Efficient Design of Perfect-Reconstruction Biorthogonal Cosine-Modulated Filter Banks Using Convex Lagrangian Relaxation and Alternating Null-Space Projections

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ABSTRACT

In essence, designing a perfect-reconstruction (PR) biorthogonal cosine-modulated filter bank (BCM) is a non-convex constrained optimization problem that can be solved in principle using general optimization solvers. However, when the number of channels is large and the order of the prototype filter (PF) is high, numerical difficulties in using those optimization solvers often occur, and the computational efficiency also becomes a concern. This paper proposes an algorithm that carries out the design in two stages. In the first stage, a convex Lagrangian relaxation technique is used to obtain a near PR (NPR) filter bank and, in the second stage, the coefficient vector of the PF obtained is alternately projected onto the null-spaces that are associated with the PR constraints, which turns the NPR filter bank into a PR filter bank. Simulation results are included to demonstrate the robustness of the proposed algorithm for designing BCM filter banks with a large number of channels and high-order PF as well as satisfactory design efficiency.

1. INTRODUCTION

Biorthogonal cosine-modulated filter banks (BCM) have played an increasingly important role in multirate signal processing because they offer reduced system delays compared to what linear-phase cosine-modulated filter banks can offer and their efficient implementation can be readily substantiated through the polyphase decomposition. In addition, the optimal synthesis of a BCM-based multirate system can be focused on the prototype filter (PF) alone.

Recent progress in the analysis and design of BCM filter banks has been reported by several authors, see, for example, [1] – [13]. Available design techniques include the quadratic-constrained least-squares (QCLS) method [4], [9], [10] that minimizes the stopband energy of the PF subject to the time-domain PR constraints; the factorization-based method [8], [11] that yields a parameterized realization in which the PR property is ensured while minimizing the stopband energy of the PF; and the sequential design method [13] that is carried out by first designing a filter bank with small number of channels and a relatively short filter length and then gradually increasing the number of channels as well as the filter length using a technique initiated in [3].

The optimization problem formulated in the time-domain is nonconvex. Although, in principle, general optimization solvers can be applied to find a solution, when the channel number is large and the order of the prototype filter (PF) is high, numerical difficulties in using those optimization solvers often occur, and the computational efficiency also becomes a concern. This paper proposes an algorithm that carries out the design in two stages. In the first stage, a convex relaxation technique is used to obtain a near PR (NPR) filter bank. The relaxation is carried out by a sequential convex approximation of the Lagrangian associated with the original (nonconvex) optimization problem, and can be viewed as an enhanced version of sequential quadratic programming (SQP) [14]. In the second stage, the coefficient vector of the PF obtained from the first stage is alternately projected onto the null-spaces that are associated with the PR constraints. The projections turn the NPR filter bank into a nearby PR filter bank with a fairly moderate increase of the stopband energy for the PF. Simulation results are included to demonstrate the robustness of the proposed algorithm for designing BCM filter banks with a large number of channels and high-order PF as well as satisfactory design efficiency.

2. DESIGN PROBLEM

2.1 BCM Filter Banks

An M-channel maximally decimated BCM filter bank is characterized by the coefficients of its analysis and synthesis filters that are given by

$$h_k(n) = 2h(n)\cos\left[\frac{p}{M}(k+\frac{1}{2})(n-\frac{D}{2}) + (-1)^k\frac{p}{4}\right] \qquad (1a)$$

and

$$f_k(n) = 2h(n)\cos\left[\frac{p}{M}(k+\frac{1}{2})(n-\frac{D}{2}) - (-1)^k\frac{p}{4}\right] \qquad (1b)$$

for $1 \le k \le M-1$ and $0 \le n \le N-1$, respectively, where $\{h(n)\}$ is the impulse response of the finite-impulse-response (FIR) PF, and D denotes the system delay. BCM filter bank structures other than that of (1) can also be obtained using different DCT modulations [10]. In this paper, however, we shall concentrate on the DCT-IV BCM filter banks as specified by (1) along with the following assumptions: (i) the channel number M is even, (ii) the filter length N assumes the form $N = 2mM$ for some positive integer m, and (iii) the system delay assumes the form $D = 2sM + d$ where s is an integer and $d = 2M - 1$. The rationale of these assumptions have been addressed in the literature [10] – [12]. The input-output relation of the system in the z-domain is given by

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$$Y(z) = T_0(z)X(z) + \sum_{l=0}^{M-1} T_l(z)X(ze^{-j2pl/M}) \qquad (2a)$$

where

$$T_0(z) = \frac{1}{M}\sum_{k=0}^{M-1} F_k(z)H_k(z) \qquad (2b)$$

$$T_l(z) = \frac{1}{M}\sum_{k=0}^{M-1} F_k(z)H_k(ze^{-j2pl/M}) \quad \text{for } l = 1, 2, ..., M-1 \qquad (2c)$$

It follows that the filter bank holds the PR property if and only if

$$T_0(e^{jw}) = e^{-jDw} \quad \text{for } w \in [0, p]$$

$$T_l(e^{jw}) = 0 \ \text{ for } w \in [0, p] \text{ and } 1 \le l \le M-1$$

In the time-domain, the PR condition can be described by the following set of quadratic equations [10]:

$$\boldsymbol{h}^T \boldsymbol{Q}_{l,n} \boldsymbol{h} = c_{l,n} \ \text{ for } 0 \le n \le 2m-2 \text{ and } 1 \le l \le M-1 \qquad (3a)$$

where $\boldsymbol{h} = [h_0 \ h_1 \cdots h_{N-1}]^T$ collects the coefficients of the PF, and

$$\boldsymbol{Q}_{l,n} = \boldsymbol{V}_{d-l}\boldsymbol{D}_n\boldsymbol{V}_l^T + \boldsymbol{V}_{d-M-l}\boldsymbol{D}_n\boldsymbol{V}_{M+l}^T \qquad (3b)$$

$$\boldsymbol{D}_n(i,j) = \begin{cases} 1 & \text{if } i+j=n \\ 0 & \text{otherwise} \end{cases} \qquad (3c)$$

$$\boldsymbol{V}_l(i,j) = \begin{cases} 1 & \text{if } i = l + 2jM \\ 0 & \text{otherwise} \end{cases} \qquad (3d)$$

and $c_{l,n} = d(n-s)/2M$. The performance of a BCM filter bank is typically measured by:

• Amplitude distortion: $e_m(w) = 1 - |T_0(e^{jw})|$

• Group-delay distortion: $e_{gd}(w) = D - \arg|T_0(e^{jw})|$

• Worst-case aliasing error: $e_a(w) = \max_{1 \le l \le M-1} |T_l(e^{jw})|$

where $w \in [0, p]$. A filter bank is said to be NPR if the above measures are uniformally small in magnitude for all frequencies. Concerning the PF, it is often desirable to construct a PR or NPR filter bank with the PF's stopband energy

$$e_2(\boldsymbol{h}) = \int_{w_s}^{p} |H(e^{jw})|^2 \, dw \qquad (4a)$$

minimized, where $H(e^{jw}) = \sum_{k=0}^{N-1} h_k e^{-jkw}$ and

$$w_s = \frac{(1+r)p}{2M} \quad \text{with } r > 0 \qquad (4b)$$

It can easily be verified that $e_2(\boldsymbol{h}) = \boldsymbol{h}^T \boldsymbol{P} \boldsymbol{h}$ where $\boldsymbol{P}$ is a symmetric positive definite Toeplitz matrix determined by its first row given by $[p - w_s \ \ -\sin w_s \ \cdots -\sin(N-1)w_s/(N-1)]$.

2.2 PR Constraints

It can be readily verified that with $d = 2M - 1$, the constraints in (3a) for $0 \le n \le 2m-2$ and $M/2 \le l \le M-1$ are identical to those for $0 \le n \le 2m-2$ and $0 \le l \le M/2-1$. Therefore, the PR constraints to be considered in this paper are given by

$$\boldsymbol{h}^T \boldsymbol{Q}_{l,n} \boldsymbol{h} = c_{l,n} \ \text{ for } 0 \le n \le 2m-2 \ \text{ and } \ 0 \le l \le M/2-1 \quad (5)$$

where $\boldsymbol{Q}_{l,n} = \boldsymbol{V}_{2M-l-1}\boldsymbol{D}_n\boldsymbol{V}_l^T + \boldsymbol{V}_{M-l-1}\boldsymbol{D}_n\boldsymbol{V}_{M+l}^T$.

2.3 Problem Formulation

The design problem can be stated in the time-domain as

$$\text{minimize} \quad e_2(\boldsymbol{h}) = \boldsymbol{h}^T \boldsymbol{P} \boldsymbol{h} \qquad (6a)$$

$$\text{subject to:} \quad \text{constraints in (5)} \qquad (6b)$$

A difference between (6) and the one in [10] is that the number of constraints involved in (6b) is a half of that in Eq. (65) of [10].

3. DESIGN METHOD

3.1 Basic Sequential Quadratic Programming

The Lagrangian of the constrained problem (6) is given by [14]

$$L(\boldsymbol{h}, l) = \boldsymbol{h}^T \boldsymbol{P} \boldsymbol{h} - \sum_{i=1}^{K} l_i a_i(\boldsymbol{h}) \qquad (7)$$

where $K = M(2m-1)/2 = (N-M)/2$ is the number of constraints in (6b), and $a_i(\boldsymbol{h}) = \boldsymbol{h}^T\boldsymbol{Q}_{l,n}\boldsymbol{h} - c_{l,n}$ with $i = nM/2 + l + 1$. It is well known that a solution of problem (6) must satisfy the following Karush-Kuhn-Tucker (KKT) condition [14]:

$$\nabla L(\boldsymbol{h}, l) = \begin{bmatrix} \nabla_h L(\boldsymbol{h}, l) \\ \nabla_l L(\boldsymbol{h}, l) \end{bmatrix} = \boldsymbol{0} \qquad (8)$$

Suppose we start with a reasonable initial PF coefficient vector $\boldsymbol{h}_0$ and an initial Lagrange multiplier vector $l_0 = \boldsymbol{0}$. In the kth iteration, $\{\boldsymbol{h}_k, l_k\}$ is updated to $\{\boldsymbol{h}_{k+1}, l_{k+1}\} = \{\boldsymbol{h}_k, l_k\} + \{d_h, d_l\}$ such that

$$\nabla L(\boldsymbol{h}_{k+1}, l_{k+1}) \approx \nabla L(\boldsymbol{h}_k, l_k) + \nabla^2 L(\boldsymbol{h}_k, l_k)\begin{bmatrix} d_h \\ d_l \end{bmatrix} = \boldsymbol{0} \qquad (9)$$

which leads to the following linear system of equations:

$$\begin{bmatrix} \boldsymbol{W}_k & -\boldsymbol{A}_k^T \\ -\boldsymbol{A}_k & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} d_h \\ d_l \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_k^T l_k - \boldsymbol{g}_k \\ \boldsymbol{f}_k \end{bmatrix} \qquad (10)$$

where $\boldsymbol{W}_k = 2(\boldsymbol{P} - \sum_{i=1}^{K} l_i \boldsymbol{Q}_i)$, $\boldsymbol{g}_k = 2\boldsymbol{P}\boldsymbol{h}_k$, $\boldsymbol{A}_k = 2[\boldsymbol{Q}_1\boldsymbol{h}_k \cdots \boldsymbol{Q}_K\boldsymbol{h}_k]^T$, and $\boldsymbol{f}_k = [a_1(\boldsymbol{h}_k) \cdots a_K(\boldsymbol{h}_k)]^T$. Equation (10) can be written as

$$\boldsymbol{W}_k d_h + \boldsymbol{g}_k = \boldsymbol{A}_k^T l_{k+1} \qquad (11a)$$

$$\boldsymbol{A}_k d_h = -\boldsymbol{f}_k \qquad (11b)$$

Note that (11a) and (11b) are the *exact* KKT conditions for the following quadratic programming (QP) problem:

$$\text{minimize} \quad \frac{1}{2} d^T \boldsymbol{W}_k d + d^T \boldsymbol{g}_k \tag{12a}$$

$$\text{subject to:} \quad \boldsymbol{A}_k d = -\boldsymbol{f}_k \tag{12b}$$

Once a solution of (12) is obtained, based on (11a) the Lagrange multiplier vector can be computed as

$$l_{k+1} = (\boldsymbol{A}_k \boldsymbol{A}_k^T)^{-1} \boldsymbol{A}_k (\boldsymbol{W}_k d_h + \boldsymbol{g}_k) \tag{13}$$

and $\boldsymbol{W}_k$, $\boldsymbol{g}_k$, and $\boldsymbol{A}_k$ can be updated to $\boldsymbol{W}_{k+1}$, $\boldsymbol{g}_{k+1}$, and $\boldsymbol{A}_{k+1}$ accordingly. The iteration continues until certain criterion, such as the norm of d_h is less than a prescribed tolerance or the number of iterations reaches a given bound, is satisfied.

3.2 Convex Relaxation of Problem (12)

In general, the objective function in problem (12) is not convex. To obtain a meaningful iterate from the approximate KKT condition in (9), a convex relaxation of (12) is desirable. This can be accomplished in two ways. Perhaps the simplest way is to replace matrix $\boldsymbol{W}_k$ with constant matrix 2$\boldsymbol{P}$. As a result, the modified problem in (12) is a *convex* QP problem that possesses a unique global minimizer. Also note that the modified Hessian matrix requires no update during the iteration process. However, because of the modification, the Lagrange multiplier l_k is no longer able to influence the Hessian and the modified algorithm usually cannot enjoy a fast convergence rate. Another way to relax the problem in (12) into a convex QP is to use a quasi-Newton update, such as the Broyden-Fletcher-Goldfarb-Shanno formula [14], [15] that replaces $\boldsymbol{W}_k$ by $\boldsymbol{Y}_k$ where $\boldsymbol{Y}_k$ is updated as follows:

$$\boldsymbol{Y}_{k+1} = \boldsymbol{Y}_k + \frac{h_k h_k^T}{d_h^T h_k} - \frac{\boldsymbol{Y}_k d_h d_h^T \boldsymbol{Y}_k}{d_h^T \boldsymbol{Y}_k d_h} \tag{14a}$$

where $\boldsymbol{Y}_0 = \boldsymbol{I}$, $d_h = \boldsymbol{h}_{k+1} - \boldsymbol{h}_k$, $h_k = q g_k + (1-q)\boldsymbol{Y}_k d_h$,

$$g_k = (\boldsymbol{g}_{k+1} - \boldsymbol{g}_k) - (\boldsymbol{A}_{k+1} - \boldsymbol{A}_k)^T l_{k+1} \tag{14b}$$

$$q = \begin{cases} 1 & \text{if } d_h^T g_k \ge 0.2 d_h^T \boldsymbol{Y}_k d_h \\ \dfrac{0.8 d_h^T \boldsymbol{Y}_k d_h}{d_h^T \boldsymbol{Y}_k d_h - d_h^T g_k} & \text{otherwise} \end{cases} \tag{14c}$$

3.3 Further Enhancements

The algorithm can be further enhanced by including a norm constraint on vector d_h and a line search step. The norm constraint is of importance because it validates the approximation (9). In doing so, the convex relaxation of problem (12) becomes

$$\text{minimize} \quad \frac{1}{2} d^T \boldsymbol{Y}_k d + d^T \boldsymbol{g}_k \tag{15c}$$

$$\text{subject to:} \quad \boldsymbol{A}_k d = -\boldsymbol{f}_k \tag{15b}$$

$$\|d\| \le b \tag{15c}$$

where b is a small positive scalar. The problem in (15) is a second-order cone programming problem [16] that can be solved using, for example, SeDuMi [17]. Having obtained the solution d, a line search is carried out by finding a positive scalar a_k that minimizes the following merit function

$$y(\boldsymbol{h}_k + ad) = e_2(\boldsymbol{h}_k + ad) + m\sum_{i=1}^{K} a_i^2(\boldsymbol{h}_k + ad) \tag{16}$$

where $m > 0$ weighs the importance of the constraints in (6b) in relative to the stopband energy. Having done this, the PF coefficient vector is updated from $\boldsymbol{h}_k$ to $\boldsymbol{h}_{k+1} = \boldsymbol{h}_k + a_k d$.

3.4 Alternating Null-Space Projections

The above method can be used to obtained a practically PR BCM filter bank when a sufficient number of iterations are carried out. Below we sketch a method that can be used to turn an NPR into a PR filter bank quickly provided that the NPR filter bank is *sufficiently* "close" to its PR counterpart.

A careful examination of the constraints in (5) shows that these equations can be expressed as either $\boldsymbol{C}_o \boldsymbol{h}_{ek} = \boldsymbol{b}_k$ or $\boldsymbol{C}_e \boldsymbol{h}_{ok} = \boldsymbol{b}_k$, where $\boldsymbol{h}_{ek}$ and $\boldsymbol{h}_{ok}$ are N/2-dimensional vectors formed by the even-indexed and odd-indexed components of $\boldsymbol{h}_k$, respectively, $\boldsymbol{C}_o$ and $\boldsymbol{C}_e$ are (N – M)/2 by N/2 matrices that are linearly determined by $\boldsymbol{h}_{ok}$ and $\boldsymbol{h}_{ek}$, respectively, and $\boldsymbol{b}_k$ is a constant vector of dimension (N – M)/2. Matrices $\boldsymbol{C}_o$ and $\boldsymbol{C}_e$ are in general of full row-rank. Consequently, for a fixed $\boldsymbol{h}_{ok}$ (or $\boldsymbol{h}_{ek}$), the null-spaces of linear operators $\boldsymbol{C}_o$ (or $\boldsymbol{C}_e$) are M/2-dimension subspaces in space $R^{N/2}$. Therefore, for a *fixed* $\boldsymbol{h}_{ok}$, if we denote a special solution of the *linear* system $\boldsymbol{C}_o \boldsymbol{h}_{ek} = \boldsymbol{b}_k$ by $\boldsymbol{h}_{es}$, then all solutions of the system can be expressed as $\boldsymbol{h}_{ek} = \boldsymbol{h}_{es} + \boldsymbol{V}_e x_e$ where $\boldsymbol{V}_e$ is a N/2 by M/2 matrix whose columns are a set of basis vectors in the null space of $\boldsymbol{C}_o$, and x_e is an M/2-dimensional "free" vector that can be determined by minimizing the stopband energy of the PF. The above process can be viewed as projecting vector $\boldsymbol{h}_k$ onto the null space so as to force the resulting coefficient vector to be PR. As such, it is expected that the change in the resulting coefficient vector will remain moderate if vector $\boldsymbol{h}_k$ is already close enough to its PR counterpart. Next, a similar projection is performed by fixing an $\boldsymbol{h}_{ek}$ and expressing the solutions of $\boldsymbol{C}_e \boldsymbol{h}_{ok} = \boldsymbol{b}_k$ as $\boldsymbol{h}_{ok} = \boldsymbol{h}_{os} + \boldsymbol{V}_o x_o$ where $\boldsymbol{V}_o$ is formed by the basis vectors of the null space of $\boldsymbol{C}_e$, and x_o is an M/2-dimensional free vector that can be determined by minimizing the stopband energy of the PF. The projection continues several times until the difference between the PF coefficient vectors before and after the projection becomes insignificant.

4. DESIGN EXAMPLES

The proposed algorithm was applied to design several BCM filter banks. In each design $r = 1$ and $m = 100$ were assumed. The algorithm was implemented using MATLAB on a Pentium III 1GHz PC. The design parameters and performance evaluation results are shown in Table I, where K_i denotes the number of iterations carried out in the first stage of the design, and Proj. # denotes the number of projections performed. As a representative of the designs, the amplitude responses of the PF and those analysis filters in the frequency range $0 \le w \le p/16$ for the 256-channel filter bank are shown in Figs. 1a and b, respectively.

Concerning the computational efficiency, note that solving the problem in (15) takes most of the CPU time in each iteration of the first design stage. The average CPU time for solving (15) in the four designs listed in Table I was 6.46, 40.60, 81.45, and 402.71 seconds, respectively. The CPU time required to carry out the second stage of the design was found insignificant in relative to that of the first stage.

Table I: Design Parameters and Performance Evaluation Results

M	32	64	128	256
N	320	640	1280	2560
D	255	511	1023	1535
$e_2(\boldsymbol{h})$	$1.04 \cdot 10^{-6}$	$5.77 \cdot 10^{-7}$	$3.01 \cdot 10^{-7}$	$2.65 \cdot 10^{-7}$
$\max\lvert e_m \rvert$	$2.68 \cdot 10^{-14}$	$4.34 \cdot 10^{-14}$	$1.24 \cdot 10^{-13}$	$2.05 \cdot 10^{-13}$
$\max\lvert e_{gd} \rvert$	$4.71 \cdot 10^{-11}$	$1.17 \cdot 10^{-10}$	$1.29 \cdot 10^{-11}$	$1.44 \cdot 10^{-11}$
$\max\lvert e_a \rvert$	$2.99 \cdot 10^{-14}$	$5.87 \cdot 10^{-14}$	$1.26 \cdot 10^{-13}$	$2.68 \cdot 10^{-13}$
K_i	100	200	550	590
Proj. #	10	10	0	0

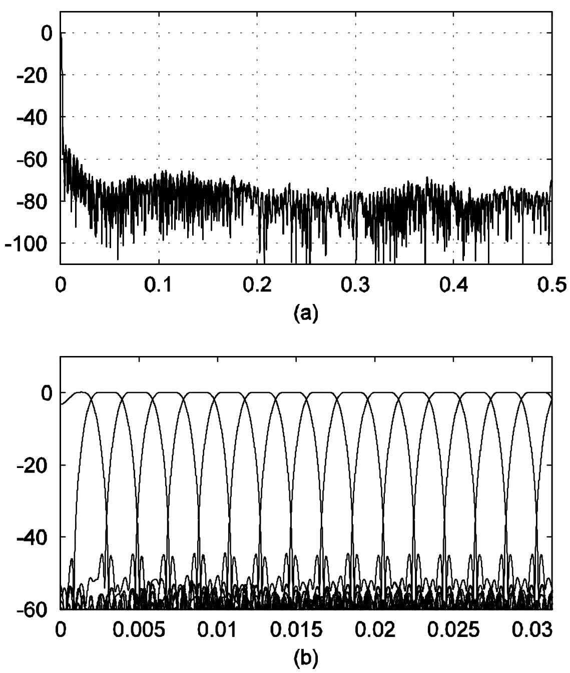

Figure 1. Amplitude responses of (a) the PF for the BCM filter bank with M = 256, N = 2560 and D = 1535; and (b) its analysis filter bank in the frequency range $0 \le w \le p/16$.

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