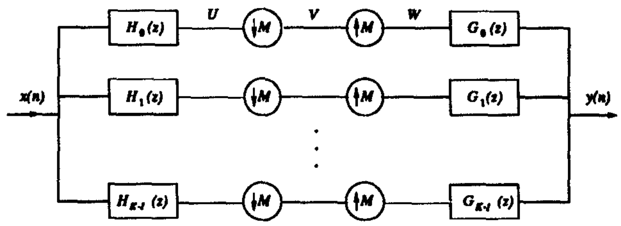

Fig. 2. An FIR partial filter bank ($K < M$).

Below, an iterative algorithm for the design of an M/K FIR partial filter bank that best approximates input discrete signals in a weighted least-squares sense is presented. The design problem is formulated as a 4th-order optimization problem in which the objective function is a weighted sum of a reconstruction-error term and an aliasing-error term. The objective function is then modified to enable one to solve the optimization problem by iteratively minimizing a pair of quadratic functions. Analytic formulas for the evaluation of these functions are derived, which lead to an efficient iterative algorithm for the design. The paper concludes with two design examples which illustrate the algorithm developed.

II. DESIGN OF PARTIAL FILTER BANKS

A. Objective Function

A downsampler is a system that will pass one input sample and discard the $M-1$ subsequent samples whereas an upsampler will insert $M-1$ consecutive zero samples after each input sample.

If $U(z)$ is the z transform of the input of a downsampler, then the z transform of the output of the downsampler is given by

$$V(z) = \frac{1}{M}\sum_{k=0}^{M-1} U(z^{1/M}e^{2\pi jk/M}) \tag{1}$$

Similarly, if $V(z)$ is the z transform of the input of an upsampler, then the z transform of the output of the upsampler is given by

$$W(z) = V(z^m) \tag{2}$$

Therefore, from (1) and (2) we have

$$W(z) = \frac{1}{M}\sum_{k=0}^{M-1} U(ze^{2\pi jk/M})$$

The input-output relation of the system depicted in Fig. 2 can be described as

$$\begin{aligned}
&Y(z)\\
&= \sum_{k=0}^{K-1} Y_k(z)\\
&= \frac{1}{M}\sum_{k=0}^{K-1} G_k(z)\sum_{l=0}^{M-1} H_k(ze^{2\pi jl/M})X(ze^{2\pi jl/M})\\
&= \left[\frac{1}{M}\sum_{k=0}^{K-1} G_k(z)H_k(z)\right]X(z)\\
&+ \left[\frac{1}{M}\sum_{l=1}^{M-1}\sum_{k=0}^{K-1} G_k(z)H_k(ze^{2\pi jl/M})\right]X(ze^{2\pi jl/M}) \quad (3)
\end{aligned}$$

The first term on the right-hand side of (3) is the z transform of the reconstructed signal while the second term is an aliasing-error term. In order to minimize the reconstruction and aliasing errors, we require that

$$\frac{1}{M}\sum_{k=0}^{K-1} G_k(z)H_k(z) \approx z^{-d}$$

$$\frac{1}{M}\sum_{l=1}^{M-1}\left|\sum_{k=0}^{K-1} G_k(z)H_k(ze^{2j\pi l/M})\right|^2 \approx 0$$

for $z = e^{jw}, 0 \le \omega \le \pi$, where d is the system delay. Then the output signal $y(n)$ would be an approximation of the input signal $x(n)$. Assuming that the analysis and synthesis FIR filters have symmetrical impulse responses and both are of odd length N, then the system delay is $d = N - 1$.

A weighted objective function can now be constructed as

$$E = E_1 + \alpha E_2 \tag{4a}$$

where α is a weight, and

$$E_1 = \int_0^\pi \left|\frac{1}{M}\sum_{k=0}^{K-1} G_kH_k - e^{-jd\omega}\right|^2 d\omega \tag{4b}$$

$$E_2 = \sum_{l=1}^{M-1}\int_0^\pi \left|\sum_{k=0}^{K-1} G_kH_k(e^{j\mu})\right|^2 d\omega \tag{4c}$$

with

$$\mu = \omega + 2\pi l/M \tag{4d}$$

B. Closed-Form Formulas for E_1 and E_2

The design problem at hand is to find $H_k(z)$ and $G_k(z)$ for $0 \le k \le K-1$ such that the objective function E in (4a) is minimized using (4b)-(4d). This turns out to be a typical unconstrained 4th-order optimization problem as will now be demonstrated. The problem can be solved by iteratively minimizing a pair of quadratic functions whose minimum points can be obtained analytically. To this end, we write

$$E_1 = \frac{1}{M^2}E_{11} - \frac{2}{M}\text{ real } E_{12} + \pi \tag{5a}$$

where

$$E_{11} = \int_0^\pi |\sum_{k=0}^{K-1} G_kH_k|^2 d\omega \tag{5b}$$

$$E_{12} = \int_0^\pi \sum_{k=0}^{K-1} G_kH_ke^{-jd\omega}d\omega \tag{5c}$$

By expressing the frequency response of filter G_i as

$$G_i(e^{j\omega}) = e^{-j\omega(N-1)/2}\mathbf{g}_i^T\mathbf{c}(\omega) \tag{5d}$$

with

$$\mathbf{g}_i^T = [g_{i,\frac{N-1}{2}} \;\; 2g_{i,\frac{N-3}{2}} \;\cdots\; 2g_{i1} \;\; 2g_{i0}] \tag{5e}$$

$$\mathbf{c}(\omega) = [1 \;\; \cos\omega \;\cdots\; \cos(N-1)\omega/2]^T \tag{5f}$$

we obtain

$$E_{11} = \mathbf{g}^T\mathbf{P_1}(\mathbf{h})\;\mathbf{g} \tag{6a}$$

$$E_{12} = \mathbf{g}^T\mathbf{b}(\mathbf{h}) \tag{6b}$$

$$E_2 = \mathbf{g}^T\mathbf{P_2}(\mathbf{h})\mathbf{g} \tag{6c}$$

where $\mathbf{g}^T$, $\mathbf{P_1}(\mathbf{h})$, $\mathbf{b}(\mathbf{h})$, and $\mathbf{P_2}(\mathbf{h})$ are defined as

$$\mathbf{g}^T = [\mathbf{g}_0^T \;\; \mathbf{g}_1^T \;\cdots\; \mathbf{g}_{K-1}^T] \tag{6d}$$

$$\mathbf{P_1}(\mathbf{h}) = \int_0^\pi W(\omega)W(\omega)^H d\omega \tag{6e}$$

$$\mathbf{b}(\mathbf{h}) = \int_0^\pi W(\omega)e^{-j(d-\frac{N-1}{2})\omega}d\omega \tag{6f}$$

$$\mathbf{P_2}(\mathbf{h}) = \sum_{l=1}^{M-1}\int_0^\pi W_l(\omega)W_l(\omega)^H d\omega \tag{6g}$$

with

$$\mathbf{C}(\omega) = \begin{bmatrix} \mathbf{c}(\omega) & 0 & \cdots & 0\\ 0 & \mathbf{c}(\omega) & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & \cdots & 0 & \mathbf{c}(\omega)\end{bmatrix} \tag{6h}$$

$$W(\omega) = C(\omega)[H_0(e^{-j\omega}) \;\cdots\; H_{K-1}(e^{-j\omega})] \tag{6i}$$

$$W_l(\omega) = C(\omega)[H_0(e^{-j\mu}) \;\cdots\; H_{K-1}(e^{-j\mu})] \tag{6j}$$

From (6a)-(6j), the objective function E in (4a) can be expressed as

$$E(\mathbf{g}) = \mathbf{g}^T \left[\frac{\mathbf{P_1(h)}}{M^2} + \alpha \mathbf{P_2(h)}\right] \mathbf{g} - \mathbf{g}^T \frac{2}{M}\mathbf{b(h)} + \pi \quad (7a)$$

For a fixed $\mathbf{h}$, $E(\mathbf{g})$ is a quadratic function. Since $\mathbf{P_1(h)}$ and $\mathbf{P_2(h)}$ are positive definite, the $\mathbf{g}$ that minimizes $E(\mathbf{g})$ is given by

$$\mathbf{g} = \frac{1}{M}\left[\frac{\mathbf{P_1(h)}}{M^2} + \alpha \mathbf{P_2(h)}\right]^{-1} \mathbf{b(h)} \quad (7b)$$

Likewise, for a fixed $\mathbf{g}$, the objective function $E(\mathbf{h})$ can be expressed as

$$E(\mathbf{h}) = \mathbf{h}^T \left[\frac{\mathbf{P_1(g)}}{M^2} + \alpha \mathbf{P_2(g)}\right] \mathbf{h} - \mathbf{h}^T \frac{2}{M}\mathbf{b(g)} + \pi \quad (8a)$$

Hence the $\mathbf{h}$ that minimize $E(\mathbf{h})$ is given by

$$\mathbf{h} = \frac{1}{M}\left[\frac{\mathbf{P_1(g)}}{M^2} + \alpha \mathbf{P_2(g)}\right]^{-1} \mathbf{b(g)} \quad (8b)$$

C. Evaluation of $\mathbf{P_1}, \mathbf{P_2}$, *and* $\mathbf{b}$

This subsection shows that matrices $\mathbf{P_1}, \mathbf{P_2}$, and vector $\mathbf{b}$ used in (7b) and (8b) can be evaluated precisely and efficiently using explicit formulas. Consequently, the use of these formulas leads to a very efficient algorithm.

Writing

$$H_i(e^{j\omega}) = e^{-j\omega(N-1)/2}\mathbf{h}_i^T \mathbf{c}(\omega)$$

with

$$\mathbf{h}_i^T = [h_{i,\frac{N-1}{2}} \;\; 2h_{i,\frac{N-3}{2}} \;\; \cdots \;\; 2h_{i1} \;\; 2h_{i0}]$$

it can be verified that

$$\mathbf{P_1(h)} = \mathbf{A}\left[\int_0^\pi |c_1(\omega)|^2 d\omega\right]\mathbf{A}^T \quad (9a)$$

$$\mathbf{b(h)} = \mathbf{A}\left[\int_0^\pi c_1(\omega) d\omega\right]\mathbf{A}^T \quad (9b)$$

$$\mathbf{P_2(h)} = \sum_{l=1}^{M-1} \mathbf{A}\left[\int_0^\pi |c_2(\omega)|^2 d\omega\right]\mathbf{A}^T \quad (9c)$$

$$c_1(\omega) = c(\omega) \otimes c(\omega)^T \quad (9d)$$

$$c_2(\omega) = c(\omega) \otimes c(\mu)^T \quad (9e)$$

where $\otimes$ denotes the Kronecker product, and A is an $(N+1)K/2 \times [(N+1)/2]^2$ matrix defined by

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_1} \\ \mathbf{A_2} \\ \vdots \\ \mathbf{A_{K-1}} \end{bmatrix}$$

with

$$\mathbf{A_i} = \begin{bmatrix} \mathbf{h}_i^T & 0 & \cdots & 0 \\ 0 & \mathbf{h}_i^T & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \mathbf{h}_i^T \end{bmatrix}$$

for $i = 1, 2, \cdots K-1$.

D. Iterative Algorithm

In the ith iteration, the iterative algorithm starts with a fixed $\mathbf{h}$, say, $\mathbf{h}^{(i-1)}$. By evaluating $\mathbf{P_1(h)}, \mathbf{P_2(h)}$, and $\mathbf{b(h)}$, $\mathbf{g}^{(i)}$ can be computed using (7b). Vector $\mathbf{g}^{(i)}$ is then updated as

$$\mathbf{g}^{(i)} = \tau \mathbf{g}^{(i)} + (1-\tau)\mathbf{g}^{(i-1)} \quad (10)$$

where τ is a smoothing parameter between 0 and 1. Likewise, with a fixed $\mathbf{g}$, *say*, $\mathbf{g}^{(i-1)}$, $\mathbf{P_1(g)}, \mathbf{P_2(g)}$, and $\mathbf{b(g)}$ can be found and $\mathbf{h}^{(i)}$ can be computed using (8b). Vector $\mathbf{h}^{(i)}$ is then updated as

$$\mathbf{h}^{(i)} = \tau \mathbf{h}^{(i)} + (1-\tau)\mathbf{h}^{(i-1)} \quad (11)$$

The algorithm terminates when $||\mathbf{g}^{(i)} - \mathbf{g}^{(i-1)}|| + ||\mathbf{h}^{(i)} - \mathbf{h}^{(i-1)}||$ is less than a prescribed tolerance ϵ. A step-by-step description of the algorithm is given below.

Algorithm

Step 1 Using a conventional method (e.g., the window method), design K linear-phase bandpass FIR filters of odd length N, and construct initial coefficient vectors $\mathbf{h}^{(0)}$ and $\mathbf{g}^{(0)}$.
Step 2 Using (9) and (7b) compute $\mathbf{g}^{(i)}$.
Step 3 Using (10) update $\mathbf{g}^{(i)}$.
Step 4 Using (9) and (8b) compute $\mathbf{h}^{(i)}$.
Step 5 Using (11) update $\mathbf{h}^{(i)}$.
Step 6 If $||\mathbf{g}^{(i)} - \mathbf{g}^{(i-1)}|| + ||\mathbf{h}^{(i)} - \mathbf{h}^{(i-1)}|| < \epsilon$, output $\mathbf{h} = \mathbf{h}^{(i)}$ and $\mathbf{g} = \mathbf{g}^{(i)}$ as the design results and stop. Otherwise, set $i = i+1$ and repeat from Step 2.

III. DESIGN EXAMPLES AND APPLICATIONS

The proposed algorithm was implemented with MATLAB and was used to design two 4/1 PFB's with design parameters $N = 9$, $\alpha = 5$, $\tau = 0.5$, $\epsilon = 10^{-6}$ and $N = 17$, $\alpha = 5$, $\tau = 0.5$, $\epsilon = 10^{-6}$. The results are listed in Tables I and II.

TABLE I
COEFFICIENTS OF THE PFB WITH FILTERS OF LENGTH 9

i	h_i	g_i
0	0.03889130449859	0.03889255700902
1	0.07774741009607	0.07774849192799
2	0.12658632770242	0.12658625011999
3	0.16865178056009	0.16865036953007
4	0.17624635428567	0.17624466282586

TABLE II
COEFFICIENTS OF THE PFB WITH FILTERS OF LENGTH 17

i	h_i	g_i
0	-0.03572653264287	-0.00342207904852
1	-0.06321433513287	-0.00073639200615
2	-0.08461305180880	0.00855882595468
3	-0.07975807832036	0.02681917508857
4	-0.00776982626837	0.05386464148861
5	0.08839613952510	0.08565181878536
6	0.20849964600198	0.11661062678619
7	0.30723149161589	0.13930506715046
8	0.33390909406061	0.14669663160158

The partial filter banks were then applied to compress bathymetric signals and electrocardiograms (ECGs). A motivation for doing so is that a typical bathymeter or electrocardiogram monitoring device can generate a huge amount of data. The approximated signals achieved with a compression ratio of 4 are illustrated in Fig. 3. The mean-square of the errors between the original and the compressed signals are given in Table III.

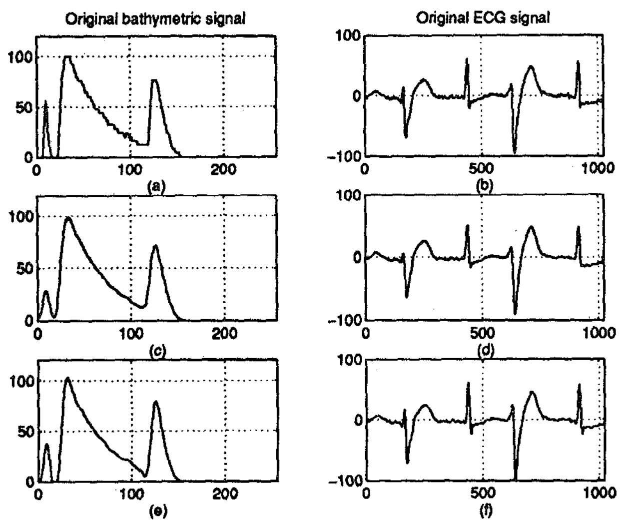

Figure 3: Test signals and their compressed versions. (a), (b) original test signals; (c), (d) outputs of the PFB with filters of length 9; (e), (f) outputs of the PFB with filters of length 17.

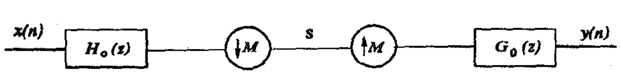

Figure 4: An $M/1$ partial filter bank.

For the purpose of depth estimation, the most important features of a bathymetric waveform are the locations of the peaks whereas for the purpose of diagnosis the locations and amplitudes of the peaks of the ECG are of importance. As can be seen by comparing Fig. 3(c) and (e) with Fig. 3(a) and Fig. 3(d) and (f) with Fig. 3(b) the main features of these waveforms are well preserved after compression.

TABLE III
MEAN-SQUARE ERROR OF COMPRESSED SIGNALS

Figure	Mean-Square Error
3(c)	0.0804
3(d)	0.0969
3(e)	0.0749
3(f)	0.0864

Note that an $M/1$ PFB system involves only one channel and hence it can be implemented efficiently.

For applications such as bathymetric data processing, an $M/1$ system can be split into two parts: the function from the input to point S (see Fig. 4) can be performed on site and can be implemented as part of the bathymeter in the aircraft in order to reduce the data storage by a factor of M. The rest of the system can be implemented at the ground station. Note that for each digitized waveform, the on-site part of the system would involve only discrete convolution and downsampling which can be readily performed in real time.

IV. CONCLUSION

An iterative algorithm for the design of FIR partial filter banks has been described. The design problem was formulated as a 4th-order optimization problem in which the objective function is a weighted sum of a reconstruction-error term and an aliasing-error term. The objective function was then converted into a pair of quadratic functions which can be minimized by iteratively minimizing the pair of quadratic functions. Explicit formulas for evaluating the minimums of the quadratic functions were derived for fast and precise implementation. The algorithm was illustrated by designing two PFBs which were then used to approximate bathymetric and ECG waveforms.

REFERENCES

[1] P. P. Vaidyanathan, "Theory and design of M-channel maximally decimated quadrature mirror filters with arbitrary M, having perfect reconstruction property, " *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 35, pp. 476-492, April 1987.

[2] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Englewood Cliffs, NJ: Prentice-Hall, 1992.

[3] M. Vetterli and J. Kovacevic, *Wavelet and Subband Coding*, Englewood Cliffs, NJ: Prentice-Hall, 1995.

[4] G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Wellesley, MA: Wellesley-Cambridge Press, 1996.