DESIGN OF FIR FILTERS WITH DISCRETE COEFFICIENTS: A SEMIDEFINITE PROGRAMMING RELAXATION APPROACH

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ABSTRACT

This paper develops a method for the design of FIR digital filters with "sum of power of two" (SP2) coefficients. It is shown that the integer programming involved in the design can be "relaxed" to a semidefinite programming (SDP) problem which is known to be solvable using efficient SDP solvers in polynomial time. Simulations demonstrate that the SDP-relaxation-based designs often yield near-optimal performance with considerably reduced computational complexity.

1. INTRODUCTION

There has been significant research interest in the design of digital filters with discrete coefficients in the last three decades [1]–[9]. Here the term "discrete coefficient" refers to a real value that can be expressed as sum of a small number of power-of-two terms. For brevity we call it a SP2 (sum of power of two) coefficient. The primary reason we are interested in digital filters with SP2 coefficients is that they admit fast implementations which require no multiplications but superposition of shifted versions of the input signal. A technical difficulty encountered in designing a general optimum filter with SP2 coefficients is its nonpolynomial-time design complexity since the design is essentially an integer programming (IP) or mixed IP problem which is known to be NP-hard [10].

In this paper, we employ a semidefinite programming (SDP) relaxation approach to the design problem. As a subclass of convex programming (CP) problems for which any local minimizer is a global minimizer, SDP problems can be solved using efficient interior-point algorithms with polynomial-time complexity. Although the solution obtained using the proposed SDP relaxation can only be claimed as suboptimal for the original design problem, simulations demonstrate that the suboptimal solutions obtained for a variety of filter types and a wide range of filter lengths are of excellent quality.

2. A DESIGN METHOD BASED ON SDP RELAXATION

There are three major steps in the proposed design method, namely, (i) design of an FIR filter with continuous coefficients that approximates a desired frequency response in a certain optimal sense; (ii) a subsequent formulation of the design of an FIR digital filter with SP2 coefficients (based on the available continuous-coefficient FIR filter) as a $\{-1,\ 1\}$-optimization problem; and (iii) an SDP relaxation of the $\{-1,\ 1\}$-optimization problem obtained in Step (ii). These steps are described in the rest of this section.

2.1. Design of Continuous-Coefficient FIR Digital Filters

Many effective methods for the design of FIR digital filters are available [11]. Recent progress in the field includes the development of several SDP-based methods for weighted least-squares and minimax designs of FIR filters to approximate arbitrary amplitude and phase responses [12][13]. Supported by efficient interior-point SDP solvers [14][15], these SDP methods offer an efficient and unified design utility for the design of a variety of digital filters. For illustration purposes, suppose we apply one of the available techniques to design an FIR digital filter of odd length N that approximates a desired frequency response $H_d(\omega)$ such that the weight least square (WLS) error

$$e = \int_0^\pi W(\omega)|H(e^{j\omega}) - H_d(\omega)|^2\, d\omega \tag{1}$$

where $W(\omega) \geq 0$ is a weighting function, is minimized, and the FIR filter obtained is represented by

$$H_c(z) = \sum_{k=0}^{N-1} h_k z^{-k} \tag{2}$$

2.2. A Weighted Least-Square $\{-1,\ 1\}$-Optimization Formulation

Given a budget of total number of power-of-two (P2) terms M, a certain number of P2 terms, m_k, is allocated to the kth coefficient of the target discrete-coefficient FIR transfer function

$$H(z) = \sum_{k=0}^{N-1} d_k z^{-k} \tag{3}$$

with

$$\sum_{k=0}^{N-1} m_k = M \tag{4}$$

The P2-term allocation can be carried out based on some criterion such as magnitudes of the coefficients in the continuous-coefficient FIR function $H_c(z)$, or something more sophisticated [7]. Typically, the P2 terms in the representation of $\{d_k\}$ are within a given range, say, between 2^{-U} and 2^{-L} with positive integers $L < U$. For example, with a given term allocation m_k, the discrete coefficient d_k in (3) can be expressed as

$$d_k = \sum_{i=1}^{m_k} b_i^{(k)} 2^{-q_i^{(k)}} \tag{5}$$

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where $b_i^{(k)} \in \{-1,\ 1\}$ and $L \le q_i^{(k)} \le U$ for $1 \le i \le m_k$ and $0 \le k \le N-1$. For a given continuous-coefficient design $H_c(z)$, a budget $\{m_k,\ k=0,\ \ldots,\ N-1\}$, and a range $\{L,\ U\}$, the *least* SP2 upper bound $\bar{d}_k$ and *largest* SP2 lower bound $\underline{d}_k$ for each continuous coefficient h_k in (2) can readily be identified such that $\underline{d}_k \le h_k \le \bar{d}_k$ with both $\bar{d}_k$ and $\underline{d}_k$ being of the form (5). It follows that, in each open interval $(\underline{d}_k,\ \bar{d}_k)$, no SP2 numbers of form (5) exist for a given budget m_k. Fig. 1 illustrates the first several coefficients h_k and their bounds.

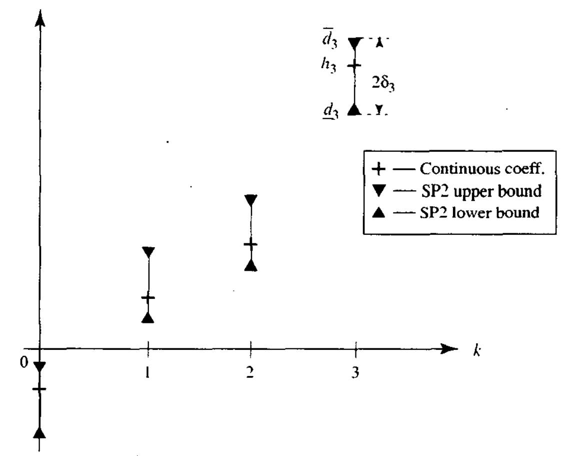

Figure 1: Continuous coefficients and their least SP2 upper bounds and largest SP2 lower bounds.

Before we proceed, note that a suboptimal SP2 design can easily be obtain as

$$H_s(z) = \sum_{k=0}^{N-1} d_k^{(s)} z^{-k} \tag{6a}$$

where coefficients $d_k^{(s)}$ are determined by

$$d_k^{(s)} = \begin{cases} \bar{d}_k & \text{if } \bar{d}_k - h_k \le h_k - \underline{d}_k \\ \underline{d}_k & \text{otherwise} \end{cases} \tag{6b}$$

In words, coefficients $\{d_k^{(s)}\}$ are obtained as SP2 bounds of form (5) that are closest to $\{h_k\}$. Simulations have indicated, however, that performance of this easy-to-obtain suboptimal design is often unsatisfactory. We shall discuss this matter in Sec. 3 with numerical examples.

Now denote the midpoint of each interval $[\underline{d}_k,\ \bar{d}_k]$ as $d_{mk} = (\bar{d}_k + \underline{d}_k)/2$ and a half of the interval length as $\delta_k = (\bar{d}_k - \underline{d}_k)/2$. It follows that the SP2 upper and lower bounds $\bar{d}_k$ and $\underline{d}_k$ can be selected as $d_{mk} + b_k\delta_k$ with $b_k = 1$ and $b_k = -1$, respectively. Thus the frequency response of discrete-coefficient FIR function $H(z)$ in (3) with

$$d_k = d_{mk} + b_k\delta_k \tag{7}$$

can be expressed as

$$H(e^{j\omega}) = H_m(e^{j\omega}) + b^T[c_\delta(\omega) - js_\delta(\omega)] \tag{8}$$

where

$$H_m(e^{j\omega}) = \sum_{i=0}^{N-1} d_{mk}e^{-j\omega} = d_m^T[c(\omega) - js(\omega)]$$

$$\begin{aligned} d_m &= [d_{m0}\ d_{m1}\ \ldots\ d_{m,N-1}]^T \\ c(\omega) &= [1\ \ \cos\omega\ \ \ldots\ \ \cos(N-1)\omega]^T \\ s(\omega) &= [0\ \ \sin\omega\ \ \ldots\ \ \sin(N-1)\omega]^T \\ c_\delta(\omega) &= [\delta_0\ \ \delta_1\cos\omega\ \ \ldots\ \ \delta_{N-1}\cos(N-1)\omega]^T \\ s_\delta(\omega) &= [0\ \ \delta_1\sin\omega \ldots\ \ \delta_{N-1}\sin(N-1)\omega]^T \end{aligned}$$

and

$$b = [b_0\ b_1\ \ \ldots\ b_{N-1}]^T \qquad \text{with } b_i \in \{-1,\ 1\}$$

In the light of (8), we can write the objective function in (1) as

$$e = b^TQb + 2b^Tq + \text{const} \tag{9a}$$

where

$$Q = \int_0^\pi W(\omega)[c_\delta(\omega)c_\delta^T(\omega) + s_\delta(\omega)s_\delta^T(\omega)]d\omega \tag{9b}$$

$$q = \int_0^\pi W(\omega)[E_r(\omega)c_\delta(\omega) + E_i(\omega)s_\delta(\omega)]d\omega \tag{9c}$$

$$E_r(\omega) = d_m^Tc(\omega) - H_{dr}(\omega) \tag{9d}$$

$$E_i(\omega) = d_m^Ts(\omega) - H_{di}(\omega) \tag{9e}$$

$$H_d(\omega) = H_{dr}(\omega) - jH_{di}(\omega) \tag{9f}$$

Remark: For the design of discrete-coefficient linear-phase FIR filters, the continuous-coefficient prototype $H_c(z)$ is a linear-phase FIR function. Assuming N is an odd integer, the symmetry of coefficients $\{h_k\}$ implies that $\{d_{mk}\}$, $\{b_k\}$, and $\{\delta_k\}$ are all symmetrical with respect to the midpoint. Hence $H_m(e^{j\omega}) = e^{-jN_1\omega}A_m(\omega)$ with $N_1 = (N-1)/2$ and $A_m(\omega)$ the real-valued magnitude-related trigonometric polynomial. In this case $H(e^{j\omega})$ in (8) is simplified to

$$H(e^{j\omega}) = e^{-jN_1\omega}[A_m(\omega) + b^Tc_\delta(\omega)] \tag{10a}$$

where

$$b = [b_0\ b_1\ \ \ldots\ b_{N_1}]^T \tag{10b}$$

$$c_\delta(\omega) = [\delta_0\ 2\delta_1\cos\omega\ \ \cdots\ \ 2\delta_{N_1}\cos N_1\omega]^T \tag{10c}$$

and Q and b in (9) become

$$Q = \int_0^\pi W(\omega)c_\delta(\omega)c_\delta^T(\omega)d\omega \tag{11a}$$

and

$$q = \int_0^\pi W(\omega)[A_m(\omega) - A_d(\omega)]c_\delta(\omega)d\omega \tag{11b}$$

where $A_d(\omega)$ is the desired amplitude response. ■

For a given continuous-coefficient transfer function $H_c(z)$, a budget $\{m_k\}$, a range $\{L,\ U\}$, a desired frequency response $H_d(\omega)$, and a weighting function $W(\omega)$, matrix $Q \in R^{N\times N}$ and $b \in R^{N\times 1}$ can be evaluated using (9b)–(9f) or (11) where Q is in general positive definite. The weighted least-square design of $H(z)$ with SP2 coefficient can then be formulated as the integer programming (IP) problem

$$\underset{b_i\in\{-1,1\}}{\text{minimize}}\ b^TQb + 2b^Tq \tag{12}$$

2.3. A Semidefinite Programming Relaxation of Problem (12)

Semidefinite programming (SDP) is concerned with a class of constrained optimization problems where a linear objective function is minimized subject to matrix constrains which affinely depend on the variable vector. Typically an SDP problem can be formulated as

$$\begin{aligned} &\text{minimize} && c^T x && (13a)\\ &\text{subject to:} && F(x) = F_0 + \sum_{i=1}^{r} x_i F_i \succeq 0 && (13b)\end{aligned}$$

where matrices F_i for $0 \le i \le r$ are symmetric and $\succeq 0$ means positive semidefinite. Since linear function $c^T x$ is always convex and the feasible region defined by the linear matrix inequality (LMI) in (13b) is convex, SDP forms an important subclass of *convex programming* problems that includes both linear and convex quadratic programming as special cases. Efficient polynomial-time interior-point algorithms have been extended to SDP [16][17] and software implementations of the algorithms are available, including the LMI Control Toolbox [14], SeDuMi [18], and SDPT3 [15]. The relevance of SDP to the design problem at hand lies in the fact that the IP problem in (12) can be *relaxed* to an SDP problem so as to obtain an approximate solution which can be solved in polynomial time. Geomans and Williamson [19] was among the first to obtain an SDP relaxation solution of the MAX-CUT problem—a well-known integer quadratic programming problem in graph theory. Following [19], SDP relaxations of various IP problems have been reported in graph optimization, network management and scheduling [20]. In what follows we present a SDP-relaxation-based solution to our design problem.

We consider the problem in (12) with vector b replaced by a more conventional notation $x = [x_1 \ \ldots \ x_N]^T$ (In the linear-phase case, the dimension of parameter vector x will be $N_1 = (N+1)/2$, and in the rest of the section the integer N is changed to N_1):

$$\begin{aligned} &\text{minimize} && x^T Q x + 2x^T q && (14a)\\ &\text{subject to:} && x_i^2 = 1 \quad \text{for } 1 \le i \le N && (14b)\end{aligned}$$

where constraints $x_i \in \{-1,\ 1\}$ have been replaced by the equality constraints in (14b). If we define matrix $X = xx^T$, then the objective function in (14a) can be written as

$$\mathrm{tr}(QX) + 2x^T q$$

where $\mathrm{tr}(\cdot)$ denotes matrix trace, and the problem in (14) is equivalent to

$$\begin{aligned} &\text{minimize} && \mathrm{tr}(QX) + 2x^T q && (15a)\\ &\text{subject to:} && X = xx^T && (15b)\\ & && x_i^2 = 1 \quad \text{for } 1 \le i \le N && (15c)\end{aligned}$$

Note that the constraints in (15b) and (15c) are equivalent to

$$\mathrm{rank}\begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} = 1,\ X_{ii} = 1,\ \text{and} \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0 \qquad (16)$$

An SDP relaxation of (15) is obtained by neglecting the rank constraint in (16) while keeping the remaining two constraints, which leads to the minimization problem

$$\begin{aligned} &\text{minimize} && \mathrm{tr}(QX) + 2x^T q && (17a)\\ &\text{subject to:} && \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0 && (17b)\\ & && X_{ii} = 1 \quad \text{for } 1 \le i \le N && (17c)\end{aligned}$$

Now let $\hat{X}$ be the matrix obtained from the symmetric matrix X with diagonal elements replaced by unity, then the problem in (17) can be simplified to

$$\begin{aligned} &\text{minimize} && \mathrm{tr}(Q\hat{X}) + 2x^T q && (18a)\\ &\text{subject to:} && \begin{bmatrix} \hat{X} & x \\ x^T & 1 \end{bmatrix} \succeq 0 && (18b)\end{aligned}$$

If we define an augmented variable vector $\hat{x}$ of dimension $\hat{N} = N(N+1)/2$ that includes the $N(N-1)/2$ independent elements of $\hat{X}$ and N elements of x, then the objective function in (18a) can be expressed (up to a constant scalar) as $\hat{c}^T \hat{x}$ for a constant vector $\hat{c} \in R^{\hat{N}\times 1}$ and (18) becomes

$$\begin{aligned} &\text{minimize} && \hat{c}^T \hat{x} && (19a)\\ &\text{subject to:} && \begin{bmatrix} \hat{X} & x \\ x^T & 1 \end{bmatrix} \succeq 0 && (19b)\end{aligned}$$

Note that the matrix in constraint (19b) is *affine* with respect to $\hat{x}$, hence (19) is an SDP problem.

Let $(x^*,\ \hat{X}^*)$ be a solution of (19), there are two approaches to generating a binary approximate solution for the original problem in (14).

The first approach is straightforward:

$$x = \mathrm{sign}(x^*) \qquad (20a)$$

where $\mathrm{sign}(\cdot)$ denotes the conventional signum function. The second approach is based on the singular value decomposition of matrix $Z^* = U\Sigma U^T$ where Z^* is the $(N+1)\times(N+1)$ matrix in (19b) at $(x^*,\ \hat{X}^*)$, and obtains the binary solution x as

$$x = \mathrm{sign}[u_1(N+1)\cdot u_1(1:N)] \qquad (20b)$$

where u_1 is the first column vector in U that corresponds to the largest singular value σ_1 of Z^*. The motivation behind the second approach is that a *perfect* solution $\hat{X}^*$ satisfying (15b) would imply that matrix Z^* has rank one, hence $\sigma_1 u_1 u_1^T$ is the best rank-one approximation of Z^* in the 2-norm sense.

3. DESIGN EXAMPLES

The SDP relaxation (SDPR) method described in Sec. 2 was applied to design linear phase FIR filters with SP2 coefficients based on corresponding WLS FIR prototype filters with continuous coefficients. Ten such filters with normalized passband edge $\omega_p = 0.225$, stopband edge $\omega_a = 0.275$, $W(\omega) \equiv 1$ on $[0,\ \omega_p]$, $W(\omega) \equiv 500$ on $[\omega_a,\ 0.5]$, $L = 0$, $U = 12$, average number of power-of-two terms per coefficient = 2.4, and filter lengths from 7 to 43 were designed and compared to the suboptimal designs characterized by (6), optimal designs with SP2 coefficients obtained using IP, and the WLS designs with continuous coefficients in terms of the WLS error e defined by (1). The results are summarized in Table 1 where these four designs are labeled as Design (6), SDPR, IP,

Table 1: Comparisons of Various Designs

N	Design (6)	SDPR	IP	WLS
7	1.5105	0.3066/ 5.4e3	0.3066/ 0.9e3	0.2941
11	0.2782	0.1500/17.7e3	0.1500/ 6.5e3	0.1154
15	0.1252	0.0591/54.2e3	0.0511/41.9e3	0.0521
19	0.2690	0.0507/14.3e4	0.0413/24.9e4	0.0247
23	0.0296	0.0153/32.9e4	0.0153/13.9e5	0.0120
27	0.0378	0.0154/73.1e4	0.0112/73.6e5	0.0059
31	0.0248	0.0041/12.8e5	0.0041/37.8e6	0.0029
35	0.0085	0.0029/22.4e5	0.0028/18.9e7	0.0014
39	0.0045	0.0019/37.3e5	0.0017/92.3e7	0.0007
43	0.0086	0.0011/59.3e5	0.0008/44.2e8	0.0004

and WLS respectively. For SDPR and IP designs, the number of floating point operations (flops) used are also enclosed as a measure of design efficiency, see the second figures in the SDPR and IP columns in Table 1.

4. CONCLUDING REMARKS

From the simulations it is observed that the designs based on (6) show considerably larger approximation error and a great deal of design instability. On the other hand, the SDPR method often yields near-optimal designs (and for filters of length $N > 20$) with much reduced computational complexity compared with that required by the IP designs. The same observations also hold for other filter types. It is worthwhile to mention that the SDPR method described in this paper is of illustrative nature as it also applies to a variety of filter designs including the weighted minimax design and WLS designs with constraints in both passbands and stopbands. A detailed account on these and other matters concerning the design of general digital filters with SP2 coefficients will be made available in a separate paper.

ACKNOWLEDGEMENT

The author is grateful to the National Sciences and Engineering Research Council of Canada for supporting this work.

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