NUMERICAL SOLUTION OF THE TWO-DIMENSIONAL LYAPUNOV EQUATIONS AND APPLICATION IN ORDER REDUCTION OF RECURSIVE DIGITAL FILTERS

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ABSTRACT

There are three different grammians, *pseudo-grammians*, *quasi-grammians*, and *structured grammians* which can be defined for a given two-dimensional digital filter. Reliable methods for the evaluation of *quasi-grammians* and *structured grammians* are not available to date. In this paper, an iterative method for the computation of 2-D quasi-grammians is developed, where each iteration involves solving two 1-D Lyapunov equations. For a stable 2-D system, the iterative algorithm converges to the desired solutions very quickly. Further, an algorithm based on unconstrained optimization is developed for the computation of structured grammians. The algorithm involves two steps. The first step is to minimize the norm of the system matrix by 2-D similarity transformations and the second step is to use a scaling technique to accomplish the optimization. Two examples are given to evaluate the performance of the reduced-order systems that are obtained from the balanced approximations using different grammians.

1. INTRODUCTION

Controllability and observability grammians have been extensively used in the analysis and synthesis of linear dynamical systems [1] and in balanced approximations for 1-D and 2-D systems [2] [3]. Three different 2-D grammians have been defined and studied in the literature, namely, *pseudo-grammians* [3], *quasi-grammians* [4] and *structured grammians* [5] [4]. Pseudo-grammians find applications in model reduction [3], round-off noise minimization [6], and filter design [6]. The computation of pseudo-grammians is addressed in [3] and very recently in [7]. A major problem with pseudo-grammians is that balanced approximations based on these grammians may lead to unstable reduced-order systems [8]. In addition, the computation required is quite extensive when the system's order is higher than 10. Quasi-grammians involve less computation, but a reliable method for their evaluation is not available to date. Structured grammians are defined by two 2-D Lyapunov inequalities. Although it has been shown that a balanced approximation based on these grammians always leads to stable reduced-order systems, methods for their numerical evaluation need to be developed.

In this paper, an iterative method for the computation of *quasi-grammians* is described. Each iteration involves solving two 1-D Lyapunov equations, which can be accomplished efficiently. If a 2-D system is stable, then the algorithm converges to the 2-D quasi-grammians very quickly. To evaluate structured grammians, an algorithm based on unconstrained optimization is developed. The algorithm involves two steps. The first step is to minimize the norm of the system matrix by 2-D similarity transformations and the second step is to use a scaling technique to accomplish the optimization. Two examples are presented to illustrate the algorithms and to evaluate the performance of the reduced-order system that are obtained from the balanced approximations using different grammians.

2. QUASI-GRAMMIANS AND COMPUTATION

Consider a 2-D digital filter system represented by Roesser's local state space model, i.e.,

$$\begin{aligned}\begin{bmatrix}\mathbf{x}^h(i+1,j)\\ \mathbf{x}^v(i,j+1)\end{bmatrix} &= \begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_3 & \mathbf{A}_4\end{bmatrix}\begin{bmatrix}\mathbf{x}^h(i,j)\\ \mathbf{x}^v(i,j)\end{bmatrix} + \begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix}\mathbf{u}(i,j)\\ &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad (1a)\\ \mathbf{y}(i,j) &= \begin{bmatrix}\mathbf{C}_1 & \mathbf{C}_2\end{bmatrix}\begin{bmatrix}\mathbf{x}^h(i,j)\\ \mathbf{x}^v(i,j)\end{bmatrix} + \mathbf{D}\mathbf{u}(i,j)\\ &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \qquad (1b)\end{aligned}$$

where $\mathbf{x}^h(i,j) \in \mathcal{R}^{(n\times 1)}$, $\mathbf{x}^v(i,j) \in \mathcal{R}^{(m\times 1)}$, $\mathbf{u}(i,j) \in \mathcal{R}^{(q\times 1)}$, and $\mathbf{y}(i,j) \in \mathcal{R}^{(p\times 1)}$.

Definition 1 2-D system (1) is called *quasi-balanced* if the solutions to the following linear equations

$$\begin{aligned}
\mathbf{A}_1\mathbf{P}_1\mathbf{A}_1^T - \mathbf{P}_1 + \mathbf{B}_1\mathbf{B}_1^T + \mathbf{A}_2\mathbf{P}_2\mathbf{A}_2^T &= 0 \quad (2a)\\
\mathbf{A}_4\mathbf{P}_2\mathbf{A}_4^T - \mathbf{P}_2 + \mathbf{B}_2\mathbf{B}_2^T + \mathbf{A}_3\mathbf{P}_1\mathbf{A}_3^T &= 0 \quad (2b)\\
\mathbf{A}_1^T\mathbf{Q}_1\mathbf{A}_1 - \mathbf{Q}_1 + \mathbf{C}_1^T\mathbf{C}_1 + \mathbf{A}_3^T\mathbf{Q}_2\mathbf{A}_3 &= 0 \quad (2c)\\
\mathbf{A}_4^T\mathbf{Q}_2\mathbf{A}_4 - \mathbf{Q}_2 + \mathbf{C}_2^T\mathbf{C}_2 + \mathbf{A}_2^T\mathbf{Q}_1\mathbf{A}_2 &= 0 \quad (2d)
\end{aligned}$$

satisfy $\mathbf{P}_1 = \mathbf{Q}_1 = \mathrm{diag}(\nu_1, \cdots, \nu_n) \geq 0$ and $\mathbf{P}_2 = \mathbf{Q}_2 = \mathrm{diag}(\kappa_1, \cdots, \kappa_m) \geq 0$. $\mathbf{P} = [\mathbf{P}_1 \oplus \mathbf{P}_2]$ and $\mathbf{Q} = [\mathbf{Q}_1 \oplus \mathbf{Q}_2]$ are called 2-D *quasi-grammians*.

The following algorithm is proposed for the computation of 2-D quasi-grammians.

Algorithm 1

Step 1: Set $\mathbf{P}_2^{(0)} = \mathbf{Q}_2^{(0)} = 0$ and $k = 1$.

Step 2: Solve the following 1-D Lyapunov equations for $\mathbf{P}_1^{(k)}$ and $\mathbf{Q}_1^{(k)}$:

$$\begin{aligned}
&\mathbf{A}_1\mathbf{P}_1^{(k)}\mathbf{A}_1^T - \mathbf{P}_1^{(k)} + \mathbf{F}_1 = 0 \quad (3a)\\
&\mathbf{A}_1^T\mathbf{Q}_1^{(k)}\mathbf{A}_1 - \mathbf{Q}_1^{(k)} + \mathbf{G}_1 = 0 \quad (3c)\\
&\mathbf{F}_1 = \mathbf{B}_1\mathbf{B}_1^T + \mathbf{A}_2\mathbf{P}_2^{(k-1)}\mathbf{A}_2^T\\
&\mathbf{G}_1 = \mathbf{C}_1^T\mathbf{C}_1 + \mathbf{A}_3^T\mathbf{Q}_2^{(k-1)}\mathbf{A}_3
\end{aligned}$$

Step 3: Solve the following 1-D Lyapunov equations for $\mathbf{P}_2^{(k)}$ and $\mathbf{Q}_2^{(k)}$:

$$\begin{aligned}
&\mathbf{A}_4\mathbf{P}_2^{(k)}\mathbf{A}_4^T - \mathbf{P}_2^{(k)} + \mathbf{F}_2 = 0 \quad (3b)\\
&\mathbf{A}_4^T\mathbf{Q}_2^{(k)}\mathbf{A}_4 - \mathbf{Q}_2^{(k)} + \mathbf{G}_2 = 0 \quad (3d)\\
&\mathbf{F}_2 = \mathbf{B}_2\mathbf{B}_2^T + \mathbf{A}_3\mathbf{P}_1^{(k)}\mathbf{A}_3^T\\
&\mathbf{G}_2 = \mathbf{C}_2^T\mathbf{C}_2 + \mathbf{A}_2^T\mathbf{Q}_1^{(k)}\mathbf{A}_2
\end{aligned}$$

Step 4: Set $k = k + 1$ and repeat Steps 2 and 3 until

$$\begin{aligned}
||\mathbf{P}_i^{(k)} - \mathbf{P}_i^{(k-1)}|| &< \epsilon, \quad (i = 1,2)\\
||\mathbf{Q}_i^{(k)} - \mathbf{Q}_i^{(k-1)}|| &< \epsilon, \quad (i = 1,2)
\end{aligned}$$

where ϵ is a prescribed tolerance. Obviously, if $\mathbf{P}_i^{(k)} \to \mathbf{P}_i$ and $\mathbf{Q}_i^{(k)} \to \mathbf{Q}_i$ for $i = 1,2$ as $k \to \infty$, then $\mathbf{P}_i$ and $\mathbf{Q}_i$ satisfy (2); therefore, they are the *quasi-grammians* of the 2-D filter.

It follows from (2) that

$$\begin{aligned}
\mathbf{A}_1\mathbf{P}_1\mathbf{A}_1^T - \mathbf{P}_1 + \mathbf{A}_2\mathbf{P}_2\mathbf{A}_2^T &= -\mathbf{B}_1\mathbf{B}_1^T\\
\mathbf{A}_4\mathbf{P}_2\mathbf{A}_4^T - \mathbf{P}_2 + \mathbf{A}_3\mathbf{P}_1\mathbf{A}_3^T &= -\mathbf{B}_2\mathbf{B}_2^T
\end{aligned}$$

so, if we let $\mathbf{T} = [\mathbf{T}_1 \oplus \mathbf{T}_2] = [\mathbf{P}_1^{1/2} \oplus \mathbf{P}_2^{1/2}]$ and $\mathbf{A} = [\mathbf{T}^{-1}\mathbf{A}\mathbf{T}]$, then

$$\begin{bmatrix} \tilde{\mathbf{A}}_1 & \tilde{\mathbf{A}}_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{A}}_1^T \\ \tilde{\mathbf{A}}_2^T \end{bmatrix} - \mathbf{I} \leq 0 \quad (4a)$$

$$\begin{bmatrix} \tilde{\mathbf{A}}_3 & \tilde{\mathbf{A}}_4 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{A}}_3^T \\ \tilde{\mathbf{A}}_4^T \end{bmatrix} - \mathbf{I} \leq 0 \quad (4b)$$

Consequently,

$$|| \begin{bmatrix} \tilde{\mathbf{A}}_1 & \tilde{\mathbf{A}}_2 \end{bmatrix} || \leq 1, \qquad || \begin{bmatrix} \tilde{\mathbf{A}}_3 & \tilde{\mathbf{A}}_4 \end{bmatrix} || \leq 1$$

and

$$|| \tilde{\mathbf{A}} || \leq 2 \quad (5)$$

3. STRUCTURED GRAMMIANS AND COMPUTATION

In what follows, we assume that the 2-D filter under consideration satisfies the 2-D Lyapunov equation

$$\mathbf{A}\mathbf{P}\mathbf{A}^T - \mathbf{P} = -\mathbf{W} \quad (6)$$

where $\mathbf{P} = [\mathbf{P}_1 \oplus \mathbf{P}_2]$ and $\mathbf{W}$ are both positive definite.

Definition 2 [5] 2-D system (1) is said to be *Q-stable* if there exist a $\mathbf{P} = [\mathbf{P}_1 \oplus \mathbf{P}_2] > 0$ and a $\mathbf{Q} = [\mathbf{Q}_1 \oplus \mathbf{Q}_2] > 0$ with $\mathbf{P}_1, \mathbf{Q}_1 \in \mathcal{R}^{(n\times n)}$ and $\mathbf{P}_2, \mathbf{Q}_2 \in \mathcal{R}^{(m\times m)}$ such that

$$\begin{aligned}
\mathbf{A}\mathbf{P}\mathbf{A}^T - \mathbf{P} + \mathbf{B}\mathbf{B}^T &< 0 \quad (7a)\\
\mathbf{A}^T\mathbf{Q}\mathbf{A} - \mathbf{Q} + \mathbf{C}^T\mathbf{C} &< 0 \quad (7b)
\end{aligned}$$

Matrices $\mathbf{P}$ and $\mathbf{Q}$ are called the *structured grammians*. They can be obtained by solving the following minimization problems [5]

$$\min_{\mathbf{P}} \rho_{\max}(\mathbf{A}\mathbf{P}\mathbf{A}^T - \mathbf{P} + \mathbf{B}\mathbf{B}^T) \quad (8a)$$

$$\min_{\mathbf{Q}} \rho_{\max}(\mathbf{A}^T\mathbf{Q}\mathbf{A} - \mathbf{Q} + \mathbf{C}^T\mathbf{C}) \quad (8b)$$

In general, if there are solutions $\mathbf{P}$ and $\mathbf{Q}$ such that

$$\begin{aligned}
\rho_{\max}(\mathbf{A}\mathbf{P}\mathbf{A}^T - \mathbf{P} + \mathbf{B}\mathbf{B}^T) &< 0\\
\rho_{\max}(\mathbf{A}^T\mathbf{Q}\mathbf{A} - \mathbf{Q} + \mathbf{C}^T\mathbf{C}) &< 0
\end{aligned}$$

then there are infinitely many solutions.

As the structured grammians $\mathbf{P}$ and $\mathbf{Q}$ are positive definite matrices, they can be written as

$$\mathbf{P} = \mathbf{T}_p\mathbf{T}_p^T, \quad \mathbf{Q} = \mathbf{T}_q\mathbf{T}_q^T \quad (9)$$

Hence, inequalities in (7) imply that

$$\begin{aligned}
&\begin{bmatrix} \mathbf{T}_p^{-1}\mathbf{A}\mathbf{T}_p & \mathbf{T}_p^{-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{T}_p^{-1}\mathbf{A}\mathbf{T}_p & \mathbf{T}_p^{-1}\mathbf{B} \end{bmatrix}^T < \mathbf{I}\\
&\begin{bmatrix} \mathbf{T}_q^{-1}\mathbf{A}^T\mathbf{T}_q & \mathbf{T}_q^{-1}\mathbf{C}^T \end{bmatrix} \begin{bmatrix} \mathbf{T}_q^{-1}\mathbf{A}^T\mathbf{T}_q & \mathbf{T}_q^{-1}\mathbf{C}^T \end{bmatrix}^T < \mathbf{I}
\end{aligned}$$

Note that

$$\| [\ \mathbf{T}_p^{-1}\mathbf{A}\mathbf{T}_p \quad \mathbf{T}_p^{-1}\mathbf{B}\ ] \| =$$

$$\| \begin{bmatrix} \mathbf{T}_p & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}_p & 0 \\ 0 & 1 \end{bmatrix} \| \quad (10a)$$

$$\| [\ \mathbf{T}_q^{-1}\mathbf{A}^T\mathbf{T}_q \quad \mathbf{T}_q^{-1}\mathbf{C}^T\ ] \| =$$

$$\| \begin{bmatrix} \mathbf{T}_q & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}^T & \mathbf{C}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}_q & 0 \\ 0 & 1 \end{bmatrix} \| (10b)$$

If $\mathbf{T}_p^T = \mathbf{Q}_p\mathbf{R}_p$ and $\mathbf{T}_q^T = \mathbf{Q}_q\mathbf{R}_q$, then

$$\mathbf{T}_p = \mathbf{L}_p\mathbf{Q}_p^T \qquad \mathbf{T}_q = \mathbf{L}_q\mathbf{Q}_q^T \tag{11}$$

where $\mathbf{Q}_p$ and $\mathbf{Q}_q$ are orthogonal matrices, $\mathbf{R}_p$ and $\mathbf{R}_q$ are upper triangular matrices, $\mathbf{L}_p$ and $\mathbf{L}_q$ are lower triangular matrices. Instead of solving the minimization problems in (8), we consider the following minimization problems

$$\min_{\mathbf{L}_p} \| [\ \tilde{\mathbf{L}}_p^{-1}\mathbf{A}_p\tilde{\mathbf{L}}_p\ ] \| < 1 \tag{12a}$$

$$\min_{\mathbf{L}_q} \| [\ \tilde{\mathbf{L}}_q^{-1}\mathbf{A}_q\tilde{\mathbf{L}}_q\ ] \| < 1 \tag{12b}$$

where

$$\tilde{\mathbf{L}}_p = \begin{bmatrix} \mathbf{L}_p & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}_p = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{L}}_q = \begin{bmatrix} \mathbf{L}_q & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}_q = \begin{bmatrix} \mathbf{A}^T & \mathbf{C}^T \\ 0 & 0 \end{bmatrix}$$

It can be shown that

$$\rho(\mathbf{T}^{-1}\mathbf{A}\mathbf{T}) < 1 \Longleftrightarrow \mathbf{A}\mathbf{P}\mathbf{A}^T - \mathbf{P} < 0 \tag{13a}$$

$$\Longleftrightarrow \mathbf{A}^T\mathbf{Q}\mathbf{A} - \mathbf{Q} < 0 \tag{13b}$$

If $\mathbf{P}$ and $\mathbf{Q}$ are scaled, it can be deduced that

$$k_p(\mathbf{A}\mathbf{P}\mathbf{A}^T - \mathbf{P}) + \mathbf{B}\mathbf{B}^T < 0 \tag{14a}$$

$$k_q(\mathbf{A}^T\mathbf{Q}\mathbf{A} - \mathbf{Q}) + \mathbf{C}^T\mathbf{C} < 0 \tag{14b}$$

Let

$$\mathbf{A}\mathbf{P}\mathbf{A}^T - \mathbf{P} = -\mathbf{W}_p = -\mathbf{U}_p\mathbf{S}_p\mathbf{U}_p^T = -\tilde{\mathbf{U}}_p\tilde{\mathbf{U}}_p^T \tag{15a}$$

$$\mathbf{A}^T\mathbf{Q}\mathbf{A} - \mathbf{Q} = -\mathbf{W}_q = -\mathbf{U}_q\mathbf{S}_q\mathbf{U}_q^T = -\tilde{\mathbf{U}}_q\tilde{\mathbf{U}}_q^T \tag{15b}$$

where $\mathbf{W}_p, \mathbf{W}_q > 0$, and $\mathbf{U}_p\mathbf{S}_p\mathbf{U}_p^T$ and $\mathbf{U}_q\mathbf{S}_q\mathbf{U}_q^T$ are the singular-value decompostions of $\mathbf{W}_p$ and $\mathbf{W}_q$, respectively. Inequalities in (14) imply that

$$k_p\mathbf{I} - \tilde{\mathbf{U}}_p^{-1}\mathbf{B}\mathbf{B}^T\tilde{\mathbf{U}}_p^{-T} < 0$$

$$k_q\mathbf{I} - \tilde{\mathbf{U}}_q^{-1}\mathbf{C}^T\mathbf{C}\tilde{\mathbf{U}}_q^{-T} < 0$$

Therefore, scaling factors k_p and k_q can be chosen as follows

$$k_p = \max\ (\text{eigenvale of } \tilde{\mathbf{U}}_p^{-1}\mathbf{B}\mathbf{B}^T\tilde{\mathbf{U}}_p^{-T})^{1/2} \tag{16a}$$

$$k_q = \max\ (\text{eigenvale of } \tilde{\mathbf{U}}_q^{-1}\mathbf{C}^T\mathbf{C}\tilde{\mathbf{U}}_q^{-T})^{1/2} \tag{16b}$$

Algorithm 2

Step 1: Find $\mathbf{L}_p$ and $\mathbf{L}_q$ by solving the following minimization problems

$$\min_{\mathbf{L}_p} \| [\ \mathbf{L}_p^{-1}\mathbf{A}\mathbf{L}_p\ ] \| < 1 \tag{17a}$$

$$\min_{\mathbf{L}_q} \| [\ \mathbf{L}_q^{-1}\mathbf{A}^T\mathbf{L}_p\ ] \| < 1 \tag{17b}$$

Step 2: Let $\mathbf{P} = \mathbf{L}_p\mathbf{L}_p^T$ and $\mathbf{Q} = \mathbf{L}_q\mathbf{L}_q^T$ and compute k_p and k_q by using (15) and (16).

Step 3: Solve the following minimization problems

$$\min_{\mathbf{L}_p} \| [\ \hat{\mathbf{L}}_p^{-1}\mathbf{A}_p\hat{\mathbf{L}}_p\ ] \| < 1 \tag{18a}$$

$$\min_{\mathbf{L}_q} \| [\ \hat{\mathbf{L}}_q^{-1}\mathbf{A}_q\hat{\mathbf{L}}_q\ ] \| < 1 \tag{18b}$$

where

$$\hat{\mathbf{L}}_p = \begin{bmatrix} k_p\mathbf{L}_p & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{\mathbf{L}}_q = \begin{bmatrix} k_q\mathbf{L}_q & 0 \\ 0 & 1 \end{bmatrix}$$

Step 4: Compute structured grammians

$$\mathbf{P} = \mathbf{L}_p\mathbf{L}_p^T, \qquad \mathbf{Q} = \mathbf{L}_q\mathbf{L}_q^T$$

4. EXAMPLES

The first example is a 2-D system considered in [8], where it is shown that the reduced model based on the pseudo-balanced approximation is unstable. The model is given by (1) with $n_1 = 2$, $n_2 = 2$, $\mathbf{D} = 0$,

$$\mathbf{A} = \begin{bmatrix} 0.9399204 & 0.0682773 & -0.1221131 & 0.2383674 \\ -0.0233407 & 0.8600796 & 0.0713971 & -0.1393688 \\ 0.4183718 & 0.2402084 & 0.9874241 & 0.0672975 \\ -0.1313337 & 0.3928846 & -0.1135699 & 0.8125759 \end{bmatrix}$$

$$\mathbf{B} = [-0.3142416\ \ -0.3879902\ \ -0.0303866\ \ -0.0008281]^T$$

$$\mathbf{C} = [-0.3528145\ \ 0.2728651\ \ -0.1139450\ \ 0.1961035]$$

We apply Algorithms 1 and 2 to compute the quasi-grammians and structured grammians, respectively, and obtain the corresponding balanced realizations

$$[\mathbf{A}_b, \mathbf{B}_b, \mathbf{C}_b, \mathbf{D}] = [\mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \mathbf{T}^{-1}\mathbf{B}, \mathbf{C}\mathbf{T}, \mathbf{D}]$$

The performance of the reduced models obtained from pseudo-, quasi-, and structurally balanced approximations is summarized in Table 1.

Table 1: Performance of Reduced Models

$n_{r1} = 2,\ n_{r2} = 1$	Stability	Error
pseudo-balanced	unstable	9.2272
quasi-balanced	stable	3.8951
structurally balanced	stable	3.4502

The second example is an example used in [4]. The model is given by (1) with $n_1 = 4$, $n_2 = 8$, $\mathbf{D} = 1.34044 \times 10^{-3}$,

$$\mathbf{A}_1 = \begin{bmatrix} 0.53700 & -0.06881 & 0.98551 & 0.50388 \\ 1.00000 & 0 & 0 & 0 \\ 0 & 0 & 0.538819 & -0.066576 \\ 0 & 0 & 1.000000 & 0 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} -0.390721 & 0.244967 & -0.483603 & -0.247260 \\ 0.251223 & -0.145125 & 0.026974 & 0.013792 \\ 1.270478 & 1.106765 & 0.198140 & 0.101307 \\ 1.796448 & 0.421991 & 0.592117 & 0.302742 \\ 0 & 0 & -0.393352 & 0.242461 \\ 0 & 0 & 0.252014 & -0.145254 \\ 0 & 0 & 1.270845 & 1.107215 \\ 0 & 0 & 1.797547 & 0.423333 \end{bmatrix}$$

$$\mathbf{A}_4 = \begin{bmatrix} 0.490714 & 1 & 0 & 0 & 0.490714 & 0 & -0.490714 & 0 \\ -0.027371 & 0 & 0 & 0 & -0.027371 & 0 & 0.027371 & 0 \\ -0.201054 & 0 & 0 & 1 & -0.201054 & 0 & 0.201054 & 0 \\ -0.600823 & 0 & 0 & 0 & -0.600823 & 0 & 0.600823 & 0 \\ 0 & 0 & 0 & 0 & 0.491231 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.028179 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.201054 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -0.600823 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}_1 = 10^{-3} \times \begin{bmatrix} 1.34044 & 0 & 1.34044 & 0 \end{bmatrix}$$

$$\mathbf{B}_{21} = \begin{bmatrix} -0.657773 & 0.036689 & 0.269501 & 0.805367 \end{bmatrix}$$

$$\mathbf{B}_{22} = \begin{bmatrix} -0.658465 & 0.037772 & 0.269501 & 0.805367 \end{bmatrix}$$

$$\mathbf{B}_2 = 10^{-3} \times \begin{bmatrix} \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

$$\mathbf{C}_1 = \begin{bmatrix} 0.983681 & 0.501644 & 0.985510 & 0.503879 \end{bmatrix}$$

$$\mathbf{C}_2 = \begin{bmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

The results are summarized in Table 2.

Table 2: Performance of Reduced Models

$n_{r1} = 4,\ n_{r2} = 4$	Stability	Error
pseudo-balanced	stable	0.1102
quasi-balanced	stable	0.2783
structurally balanced	stable	1.0682

The structurally balanced approximation produces the most stable reduced-order system for both examples 1 and 2. Pseudo-balanced approximation results in a stable reduced-order system for example 2, but, an unstable one for example 1. On the other hand, the quasi-balanced approximation results in a stable reduced-order system for example 1 and a marginally stable one for example 2. As long as the approximation error is concerned, the structurally balanced approximation gives the smallest maximum error for example 1, but the largest maximum error for example 2.

5. CONCLUSIONS

An iterative method for the computation of 2-D *quasi-grammians* has been developed. Each iteration solves two 1-D Lyapunov equations. For a 2-D stable system, the iterative algorithm is quite fast. An algorithm based on unconstrained optimization has been developed to compute *structured grammians*. As can be seen from Definition 2, there are infinitely many solutions which satisfy (8). Finding the structured grammians that would lead to minimal error for the reduced model still remains an open problem. A great deal of research is also needed to clarify the merits and demerits associated with the various types of grammians.

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