A NEAR-OPTIMAL MULTIUSER DETECTOR FOR CDMA CHANNELS USING SEMIDEFINITE PROGRAMMING RELAXATION

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ABSTRACT

A semidefinite-programming based multiuser detector is proposed. It is shown that maximum likelihood (ML) detection can be carried out by 'relaxing' the associated integer programming problem to a semidefinite-programming problem, which leads to a detector of polynomial complexity. Computer simulations presented demonstrate that the proposed detector offers near-optimal performance with much reduced computational complexity compared with that of the ML detector.

1. INTRODUCTION

Demodulation in the presence of multiple access interference (MAI) is of central importance in multiuser wireless communications. The optimal maximum likelihood (ML) multiuser detector for direct-sequence code-division multiple-access (DS-CDMA) channels was proposed by Verdù in [1]. The detection is carried out by solving a combinatorial optimization problem involving a quadratic objective function of a binary variable vector. Because of the exponential computational complexity of the ML detector, which becomes prohibitive even for a moderate number of users, several linear but suboptimal detectors were proposed following the pioneering work of Verdù. A detailed account of many useful multiuser detectors can be found in [2].

In this paper, a semidefinite-programming (SDP) based algorithm for multiuser detection is proposed. It is shown that the maximization of the ML function can be carried out by 'relaxing' the associated integer programming problem to an SDP problem where both the objective function and constraint functions are convex functions of continuous variables. This leads to a detection algorithm of polynomial complexity. Computer simulations are presented which demonstrate that the proposed detector offers near-optimal performance with a considerably reduced computational complexity compared with that of the ML detector. Comparisons with well-known linear detectors such as the decorrelating and minimum mean-square-error (MMSE) detectors are also presented.

2. CHANNEL MODEL AND REVIEW OF MULTIUSER DETECTION METHODS

2.1. CDMA Channel Model

We consider a synchronous DS-CDMA system, where K users transmit data packets through a single-path, frequency-nonselective, slowly Rayleigh fading channel. The bit interval of each user is T_b seconds and each information bit belongs to the set $\{1, -1\}$. Each signal is assigned a signature waveform (often called spreading sequence) $s(t)$ given by

$$s(t) = \sum_{i=1}^{N} (-1)^{c_i} p_{T_c}[t-(i-1)T_c] \quad \text{for } t \in [0,\ T_b] \tag{1}$$

where $p_{T_c}(t)$ is a rectangular pulse which takes a value of one for $0 \le t \le T_c$ and zero elsewhere, $\{c_1, c_2,\ \ldots,\ c_N\}$ is a binary sequence, and $N = T_b/T_c$ is the length of the signature waveform, which is often referred to as the *spreading gain*. The signature waveforms are normalized to have unit energy, i.e., $||s_k(t)||^2 = 1$, for $1 \le k \le K$. The received baseband signal is given by

$$y(t) = \sum_{i=0}^{\infty} \sum_{k=1}^{K} A_k^i b_k^i s_k(t - iT_b - \tau_k) + n(t) \tag{2}$$

where b_k^i is an information bit, τ_k is the transmission delay, A_k^i is the signal amplitude of the kth user, and $n(t)$ is additive white Gaussian noise (AWGN) with variance σ^2. In a synchronous DS-CDMA system, τ_k in (2) is set to zero for $1 \le k \le K$, and thus

$$y(t) = \sum_{k=1}^{K} A_k b_k s_k(t) + n(t) \tag{3}$$

where t is in the bit interval $[0,\ T_b]$.

The demodulation begins by filtering the received signal $y(t)$ with a bank of matched filters. The filter bank consists of K filters each matched to each signature waveform, and the filtered signals are sampled at the end of each bit interval. The outputs of the matched filters are given by

$$y_k = \int_0^{T_b} y(t) s_k(t)\, dt \qquad \text{for } 1 \le k \le K \tag{4}$$

Using (3), (4) can be expressed as

$$y_k = A_k b_k + \sum_{j \ne k} A_j b_j \rho_{jk} + n_k \qquad \text{for } 1 \le k \le K \tag{5}$$

where $\rho_{jk} = \int_0^{T_b} s_j(t) s_k(t)\, dt$ and $n_k = \int_0^{T_b} n(t) s_k(t)\, dt$. The discrete-time synchronous model in (5) can be described in matrix form as

$$\mathbf{y} = \mathbf{RAb} + \mathbf{n} \tag{6}$$

where $\mathbf{y} = [y_1\ y_2\ \cdots\ y_K]^T$, $\mathbf{A} = \text{diag}\,\{A_1,\ A_2,\ \ldots,\ A_K\}^T$, $\mathbf{b} = [b_1\ b_2\ \cdots\ b_K]^T$, $\mathbf{R}_{ij} = \rho_{ij}$, and $\mathbf{n} = [n_1\ n_2\ \cdots\ n_K]^T$.

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Since $n(t)$ in (2) is an AWGN with variance σ^2, $\mathbf{n}$ in (6) is a zero-mean Gaussian noise vector with covariance matrix $\sigma^2\mathbf{R}$.

The objective of multiuser detection is to detect the transmitted information vector $\mathbf{b}$ from $y(t)$, which is the superposition of the received user signals and noise. In the rest of this section, we briefly review three well-known multiuser detection methods for DS-CDMA channels.

2.2. Maximum Likelihood Multiuser Detector

The goal of the optimal multiuser detection is to generate an estimate of the information vector $\mathbf{b}$ in (6) that maximizes the likelihood function defined by

$$f(\mathbf{b}) = \exp\left(-\frac{1}{2\sigma^2}\int_0^{T_b}[y(t) - \sum_{k=1}^{K} A_k b_k s_k(t)]^2\,dt\right) \quad (7)$$

which is equivalent to maximizing the quadratic function

$$\Omega(\mathbf{b}) = 2\mathbf{b}^T\mathbf{A}\mathbf{y} - \mathbf{b}^T\mathbf{A}\mathbf{R}\mathbf{A}\mathbf{b} \quad (8)$$

By defining the unnormalized crosscorrelation matrix $\mathbf{H} = \mathbf{ARA}$ and $\mathbf{p} = -2\mathbf{A}y$, the maximum likelihood detector (MLD) is characterized by the solution of the combinatorial optimization problem [1]

$$\text{minimize} \quad \mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{p} \quad (9a)$$

$$\text{subject to :} \quad \mathbf{x}_i \in \{1, -1\} \qquad \text{for } i = 1, 2, \ldots, K \quad (9b)$$

Because of the binary constraints in (9b), the optimization problem in (9) is an integer programming problem. Its solution can be obtained by exhaustive evaluation of the objective function over 2^K possible $\mathbf{x}$. However, the computation involved becomes prohibitive even for a moderate number of users K.

The lower and upper bounds of the MLD are given in [1] as

$$P_k^l(\sigma) = 2^{1-w_{k,min}}Q\left(\frac{d_{k,min}}{\sigma}\right)$$

$$P_k^u(\sigma) = \sum_{\varepsilon \in E_k} 2^{-w(\varepsilon)}Q\left(\frac{||\mathbf{S}(\varepsilon)||}{\sigma}\right)$$

respectively, where $Q(\cdot)$ denotes the complementary unit cumulative Gaussian distribution, ε is a K-dimensional vector with $\varepsilon_k \in \{-1, 0, 1\}$, $E_k = \{\varepsilon : \varepsilon_k \neq 0\}$, $w(\varepsilon)$ denotes the number of nonzero components in ε, $w_{k,min}$ is the smallest possible value of $w(\varepsilon)$, $||\mathbf{S}(\varepsilon)||$ is the energy of the error vector ε, and $d_{k,min}$ is the minimum value of $||\mathbf{S}(\varepsilon)||$.

2.3. Decorrelating Multiuser Detector

It follows from (6) that

$$\mathbf{A}b = \mathbf{R}^{-1}\mathbf{y} - \mathbf{R}^{-1}\mathbf{n} = \mathbf{R}^{-1}\mathbf{y} + \hat{\mathbf{n}} \quad (10)$$

where $\hat{\mathbf{n}} = -\mathbf{R}^{-1}\mathbf{n}$ is a noise term with covariance matrix $\mathbf{E}(\hat{\mathbf{n}}\hat{\mathbf{n}}^T) = \sigma^2\mathbf{R}^{-1}$. Since $\text{sgn}(\mathbf{A}b) = \mathbf{b}$, (10) implies that

$$\mathbf{b} = \text{sgn}(\mathbf{A}b) = \text{sgn}(\mathbf{R}^{-1}\mathbf{y} + \hat{\mathbf{n}}) \quad (11)$$

If the noise term $\hat{\mathbf{n}}$ in (11) is neglected, then one can obtain an estimate of vector $\mathbf{b}$ as

$$\hat{\mathbf{b}} = \text{sgn}(\mathbf{R}^{-1}\mathbf{y}) \quad (12)$$

A detector based on (12) is referred to as a *decorrelating detector* (DD) [3]. An important feature of the DD is that it completely eliminates the MAI because the detection error for the kth user is only caused by the noise term $\hat{\mathbf{n}}$, which is independent of the signals of other users. The main problem with the DD is that neglecting the noise term in the estimate $\hat{\mathbf{b}}$ leads to an estimation error whose covariance is always larger than $\sigma^2\mathbf{I}$. The error probability of the decorrelating detector for the kth user is given by [2] as

$$P_k(\sigma) = Q\left[\sqrt{\frac{A_k}{\sigma\left(\mathbf{R}^{-1}\right)_{k,k}}}\right]$$

2.4. MMSE Multiuser Detector

When the SNR is low, neglecting $\hat{\mathbf{n}}$ in (11) introduces significant error in the DD and its performance becomes unsatisfactory. For a synchronous DS-CDMA channel modeled by (6), the minimum mean-square-error (MMSE) detector [4][5] is a linear detector represented by $\hat{\mathbf{b}} = \text{sgn}(\mathbf{P}y)$, where the transformation matrix $\mathbf{P}$ minimizes the mean-squared error $E[||\mathbf{b} - \mathbf{P}\mathbf{y}||^2]$, which is given by $\mathbf{P} = \mathbf{A}^{-1}[\mathbf{R} + \sigma^2\mathbf{A}^{-2}]^{-1}$. This leads to the estimate

$$\hat{\mathbf{b}} = \text{sgn}\left[\left(\mathbf{R} + \sigma^2\mathbf{A}^{-2}\right)^{-1}\mathbf{y}\right] \quad (13)$$

where $\sigma^2\mathbf{A}^{-2}$ is the SNR of the users. Evidently, the transformation $\mathbf{R}^{-1}$ in the DD is now replaced by $\left(\mathbf{R} + \sigma^2\mathbf{A}^{-2}\right)^{-1}$. From (12) and (13), we note that for small σ^2, the MMSE detector behaves like the DD. On the other hand, if the noise level becomes dominant, then the MMSE detector performs more like the conventional receiver. Therefore, unlike the DD which eliminates the MAI but ignores the channel noise, the MMSE detector offers a compromise solution that combats both the MAI and channel noise. The error probability of the MMSE detector for the kth user is given in [5] as

$$P_k = 2^{1-K}\sum_{\substack{b_i \in \{1,-1\} \\ i \neq k}} Q\left[\frac{A_k(\mathbf{P}R)_{kk}}{\sigma\sqrt{(\mathbf{P}RP)_{kk}}}\left(\frac{\sum_{i=1}^{K}\beta_i b_i}{\beta_k b_k}\right)\right]$$

where $B_i = A_i(\mathbf{P}R)_{ik}$ and $\beta_i = B_i/B_k$ for $i = 1, 2, \ldots, K$.

An additional advantage of the MMSE multiuser detector is that the channel estimation and matrix inversion can be eliminated by adaptive implementations [2].

3. NEW MULTIUSER DETECTOR USING SEMIDEFINITE PROGRAMMING RELAXATION

3.1. Semidefinite Programming

SDP is a class of numerical methods for the solution of optimization problems where the objective function is linear and the constraints are linear matrix inequalities (LMI) or equalities. A typical SDP problem formulation is given by

$$\text{minimize} \quad \mathbf{c}^T\mathbf{x} \quad (14a)$$

$$\text{subject to :} \quad \mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + \sum_{i=1}^{n} x_i\mathbf{F}_i \succeq \mathbf{0} \quad (14b)$$

where $\mathbf{x} = [x_1\ x_2\ \cdots\ x_n]^T$ denotes the variable vector, $\mathbf{c} \in R^{n\times 1}$, $\mathbf{F}_i \in R^{m\times m}$ for $0 \leq i \leq n$ are given and are symmetric matrices. The notation $\mathbf{F}(\mathbf{x}) \succeq \mathbf{0}$ denotes that matrix $\mathbf{F}(\mathbf{x})$ is positive semidefinite. Many important analysis and design problems in engineering can be formulated as SDP problems [6], and efficient interior-point optimization algorithms for the solution of SDP problems have been developed in the past several years [7]-[9].

3.2. SDP Relaxation of MAX-CUT Problem

The MAX-CUT problem is a well-known integer quadratic programming (IQP) problem in graph theory. It can be formulated as [2]

$$\text{maximize} \quad \frac{1}{2}\sum_{i<j}\sum w_{ij}(1 - x_i x_j) \tag{15a}$$

$$\text{subject to}: \quad x_i \in \{1, -1\} \quad \text{for } 1 \le i \le n \tag{15b}$$

The constraints in (15b) can be expressed as $x_i^2 = 1$ for $1 \le i \le n$. If we define a symmetric matrix $\mathbf{W} = (w_{ij})$ with $w_{ii} = 0$ for $1 \le i \le n$, then the objective function in (15a) can be expressed as

$$\frac{1}{2}\sum_{i<j}\sum w_{ij}(1 - x_i x_j) = \frac{1}{4}\sum\sum w_{ij} - \frac{1}{4}\text{tr}(\mathbf{W}X)$$

where tr(·) denotes the trace of the matrix and $\mathbf{X} = \mathbf{x}x^T$ with $\mathbf{x} = [x_1\ x_2\ \cdots\ x_n]^T$. Note that the set of matrices $\{\mathbf{X} : \mathbf{X} = \mathbf{x}x^T$ with $x_i^2 = 1$ for $1 \le i \le n\}$ can be characterized as $\{\mathbf{X} : x_{ii} = 1$ for $1 \le i \le n$, $\mathbf{X} \succeq \mathbf{0}$, and $\text{rank}(\mathbf{X}) = 1\}$. The above observations suggest the SDP problem

$$\text{minimize} \quad \text{tr}(\mathbf{W}X) \tag{16a}$$

$$\text{subject to}: \quad \mathbf{X} \succeq \mathbf{0} \tag{16b}$$

$$\mathbf{X}_{ii} = 1 \quad \text{for } 1 \le i \le n \tag{16c}$$

as a replacement of the problem in (15). Note that the only difference between the formulations in (15) and (16) is that the rank constraint $\text{rank}(\mathbf{X}) = 1$ is neglected in (16). The problem in (16) is said to be an SDP relaxation of the integer programming problem in (15). Following Geomans and Williamson [10], SDP relaxations of various integer programming problems have been used in graph optimization, network management, and scheduling [6][9]. In what follows, we present an SDP relaxation based multiuser detection algorithm.

3.3. SDP-Relaxation Based Multiuser Detector

By using the property that $\text{tr}(\mathbf{A}B) = \text{tr}(\mathbf{B}A)$, the objective function in (9a) can be expressed as $\mathbf{x}^T\mathbf{H}x + \mathbf{x}^T\mathbf{p} = \text{tr}(\mathbf{X}H) + \mathbf{x}^T\mathbf{p}$, where $\mathbf{X} = \mathbf{x}x^T$. Hence an SDP relaxation of the integer-programming problem in (9) can be formulated as

$$\text{minimize} \quad \text{tr}(\mathbf{X}H) + \mathbf{x}^T\mathbf{p} \tag{17a}$$

$$\text{subject to:} \quad \begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix} \succeq \mathbf{0} \tag{17b}$$

$$\mathbf{X}_{ii} = 1 \quad \text{for } i = 1, 2, \ldots, \mathrm{K} \tag{17c}$$

Once the solution $(\mathbf{X}^*, \mathbf{x}^*)$ of this problem is obtained, the binary solution of the original optimization problem in (9) can be obtained using two approaches. The first approach is simply to apply sgn(·) to $\mathbf{x}^*$: $\hat{\mathbf{b}}^* = \text{sgn}(\mathbf{x}^*)$. A better approach is based on the singular-value decomposition (SVD) of matrix $\mathbf{X}^*$, i.e., $\mathbf{X}^* = \mathbf{U}SV^T$, where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices and $\mathbf{S}$ is a diagonal matrix with the singular values of $\mathbf{X}^*$ on its diagonal in decreasing order [11]. Since $\mathbf{X}^*$ is symmetric, we have $\mathbf{U} = \mathbf{V}$. It is well known that an optimal rank-one approximation of $\mathbf{X}^*$ in the 2-norm is given by $\sigma_1\mathbf{u}_1\mathbf{u}_1^T$, where $\sigma_1 > 0$ is the largest singular value of $\mathbf{X}^*$ and $\mathbf{u}_1$ is its associated singular vector. Hence, the second approach generates the binary solution as $\hat{\mathbf{b}} = \text{sgn}(\mathbf{u}_1)$. Since $-\mathbf{u}_1$ is also a singular vector associated with σ_1, an alternative binary solution is given by $\hat{\mathbf{b}}' = \text{sgn}(-\mathbf{u}_1)$. The better solution between $\hat{\mathbf{b}}$ and $\hat{\mathbf{b}}'$ may be chosen by evaluating function $\Omega(\mathbf{b})$ in (8) to see which one yields a larger value.

Because of the relaxation involved, the proposed detector is *suboptimal*. As demonstrated in our simulations in Sec. 4, the new detector has a performance that is comparable with that of the optimal MLD but with considerably reduced computational complexity.

4. SIMULATION RESULTS

Computer simulations were conducted to evaluate the performance of the SDP-relaxation (SDPR) based multiuser detector in terms of bit-error rate (BER) and computational complexity. Comparisons with the multiuser detectors reviewed in Sec. 2 were also carried out.

The SDPR detectors were implemented using the MATLAB SDP toolbox [7]. Since the ultimate goal of the detector is to estimate the *binary-valued* information vector $\mathbf{b}$, a reasonably large tolerance ε was used for convergence verification. In order to further reduce the computational complexity, a suitable upper bound K_u was used to limit the number of iterations. In the simulation results reported in this section, the values $\varepsilon = 10^{-3}$ and $K_u = 5$ were used. The initial $(\mathbf{X}_0, \mathbf{x}_0)$ used for the SDP problem in (17) is given by

$$\mathbf{X}_0 = \mathbf{x}_m\mathbf{x}_m^T + \alpha\mathbf{I} \quad \text{and} \quad \mathbf{x}_0 = \mathbf{x}_m \tag{18}$$

where $\mathbf{x}_m$ is the information vector demodulated by the MMSE detector and α is a small positive scalar to assure the positive definiteness of matrix $\mathbf{X}_0$.

Due to the extremely high computational complexity of the MLD, the BER performance of the MLD was evaluated for a synchronous DS-CDMA system with six equal-power users. The user signatures used in the transmission were 16-chip Gold sequences. Over 10^4 runs were performed to evaluate the average BER of the MLD, DD, MMSE, matched-filter (MF) based, and SDPR detectors, and the results obtained are depicted in Fig. 1.

The second set of simulations deals with a synchronous DS-CDMA system with eight unequal-power users in a frequency-flat Rayleigh fading channel. The user signatures used were the same as those in the first example and the signal amplitudes of the eight users were set to 2.24, 1.73, 1.34, 1.00, 0.77, 0.55, 0.45, and 0.32, respectively, which remained unchanged throughout the transmission. The fourth user with unit power was designated as the desired one and the coherent time of the fading channel was set to $10T_b$. The signal-to-noise ratio (SNR) for the MMSE detectors was assumed to be known. The average BER (over 10^4 runs) of the MLD, DD, MMSE, matched-filter (MF) based, and SDPR detectors are plotted in Fig. 2.

From the results presented in the above examples, it is observed that the BER of the SDPR detector is very close to that of the MLD and superior to that of the DD and MMSE detectors. The computational complexity of the detectors was evaluated in terms of number of floating-point operations (flops) for different numbers of users. It is noted that the computational complexity of the MLD increases exponentially with the number of users, whereas the SDPR detector requires a considerably reduced amount of computation, which grows with the number of users in a nonexponential manner. Table 1 gives the amount of computation in terms of flops required

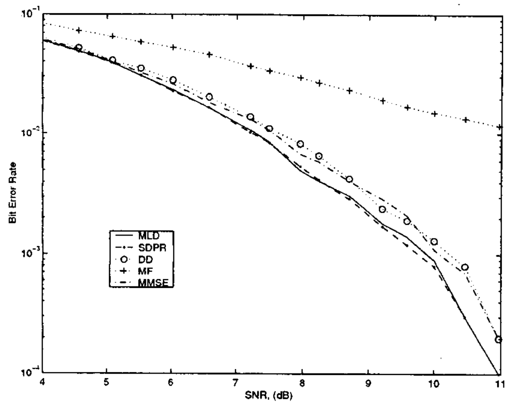

Figure 1: BER of six equal-power users.

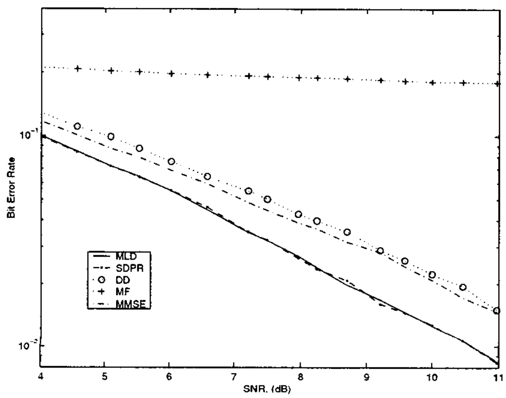

Figure 2: BER of eight unequal-power users.

to detect the *first* information bit for different detectors. For the DD and MMSE detectors with time-invariant SNR, the inverse of the crosscorrelation matrix needs to be evaluated only once, which means that the amount of computation required by the linear detectors to detect subsequent information bits is reduced to just a few multiplications.

Although the SDPR detector has demonstrated a performance comparable with that of the MLD with a considerably reduced amount of computation, the computational burden is still much heavier than that for the DD and MMSE detectors, and to alleviate this problem, we are presently examining a number of approaches to improve the computational efficiency of the SDPR detector.

5. CONCLUSIONS

A semidefinite programming based multiuser detection algorithm has been proposed. It has been shown that by neglecting the rank constraint on the positive semidefinite matrix, the integer programming problem for the design of ML detector can be relaxed into an SDP problem. The major advantages of the proposed detection algorithm are its considerably reduced computational complexity and near-optimal BER performance compared with that of the ML detector. These technical advantages have been demonstrated by

Table 1: Comparisons of computational complexity for detecting one information bit.

Number of Users	Number of Flops Used			
	DD	MMSE	MLD	SDPR
10	2.8×10^3	2.8×10^3	4.7×10^5	5.5×10^4
11	3.6×10^3	3.6×10^3	1.0×10^6	7.0×10^4
12	4.5×10^3	4.6×10^3	2.6×10^6	8.6×10^4
13	5.6×10^3	5.7×10^3	6.2×10^6	1.0×10^5
14	6.9×10^3	7.0×10^3	1.4×10^7	1.3×10^5
15	8.4×10^3	8.5×10^3	3.6×10^7	1.5×10^5
16	1.0×10^4	1.0×10^4	—	1.7×10^5
17	1.2×10^4	1.2×10^4	—	2.1×10^5
18	1.4×10^4	1.4×10^4	—	2.4×10^5
19	1.6×10^4	1.6×10^4	—	2.8×10^5
20	1.9×10^4	1.9×10^4	—	3.2×10^5
21	2.2×10^4	2.2×10^4	—	3.6×10^5
22	2.5×10^4	2.5×10^4	—	4.1×10^5
23	2.8×10^4	2.8×10^4	—	4.7×10^5
24	3.2×10^4	3.2×10^4	—	5.2×10^5
25	3.6×10^4	3.6×10^4	—	5.9×10^5

computer simulations.

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