DESIGN OF 2-D DIGITAL FILTERS WITH ARBITRARY AMPLITUDE AND PHASE RESPONSES BY USING THE SINGULAR VALUE DECOMPOSITION

W.-S. Lu and A. Antoniou

Department of Electrical and Computer Engineering
University of Victoria
Victoria, B.C. Canada V8W 3P6

Abstract

The SVD has been applied for the design of IIR filters with guadrantally symmeric amplitude response and for the design of FIR filters with arbitrary amplitude response and linear phase response. In this paper the SVD method is extended to include the design of FIR and IIR 2-D filters with arbitrary amplitude and phase responses.

The method is relatively simple to apply and leads to a parallel arrangement of pairs of cascaded 1-D filter sections. Structures of this type allow a large amount of concurrent processing and, further, they can be implemented in terms of systolic arrays. Therefore, they are amenable to VLSI implementation.

Introduction

The singular-value decomposition (SVD) [1][2] is a numerically reliable matrix decomposition that has found numerous scientific and engineering applications in the past[3]-[5]. An important application of the SVD is concerned with the design of two-dimensional (2-D) digital filters. In [6] the SVD has been applied for the design of infinite-impulse response (IIR) filters with arbitrary amplitude responses; in [7] it has been used for the design of finite-impulse response (FIR) filters with arbitrary amplitude responses and linear phase responses; in [8] it has been used in conjunction with a balanced approximation for the design of IIR filters with arbitrary amplitude responses and linear phase responses. Other related methods are detailed in [9]-[10].

In this paper, the SVD method is extended to include the design of FIR and IIR 2-D filters with arbitrary amplitude and phase responses. The design starts by sampling the desired frequency response in order to generate a *complex* matrix that represents both the amplitude and phase responses. The SVD is then applied to this matrix to obtain an outer-product sum expression. It is shown in Section II that each vector in a outer-product term can be interpreted as the frequency response of a 1-D digital filter and, therefore, the design task can be completed if methods for the design of 1-D digital filters with arbitrary amplitude *and* phase responses are available. Two such methods for the design of 1-D FIR filters based on least-squares and least-pth optimization are presented in Section III.

II. The SVD of A Complex Matrix and Its Interpretation

2.1 The SVD of A Complex Matrix

If $\mathbf{F}$ is an arbitrary complex matrix of dimension $N_1 \times N_2$, with $N_1 \geq N_2$, then there exist unitary matrices $\mathbf{U} \in C^{N_1 \times N_1}$ and $\mathbf{V} \in C^{N_2 \times N_2}$, and a real matrix

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ & & \mathbf{0} & & & \end{bmatrix} \in R^{N_1 \times N_2}$$

such that

$$\mathbf{F} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H \tag{1}$$

where $\mathbf{V}^H$ denotes the complex conjugate transpose of $\mathbf{V}$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ are the singular values and r is the rank of $\mathbf{F}$.

Note that (1) can also be written as

$$\mathbf{F} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^H \tag{2}$$

or

$$\mathbf{F} = \sum_{i=1}^{r} \tilde{\mathbf{u}}_i \tilde{\mathbf{v}}_i^H \tag{3}$$

where the $\mathbf{u}_i$'s and $\mathbf{v}_i$'s are singular vectors of $\mathbf{F}$, and $\tilde{\mathbf{u}}_i = \sigma_i^{\frac{1}{2}} \mathbf{u}_i, \tilde{\mathbf{v}}_i = \sigma_i^{\frac{1}{2}} \mathbf{v}_i$ are the weighted singular vectors of $\mathbf{F}$. The matrix decomposition (1), (2) or (3) has been known for many years as the SVD of matrix $\mathbf{F}$ [1][2][11]. However, it is only recently that it has been utilized for the design of 2-D digital filters [6]-[9].

As is mentioned in the introduction, although the SVD has been used to design a large class of 2-D digital filters, the case where an arbitrary phase response is

CH 3006-4/91/0000 – 0618 $1.00 © IEEE

required has not been considered [6]-[10]. As a matter of fact, the presently available SVD design approaches always begin with the SVD of a sampled amplitude response matrix. In what follows, it will be shown that for a properly formed sampled frequency response matrix $\mathbf{F}$, there exists an SVD of $\mathbf{F}$ such that the resulting singular vectors associated with the nonzero singular values can be interpreted as frequency responses of causal 1-D digital filters.

2.2 A Fundamental Property of a Sampled Frequency Response Matrix

Let $H_d(e^{j\omega_1 T_1}, e^{j\omega_2 T_2})$ be a desired 2-D frequency response. For the sake of simplicity, let $\mu = \omega_1 T_1$ and $\nu = \omega_2 T_2$ and write the frequency response as

$$H_d(e^{j\mu}, e^{j\nu}) = m_d(\mu, \nu) e^{j\theta_d(\mu,\nu)} \tag{4}$$

Obviously, for any integers k and l we have

$$m_d(\mu,\nu) = m_d(-\mu,\nu) \tag{5}$$
$$\theta_d(\mu,\nu) = -\theta_d(-\mu,\nu) \tag{6}$$

If $\mathbf{M}$ and $\boldsymbol{\Theta}$ are matrices of dimension $N_1 \times N_2$ obtained by sampling m_d and θ_d over the square $\{(\mu,\nu): -\pi \le \mu \le \pi, -\pi \le \nu \le \pi\}$, then matrix $\mathbf{F}$ defined by

$$\mathbf{F} = \mathbf{M} \circ \boldsymbol{\Theta} \tag{7}$$

is a discrete version of the desired frequency response, where $\mathbf{M} \circ \boldsymbol{\Theta}$ denotes the entrywise product (usually called Schur product) of $\mathbf{M}$ and $\boldsymbol{\Theta}$.

For the sake of simplicity, let us assume that $N_1 = N_2 = 2N$. From (5)-(7) it follows that the entries of matrix $\mathbf{F}$ satisfy the relation

$$f_{ij} = \bar{f}_{2N+1-i,2N+1-j} \quad \text{for} \quad 1 \le i,j \le 2N \tag{8}$$

where $\bar{f}$ denotes the complex conjugate of f. A fundamental property of such a sampled frequency response matrix can now be stated in terms of the following theorem.

Theorem 1

If $F \in C^{2N \times 2N}$ is a sampled frequency response matrix obtained from (7) with rank r and $N_1 = N_2 = 2N$, then there exists a SVD of $\mathbf{F}$

$$\mathbf{F} = \sum_{i=1}^{r} \sigma_i \hat{\mathbf{u}}_i \hat{\mathbf{v}}_i^H = \sum_{i=1}^{r} \tilde{\mathbf{u}}_i \tilde{\mathbf{v}}_i^H \tag{9}$$

such that $\hat{\mathbf{u}}_i, \hat{\mathbf{v}}_i$, and therefore $\tilde{\mathbf{u}}_i, \tilde{\mathbf{v}}_i$, are mirror-image complex-conjugate symmetric for $1 \le i \le r$. That is,

$$\hat{\mathbf{u}}_i = \begin{bmatrix} \hat{\mathbf{u}}_{i1} \\ \hat{\mathbf{I}}\bar{\hat{\mathbf{u}}}_{i1} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{v}}_i = \begin{bmatrix} \hat{\mathbf{v}}_{i1} \\ \hat{\mathbf{I}}\bar{\hat{\mathbf{v}}}_{i1} \end{bmatrix} \quad 1 \le i \le r$$

where $\hat{\mathbf{u}}_{i1}, \hat{\mathbf{v}}_{i1} \in C^{N\times 1}$, $\bar{\hat{\mathbf{u}}}_{i1}$ denotes the complex conjugate of $\hat{\mathbf{u}}_{i1}$, and $\hat{\mathbf{I}} \in R^{N\times N}$ is defined by

$$\hat{\mathbf{I}} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}$$

Proof

Let $\mathbf{I}$ be the $N \times N$ identity matrix and define

$$\tilde{\mathbf{I}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{I}} \end{bmatrix}, \quad \mathbf{P} = \tilde{\mathbf{I}}\mathbf{F}\tilde{\mathbf{I}} \tag{10}$$

From (8), matrix $\mathbf{P}$ can be written as

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \bar{\mathbf{P}}_2 & \bar{\mathbf{P}}_1 \end{bmatrix} \tag{11}$$

and from (2) and (10)

$$\mathbf{F} = \tilde{\mathbf{I}}\mathbf{P}\tilde{\mathbf{I}} = [\mathbf{u}_1 \ldots \mathbf{u}_{2N}]\boldsymbol{\Sigma}[\mathbf{v}_1 \ldots \mathbf{v}_{2N}]^H \tag{12}$$

Hence

$$\mathbf{F}\mathbf{F}^H \mathbf{u}_i = \sigma_i \mathbf{u}_i, \quad 1 \le i \le r$$

and

$$\mathbf{P}\mathbf{P}^H \tilde{\mathbf{I}}\mathbf{u}_i = \sigma_i \tilde{\mathbf{I}}\mathbf{u}_i \tag{13}$$

If we let

$$\mathbf{u}_i = \begin{bmatrix} \mathbf{u}_{i1} \\ \mathbf{u}_{i2}, \end{bmatrix}, \quad \tilde{\mathbf{I}}\mathbf{u}_i = \begin{bmatrix} x_{i1} \\ x_{i2}, \end{bmatrix} \tag{14}$$

then (13) becomes

$$\mathbf{P}\mathbf{P}^H \begin{bmatrix} \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \end{bmatrix} = \sigma_i \begin{bmatrix} \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \end{bmatrix} \tag{15}$$

From (11) and (15)

$$\mathbf{P}\mathbf{P}^H \begin{bmatrix} \bar{\mathbf{x}}_{i2} \\ \bar{\mathbf{x}}_{i1} \end{bmatrix} = \sigma_i \begin{bmatrix} \bar{\mathbf{x}}_{i2} \\ \bar{\mathbf{x}}_{i1} \end{bmatrix} \tag{16}$$

Consequently,

$$\begin{bmatrix} \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{\mathbf{x}}_{i2} \\ \bar{\mathbf{x}}_{i1} \end{bmatrix}$$

are eigenvectors of matrix $\mathbf{P}\mathbf{P}^H$ associated with the same eigenvalue σ_i and, therefore, the two normalized vectors are linearly dependent, i.e.

$$\begin{bmatrix} \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \end{bmatrix} = e^{j\alpha_i} \begin{bmatrix} \bar{\mathbf{x}}_{i2} \\ \bar{\mathbf{x}}_{i1} \end{bmatrix} \tag{17}$$

for some α_i. It now follows from (14) and (17) that

$$\begin{aligned} \mathbf{u}_i = \tilde{\mathbf{I}} \begin{bmatrix} \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \end{bmatrix} &= \begin{bmatrix} \mathbf{x}_{i1} \\ e^{j\alpha_i}\hat{\mathbf{I}}\bar{\mathbf{x}}_{i1} \end{bmatrix} = e^{j\frac{\alpha_i}{2}} \begin{bmatrix} e^{-j\frac{\alpha_i}{2}}\mathbf{x}_{i1} \\ \hat{\mathbf{I}}\,\overline{\left(e^{-j\frac{\alpha_i}{2}}\mathbf{x}_{i1}\right)} \end{bmatrix} \\ &\equiv e^{j\frac{\alpha_i}{2}} \begin{bmatrix} \hat{\mathbf{u}}_{i1} \\ \hat{\mathbf{I}}\bar{\hat{\mathbf{u}}}_{i1} \end{bmatrix} \equiv e^{j\frac{\alpha_i}{2}}\hat{\mathbf{u}}_i \end{aligned} \tag{18}$$

where

$$\hat{\mathbf{u}}_i = \begin{bmatrix} \hat{\mathbf{u}}_{i1} \\ \mathbf{I}\bar{\hat{\mathbf{u}}}_{i1} \end{bmatrix} \tag{19}$$

with $\hat{\mathbf{u}}_{i1} = e^{-j\frac{\alpha_i}{2}}\mathbf{x}_{i1}$, Furthermore, from (12) and (18)

$$\mathbf{v}_i = \sigma_i^{-1}\tilde{\mathbf{I}}\mathbf{P}^H\tilde{\mathbf{I}}\mathbf{u}_i = \sigma_i^{-1}e^{j\frac{\alpha_i}{2}}\tilde{\mathbf{I}}\mathbf{P}^H \begin{bmatrix} \hat{\mathbf{u}}_{i1} \\ \mathbf{I}\bar{\hat{\mathbf{u}}}_{i1} \end{bmatrix} = e^{j\frac{\alpha_i}{2}}\hat{\mathbf{v}}_i \tag{20}$$

where

$$\hat{\mathbf{v}}_i = \begin{bmatrix} \hat{\mathbf{v}}_{i1} \\ \mathbf{I}\bar{\hat{\mathbf{v}}}_{i1} \end{bmatrix} \tag{21}$$

with $\hat{\mathbf{v}}_{i1} = \sigma_i^{-1}(\mathbf{P}_1^H\hat{\mathbf{u}}_{i1} + \bar{\mathbf{P}}_2^H\bar{\hat{\mathbf{u}}}_{i1})$. Equations (18)-(21) now imply that

$$\mathbf{F} = \sum_{i=1}^{r}\sigma_i\mathbf{u}_i\mathbf{v}_i^H = \sum_{i=1}^{r}\sigma_i\hat{\mathbf{u}}_i\hat{\mathbf{v}}_i^H = \sum_{i=1}^{r}\sigma_i\tilde{\mathbf{u}}_i\tilde{\mathbf{v}}_i^H$$

where $\tilde{\mathbf{u}}_i = \sigma_i^{\frac{1}{2}}\hat{\mathbf{u}}_i, \tilde{\mathbf{v}}_i = \sigma_i^{\frac{1}{2}}\hat{\mathbf{v}}_i$, and $\hat{\mathbf{u}}_i, \hat{\mathbf{v}}_i$ given by (19), (21) are mirror-image complex-conjugate symmetric. □

III. Design Issues

The SVD given by (9) can be used to design 2-D digital filters that have arbitrary desired amplitude and phase responses in terms of r pairs of 1-D digital filters, where the sampled frequency response of each 1-D filter approximates one of the weighted singular vectors $\tilde{\mathbf{u}}_i$ and $\tilde{\mathbf{v}}_i$ in (9). The 2-D digital filter obtained by this design approach can be a finite impulse response or infinite impulse response filter depending on whether the 1-D filters used are FIR or IIR filters. Moreover, the 2-D filter designed is stable if all 1-D filters used are stable. Like what we have experienced in our previous research [6]-[8], the amplitude as well as phase characteristics of the weighted singular vectors $\tilde{\mathbf{u}}_i$ and $\tilde{\mathbf{v}}_i$ are quite irregular even for 2-D filters with regular amplitude and phase responses such as linear-phase circular-symmetric lowpass or highpass filters. In what follows, two methods are proposed for the design of 1-D FIR digital filters with arbitrary amplitude and phase responses. Filters of this type can be used if the design of corresponding 2-D FIR filters.

3.1 Design of 1-D FIR Filters by Least-Squares Optimization

It is observed from Theorem 1 that the entire information of a weighted singular vector, say $\tilde{\mathbf{v}}_i$, is contained in its bottom N entries which can be intepreted as a 1-D frequency response over frequency range $[0, \pi]$. So let us assume that $G_d(\nu)$ is the desired frequency response over $\nu \in [0, \pi]$ and that its sampled version is represented by an N-dimensional complex vector $\mathbf{v}_d$. A 1-D FIR filter with transfer function

$$G(z_1) = \sum_{i=0}^{M-1} g_i z_1^{-i} \tag{22}$$

is sought such that the least-square error

$$e_2(\mathbf{g}) = \int_0^{\pi} |G(e^{j\nu}) - G_d(\nu)|^2 d\nu \tag{23}$$

is minimized where $\mathbf{g} = [g_0 \dots g_{M-1}]^T$ is the real parameter vector to be determined in the optimization.
By writing

$$G(e^{j\nu}) = \mathbf{g}^T\mathbf{q}_M$$

with

$$\mathbf{q}_M = \begin{bmatrix} 1 & e^{j\nu} & \cdots & e^{-j(M-1)} \end{bmatrix}^T$$

the least-squares error can be expressed as

$$e_2(\mathbf{g}) = \mathbf{g}^T\hat{\mathbf{Q}}\mathbf{g} - 2\mathbf{g}^T\mathbf{b} + c \tag{24}$$

where

$$\hat{\mathbf{Q}} = \int_0^{\pi}\mathbf{q}_M\mathbf{q}_M^T d\nu \text{ and } \mathbf{b} = \text{Re}\left[\int_0^{\pi}\bar{\mathbf{q}}_M G_d(\nu)d\nu\right]$$

Since $e_2(\mathbf{g})$ is real, (24) implies that

$$\begin{aligned} e_2(\mathbf{g}) &= \mathbf{g}^T\left[\frac{\hat{Q}+\bar{\hat{Q}}}{2}\right]\mathbf{g} - 2\mathbf{g}^T\mathbf{b} + c \\ &\equiv \mathbf{g}^T\mathbf{Q}\mathbf{g} - 2\mathbf{g}^T\mathbf{b} + c \end{aligned}$$

where $\mathbf{Q} = \pi\mathbf{I}$.

Hence, the vector $\mathbf{g}$ that minimizes $e_2(\mathbf{g})$ is given by

$$\mathbf{g} = \mathbf{b}/\pi = \frac{1}{\pi}\text{Re}\left[\int_0^{\pi}\bar{\mathbf{q}}_M G_d(\nu)d\nu\right]$$

which implies that

$$g_i \approx \text{Re}\left[\frac{1}{N}\sum_{k=1}^{N} v_{dk}e^{j(i-1)(k-1)\pi/N}\right], \quad 1 \le i \le M$$

where v_{dk} denotes the kth entry of $\mathbf{v}_d$. In effect, the best $\mathbf{g}$ is approximately equal to the real part of the discrete Fourier transform of $\mathbf{v}_d$.

3.2 Design of 1-D FIR Filters by Least-pth Optimization

An 1-D FIR filter with the transfer function in (22) can be designed such that its frequency response $G(\nu)$ best approximates the desired frequency response $G_d(\nu)$ with respect to the L_p-norm

$$e_p(\mathbf{g}) = \int_0^{\pi} |G(e^{j\nu}) - G_d(\nu)|^p \, d\nu, \quad p > 0 \text{ even}$$

Theorem 2

The objective function $e_p(\mathbf{g})$ is globally convex over the entire paramenter space $R^{M\times 1}$ for any even integer $p > 0$.

Proof

By writing $e_p(\mathbf{g})$ as

$$\begin{aligned} e_p(\mathbf{g}) &= \int_0^{\pi}\left[|G(e^{j\nu}) - G_d(\nu)|^2\right]^{p'} d\nu \\ &= \int_0^{\pi}[\mathbf{g}^T\tilde{\mathbf{Q}}\mathbf{g} - 2\mathbf{g}^T\tilde{\mathbf{b}} + \tilde{c}]^{p'} d\nu \end{aligned}$$

where $p' = p/2, \tilde{\mathbf{Q}} = \mathbf{q}_M \mathbf{q}_M^H, \tilde{\mathbf{b}} = \mathrm{Re}[\bar{\mathbf{q}}_M G_d(\nu)]$, and $\tilde{c} = |G_d(\nu)|^2$, we can express the Hessian matrix $\mathbf{H}$ of the objective function as

$$\begin{aligned} \mathbf{H} &= \frac{\partial^2 e_p(\mathbf{g})}{\partial \mathbf{g} \partial \mathbf{g}^H} \\ &= (p'-1) \int_0^\pi |G_e^{j\nu} - G_d(\nu)|^{p'-2} \mathbf{Y} \, d\nu \\ &\quad + \int_0^\pi |G(e^{J\nu}) - G_d(\nu)|^{p'-1} \tilde{\mathbf{Q}} \, d\nu \end{aligned} \tag{25}$$

where $\mathbf{Y} = (\mathbf{g}^T \tilde{\mathbf{Q}} - \tilde{\mathbf{b}})(\mathbf{g}^T \tilde{\mathbf{Q}} - \tilde{\mathbf{b}})^H$.

It follows that $\mathbf{H} \geq \mathbf{0}$ and, therefore, $e_p(\mathbf{g})$ is globally convex. □

For FIR filters of moderate order, the classic Newton method might be the best choice for the minimization of objective function $e_p(\mathbf{g})$ owing to its fast convergence rate and computational efficiency. A major problem with the Newton method is that the Hessian matrix $\mathbf{H}$ in (25) must remain possitive definite during the optimization process. This is a fairly restrictive condition because as vector $\mathbf{g}$ approaches its optimal value, $|G(e^{j\nu}) - G_d(\nu)|$ becomes small; hence all entries of $\mathbf{H}$ become small as is seen from (25). Consequently, matrix $\mathbf{H}$ is likely to become ill-conditioned, if not singular, during the final stage of the optimization and may result in an unreliable solution. Quasi-Newton methods such as the Davidon-Fletcher-Powell and the Broyden-Fletcher-Goldfarb-Shanno methods often exhibit fast conversengence and do not require of the Hessian matrix. However, if the problem at hand is convex and the number of independent variables is large, as is the case, excellent results can be obtained by using the nonquadratic extension of the well-known conjugate gradient algorithm, often called the quadratic approximation (QA) method [12].

The QA method starts with an intial estimate of the parameter vector $\mathbf{g}$, say $\mathbf{g}_0$, which may be obtained, for example, by the least-squares method described in Sec. 3.1. Then improved estimates of $\mathbf{g}$, say $\mathbf{g}_1, \mathbf{g}_2, \cdots$ are obtained by using the recursive relations

$$\begin{aligned} d_0 &= -\nabla^T e_p(\mathbf{g}_0) \\ \alpha_k &= \frac{-\nabla^T e_p(\mathbf{g}_k)\mathbf{d}_k}{\mathbf{d}_k^T \mathbf{H}(\mathbf{g}_k)\mathbf{d}_k} \\ \mathbf{g}_{k+1} &= \mathbf{g}_k + \alpha_k \mathbf{d}_k \\ \mathbf{d}_{k+1} &= -\nabla e_p(\mathbf{g}_{k+1}) + \beta_k \mathbf{d}_k \\ \beta_k &= \frac{\nabla^T e_p(\mathbf{g}_{k+1})\mathbf{H}(\mathbf{g}_k)\mathbf{d}_k}{\mathbf{d}_k^T \mathbf{H}(\mathbf{g}_k)\mathbf{d}_k} \end{aligned}$$

IV. Conclusions

The SVD has been applied in conjunction with 1-D techniques for the design of 2-D digital filters with arbitrary amplitude and phase responses.

The method is relatively simple to apply and leads to a parallel arrangement of pairs of cascaded 1-D filter sections. Structures of this type allow a large amount of concurrent processing, and, further, they can be implemented in terms of systolic arrays. Therefore, they are amenable to VLSI implementation.

Since the 2-D design is accomplished by designing a set of 1-D filters, low approximation error can easily be achieved and stability can easily be assured.

Reference

[1] C. Eckart and G.Young, "The approximation of one matrix by another of lower rank", Psychometrika, vol. 1, pp. 211-218, 1936.

[2] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 2nd Ed., Baltimore, MD: The Johns Hopkins University Press, 1989.

[3] V. C. Klema and A. Laub, "The singlar value decomposition. Its computation and some applications", *IEEE Trans Automat. contr.* vol. 25, pp. 164-176, April 1980.

[4] H. C. Andrews and B. R. Hunt, *digital Image Restoration*, Englewood Cliffs, N. Y.: Prentice-Hall, 1977.

[5] B. A. Francis, *A Course in H_∞ Control Theory*, NewYork: Springer-Verlay, 1987.

[6] A. Antoniou and W.-S. Lu, "Design of 2-D digital filters by using the singular value decomposition", *IEEE Trans. Circuits Syst.*, vol. 34, pp. 1191-1198, Oct. 1987.

[7] W.-S. Lu, H.-P. Wang, and A. Antoniou, "Design of 2-D FIR digital filters by using the singular value decomposition", *IEEE Trans. Circuits Syst.*, vol. 37, pp. 35-46, Jan. 1990.

[8] W.-S. Lu, H.-P. Wang, and A. Antoniou, "Design of 2-D digital filters using the singular value decomposition and balanced approximation method", *Proc. 1989 IEEE Int. Symp. on Circuits Syst.*, Portland, pp. 1656-1659, 1989.

[9] D. Shpak, "A weighted-least-squares matrix decomposition method with application to the design of two-dimensional digital filters", *Proc. 33rd Midwest Symp. on Circuits Syst.*, Calgary, August 1990.

[10] T. Deng and M. Kawamata, "Reduced-dimensional decomposition design of 2-D digital filters based on the iterative singular value decomposition", 1990 Spring National Convention Record, IEICE Japan, pp. A-158, March 1990.

[11] G. W. Stewart, *Introduction to Matrix Computations*, Orlando: Academic Press, 1973.

[12] D. G. Luenberger, *Linear and Nonlinear Programming*, 2nd Ed., Reading, MA: Addison-Wesley, 1984.