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Application of the Singular-Value Decomposition for the Design of 2-D Digital Filters

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Abstract

A method for the design of 2-D digital filters based on the singular value decomposition (SVD) received considerable attention in recent years. The method can now be used to design recursive or nonrecursive, 2-D or 3-D digital filters in spatial or frequency domain and filters that satisfy arbitrary amplitude and phase responses, including linear phase responses, can be easily obtained.

This paper provides a survey of the evolution of the method as well as several important developments that have led to its present status.

I. Introduction

The singular-value decomposition (SVD) is a numerically reliable matrix decomposition method that has found numerous scientific and engineering applications in the past. An important application of the SVD is concerned with the design of two-dimensional (2-D) and three-dimensional (3-D) digital filters [1]-[7]. The method can now be used to design recursive or nonrecursive, 2-D or 3-D digital filters, in spatial or frequency domain and filters that satisfy arbitrary amplitude and phase responses can be obtained. The SVD approach offers several advantages. The filters obtained are always stable and form parallel structures which allow extensive pipelining. Moreover, a quantitative error analysis is available that can facilitate the determination of the number of parallel sections to be used and the required design accuracy for each of the parallel 1-D subfilters to achieve a desired approximation error.

In this paper, the SVD design method is surveyed and several important developments that have led to the present status of the method are described.

II. SVD of a Nonnegative Matrix and Its Application

2.1 SVD of a Nonnegative Matrix

If $\mathbf{A}$ is a nonnegative matrix of dimension $N_1 \times N_2$ with $N_1 \geq N_2$, then there exist orthogonal matrices $\mathbf{U} \in R^{N_1 \times N_1}$ and $\mathbf{V} \in R^{N_2 \times N_2}$, and a real matrix

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \\ & & \mathbf{0} & & & \end{bmatrix} \in R^{N_1 \times N_2} \tag{1}$$

such that

$$\mathbf{A} = \mathbf{U\Sigma V}^T \tag{2}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the nonzero singular values of $\mathbf{A}$ and r is its rank [8]. By writing

$$\begin{aligned} \mathbf{U} &= [\mathbf{u}_1\, \mathbf{u}_2 \ldots \mathbf{u}_{N1}] \\ \mathbf{V} &= [\mathbf{v}_1\, \mathbf{v}_2 \ldots \mathbf{v}_{N2}] \end{aligned}$$

(2) can be expressed as

$$\mathbf{A} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T \tag{3}$$

or

$$\mathbf{A} = \sum_{i=1}^{r} \tilde{\mathbf{u}}_i \tilde{\mathbf{v}}_i^T \tag{4}$$

where the $\mathbf{u}_i$'s and $\mathbf{v}_i$'s are the singular vectors of $\mathbf{A}$ and $\tilde{\mathbf{u}}_i = \sigma_i^{1/2}\mathbf{u}_i$, $\tilde{\mathbf{v}}_i = \sigma_i^{1/2}\mathbf{v}_i$ are the *weighted* sigular vectors of $\mathbf{A}$.

2.2 Method of Twogood and Mitra

An important property of the SVD can be stated as [8]

$$||\mathbf{A} - \sum_{i=1}^{k} \tilde{\mathbf{u}}_i \tilde{\mathbf{v}}_i^T = \min_{\alpha_i, \beta_i} ||\mathbf{A} - \sum_{i=1}^{k} \boldsymbol{\alpha}_i \boldsymbol{\beta}_i^T|| \tag{5}$$

where $|| \cdot ||$ can be the L_2-norm or the Frobenius norm. Furthermore, for a nonnegative matrix $\mathbf{A}$, it is well-known that $\tilde{\mathbf{u}}_1$ and $\tilde{\mathbf{v}}_1$ i.e. the first pair of the singular vectors of $\mathbf{A}$, are nonnegative. This is an immediate consequence of the well-known Perron theorem on nonnegative matrices [9].

Based on the above observations, Twogood and Mitra [1] proposed a method for the design of separable 2-D digital filters with transfer function $H_1(z_1, z_2) = f_1(z_1)g_1(z_2)$ where $|f_1(e^{j\omega_1 T_1})|$ and $|g_1(e^{j\omega_2 T_2})|$ approximate the weighted singular vectors $\tilde{\mathbf{u}}_1$ and $\tilde{\mathbf{v}}_1$, respectively.

2.3 Improved SVD Method

Since $\tilde{\mathbf{u}}_i$ and $\tilde{\mathbf{v}}_i$ with $i \geq 2$ may have negative components, a careful treatment of the terms in (4) is necessary [2]. If u_2^- and v_2^- are the absolute values of the most negative components of $\tilde{\mathbf{u}}_2$ and $\tilde{\mathbf{v}}_2$, respectively, then all components of

$$\hat{\mathbf{u}}_2 = \tilde{\mathbf{u}}_2 + u_2^- \mathbf{e}_{N1} \quad \text{and} \quad \hat{\mathbf{v}}_2 = \tilde{\mathbf{v}}_2 + v_2^- \mathbf{e}_{N2}$$

are nonnegative, where $\mathbf{e}_k = [1 \ \cdots \ 1]^T \in R^{k \times 1}$ and $k = N_1$ or N_2. Assume that $\tilde{f}_2(z_1)$ and $\tilde{g}_2(z_2)$ are zero-phase transfer functions such that $\tilde{f}_2(e^{j\omega_1 T_1})$ and $\tilde{g}_2(e^{j\omega_2 T_2})$ approximate $\hat{\mathbf{u}}_2$ and $\hat{\mathbf{v}}_2$, respectively. Under these circumstances, the transfer functions

$$f_2(z_1) = \tilde{f}_2(z_1) - u_2^-$$

and

$$g_2(z_2) = \tilde{g}_2(z_2) - v_2^-$$

evaluated on the unit circles $z_1 = e^{j\omega_1 T_1}$ and $z_2 = e^{j\omega_2 T_2}$ approximate $\tilde{\mathbf{u}}_2$ and $\tilde{\mathbf{v}}_2$. Consequently, if $f_1(z_1)$ and $g_1(z_2)$ are zero-phase transfer function which approximate $\tilde{\mathbf{u}}_1$ and $\tilde{\mathbf{v}}_1$, respectively, then the amplitude response of the filter represented by

$$H_2(z_1, z_2) = f_1(z_1)g_1(z_2) + f_2(z_1)g_2(z_2)$$

approximates $\tilde{\mathbf{u}}_1\tilde{\mathbf{v}}_1^T + \tilde{\mathbf{u}}_2\tilde{\mathbf{v}}_2^T$. Through the use of data $\tilde{\mathbf{u}}_i$ and $\tilde{\mathbf{v}}_i$ ($i = 3, \ldots, k$, $k \leq r$) given in (4), vectors $\hat{\mathbf{u}}_i$ and $\hat{\mathbf{v}}_i$ can be found, and correction sections characterized by $f_i(z_1)g_i(z_2)$ can then be designed in a similar manner. When k sections ($k \leq r$) are designed, $H_k(z_1, z_2)$ can be found as

$$H_k(z_1, z_2) = \sum_{i=1}^{k} f_i(z_1)g_i(z_2)$$

and from (5), we have

$$\begin{aligned} ||\mathbf{A} - |H_k(e^{j\omega_1 T_1}, e^{j\omega_2 T_2})|\,|| &\approx ||\mathbf{A} - |\sum_{i=1}^{k} \tilde{\mathbf{u}}_i\tilde{\mathbf{v}}_i^T|\,|| \\ \leq ||\mathbf{A} - \sum_{i=1}^{k} \tilde{\mathbf{u}}_i\tilde{\mathbf{v}}_i^T|| &= \min_{\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i} ||\mathbf{A} - \sum_{i=1}^{k} \boldsymbol{\alpha}_i\boldsymbol{\beta}_i^T|| \end{aligned}$$

2.4 Design of FIR Filters with Arbitrary Amplitude Responses

A limitation of the SVD method as reported in [2] and [3] is that the amplitude response of the filter is required to have quadrantal symmetry with respect to the (ω_1, ω_2) plane. In addition, zero phase is required for each subfilter. This necessitates data transpositions at the inputs and outputs of subfilters and, as a result, the usefulness of these designs is limited to nonreal-time applications, where the delay introduced in the proprocessing is unimportant.

Consider an FIR filter characterized by the transfer function

$$H(z_1, z_2) = \sum_{n_1=-N_1/2}^{N_1/2} \sum_{n_2=-N_2/2}^{N_2/2} h(n_1, n_2) z_1^{-n_1} z_2^{-n_2} \tag{6}$$

If $h(n_1, n_2) = h(-n_1, -n_2)$ and $h(n_1, n_2)$ is real, then the frequency response denoted by

$$X(\omega_1, \omega_2) = H(e^{j\omega_1 T_1}, e^{j\omega_2 T_2})$$

is a real function which is symmetric with respect to the origin of the (ω_1, ω_2) plane, i.e.

$$X(\omega_1, \omega_2) = X(-\omega_1, -\omega_2) \tag{7}$$

Now assume that matrix $\mathbf{A}$ represents a desired arbitrary frequency response such that (7) is satisfied, and the SVD of $\mathbf{A}$ is given by (4). An important property of this class of FIR filters is stated in terms of the following theorem [6].

Theorem 1

If the frequency response of an FIR filter satisfies (7), then vectors $\tilde{\mathbf{u}}_i$ and $\tilde{\mathbf{v}}_i$ in (4) are either mirror-image symmetric or antisymmetric simultaneously for $i = 1, 2, \ldots, r$.

By designing the 1-D FIR filters characterized by $f_i(z_1)$ and $g_i(z_2)$ as zero-phase or $\frac{\pi}{2}$-phase filters that approximate $\tilde{\mathbf{u}}_i$ and $\tilde{\mathbf{v}}_i$ ($1 \leq i \leq k$, $k \leq r$), respectively, a zero-phase 2-D FIR filter with transfer function

$$H_k(z_1, z_2) = \sum_{i=1}^{k} f_i(z_1)g_i(z_2)$$

is obtained whose amplitude response is a good approximation to the desired amplitude response in the minimal mean-square-error sense. A 2-D causal linear-phase filter can readily be obtained by multiplying $f_i(z_1)$ and $g_i(z_2)$ by $z_1^{-N_1/2}$ and $z_2^{-N_2/2}$, respectively.

2.5 Design of IIR filters with Linear Phase Response

Separable IIR 2-D digital filters with linear phase response can be designed by applying the balanced approximation [10] to high-order FIR 2-D filters [4]. More precisely, the design can be obtained by following the steps listed below.

Step1 Design a high-order linear-phase FIR filter that has the desired amplitude response.

Step2 Find a state-space realization for the FIR filter designed in Step 1.

Step3 Compute the controllability and observability gramians of the state-space model obtained in Step 2.

Step4 Obtain a low-order balanced approximation of the state space model using the techniques proposed in [10].

Step5 Obtain the transfer function of the 2-D IIR filter from the reduced order state-space model found in Step 4.

The features of this design method include the following:

1) The balanced approximation always yields a stable and separable IIR filter that can be realized by a parallel structure.

2) The maximum error introduced by the balanced approximation is well controlled by the Hankel singular values of the original FIR filters. Extensive, experimental results have indicated that there is a large number of negligible Hankel singular values for a high-order FIR filter and, therefore, the original transfer function can be approximated by a much lower order IIR transfer function.

3) If $\varphi_r(\omega_1, \omega_2)$ and $\varphi(\omega_1, \omega_2)$ are the phase responses of the reduced and the original filter, respectively, and Ω_p denotes the passband region of the original filter, then

$$\max_{\Omega_p} |\varphi_r(\omega_1, \omega_2) - \varphi(\omega_1, \omega_2)| \approx \delta_m \leq \frac{2e_r}{1 - \varepsilon_p} \tag{8}$$

provided that $e_r \ll 1$, where e_r is defined by

$$e_r = \max_{|z_i|=1, i=1,2} |H_r(z_1, z_2) - H(z_1, z_2)|$$

and $H_r(z_1, z_2)$ is the transfer function of the reduced order IIR filter.

Inequality (8) implies that the phase response linearity of the original digital filter will be preserved in the reduced filter over the passband.

4) The reduced filter is nearly balanced [4] and, therefore, it has low roundoff noise and low sensitivity to coefficient quantization.

III. SVD of a Complex Matrix and Its Application

It is noted that the SVD approach developed in [1]-[4] always begins with a sampled *amplitude* response which is a real matrix with nonnegative entries. That is, the equally important information about phase response is neglected. In [5] the SVD method is extended to include the design of FIR or IIR 2-D filters with *arbitrary amplitude and phase responses.* The design starts by sampling a desired frequency response to generate a *complex* matrix that represents both the amplitude and phase responses. The SVD is then applied to the matrix to obtain an outer-product sum expression of the type given in (4). It is shown that each weighted singular vector in (4) can be interpreted as the frequency response of a 1-D digital filter and, therefore, an FIR or IIR 2-D filter can be obtained by designing r pairs of FIR or IIR 1-D digital filters.

This extention of the SVD method is based on the following theorem.

Theorem 2

If $\mathbf{M} \in R^{2N\times 2N}$ *and* $\Theta \in C^{2N\times 2N}$ *are sampled amplitude- and phase-response matrices obtained by sampling a desired frequency response, and* $\mathbf{F}$ *is the entrywise product of* $\mathbf{M}$ *and* Θ, *i.e.*

$$\mathbf{F} = \mathbf{M} \circ \Theta$$

then matrix $\mathbf{F}$ *has an SVD of the form*

$$\mathbf{F} = \sum_{i=1}^{r} \tilde{\mathbf{u}}_i \tilde{\mathbf{v}}_i^H \quad (9)$$

where r *is the rank of* $\mathbf{F}$, *such that the weighted singular vectors in (8) are mirror-image complex-conjugate symmetric.*

Experience has shown that the required 1-D filters have irregular amplitude and phase responses even for 2-D filters with regular frequency responses such as circularly symmetric lowpass or highpass filters. Two methods for the design of 1-D digital filters with *arbitrary* frequency responses using least-squares and least-pth optimization are proposed in [5].

IV. Variations of the SVD Approach

In what follows, we briefly describe two variations of the SVD approach, which lead to some improvements in the design.

4.1 Weighted-Least-Squares Matrix Decomposition Method [6]

In the SVD approach described in Sections II and III, the relative error in the different bands cannot be controlled. A weighted-least-squares matrix decomposition method which can ease this restriction has been proposed in [9]. In this method, an ideal sampled 2-D amplitude response of the form $\mathbf{A} = \{a_{ij}\} \in R^{N\times N}$ is assumed and an outer product of vectors $\mathbf{u}_1$ and $\mathbf{v}_1$, i.e. $\mathbf{u}_1\mathbf{v}_1^T$ is sought, such that

$$e_1 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}(u_{1i}v_{1j} - a_{ij})^2$$

is minimized, where w_{ij} is the error weighting at point (i, j). The optimization problem at hand can be solved by using the conjugate gradient method [11]. As in the SVD method, the vectors $\mathbf{u}_1$ and $\mathbf{v}_1$ obtained are interpreted as the desired amplitude responses for the design of the first pair of cascaded subfilters. The desired magnitude response for the next filter section is computed as $\mathbf{A}_1 = \mathbf{A} - \hat{\mathbf{u}}_1\hat{\mathbf{v}}_1^T$ where $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{v}}_1$ are the sampled amplitude responses of the first pair of filters. A second outer product $\mathbf{u}_2\mathbf{v}_2^T$ is then sought such that

$$e_2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}(u_{2i}v_{2j} - a_{ij}^{(1)})^2$$

is minimized, where $a_{ij}^{(1)}$ is the (i, j) entry of matrix $\mathbf{A}_1$, and the second section of cascaded subfilters can be designed on the basis of vectors $\mathbf{u}_2$ and $\mathbf{v}_2$ obtained. Proceeding as in the SVD approach, a good 2-D filter can be obtained.

4.2 The Iterative SVD Method [7]

As is noted in [3], the weighted singular vectors $\tilde{\mathbf{u}}_i$ and $\tilde{\mathbf{v}}_i$ in (4) with $i \geq 2$ have, in general, negative components and, therefore, they cannot be treated as sampled amplitude responses directly. Instead of using the method described in Section 2.3 to modify these singular vectors, a so-called iterative SVD approach has been proposed in [7]. In this method, $\mathbf{A} \in R^{N\times N}$ is assumed to be a sampled amplitude matrix and $\tilde{\mathbf{u}}_1, \tilde{\mathbf{v}}_1$ is the first pair of the weighted singular vectors. A matrix $\mathbf{A}_2$ is constructed as $\mathbf{A}_2 = \mathbf{A} - \mathbf{u}_1\mathbf{v}_1^T$ and is then decomposed as

$$\mathbf{A}_2 = \mathbf{A}_2^+ - \mathbf{A}_2^-$$

where

$$\begin{aligned} A_2^+(i,j) &= \begin{cases} A_2(i,j) & \text{if } A_2(i,j) > 0 \\ 0 & \text{otherwise} \end{cases} \\ A_2^-(i,j) &= \begin{cases} -A_2(i,j) & \text{if } A_2(i,j) < 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Since both matrices $\mathbf{A}_2^+$ and $\mathbf{A}_2^-$ are nonnegative, the SVD of $\mathbf{A}_2^+$ and $\mathbf{A}_2^-$ leads to

$$\begin{aligned} \mathbf{A}_2^+ &= \tilde{\mathbf{u}}_2^+\tilde{\mathbf{v}}_2^{+T} + \cdots \\ \mathbf{A}_2^- &= \tilde{\mathbf{u}}_2^-\tilde{\mathbf{v}}_2^{-T} + \cdots \end{aligned}$$

where the weighted singular vectors $\tilde{\mathbf{u}}_2^+, \tilde{\mathbf{u}}_2^-, \tilde{\mathbf{v}}_2^+$ and $\tilde{\mathbf{v}}_2^-$ are all nonnegative and, therefore, can be treated as sampled amplitude responses of 1-D filters.

V. Realization

From (4) it follows that a total of r parallel sections is required to obtain a 2-D filter whose amplitude response is a good approximation of the sampled amplitude response $\mathbf{A}$. Experience has shown that for commonly used 2-D filters such as lowpass, highpass, bandpass, bandstop or fan filters, matrix $\mathbf{A}$ contains a large number of nonzero but negligible singular values. Evidently, when r_0 small singular values are neglected from (3) and (4), the 2-D filter can be realized by using only $r - r_0$ parallel sections. However, at the same time, additional approximation error is introduced. A detailed error analysis which addresses this issue can be found in [3].

Economical FIR realizations can be obtained through the following procedure:

Step1 Design r sets of cascaded 1-D filters based on (4) and denote the transfer functions obtained by $f_i(z_1), g_i(z_2), i = 1, \ldots, r$.

Step2 Write the 2-D transfer function as

$$\begin{aligned} H(z_1, z_2) &= \sum_{i=1}^{r} f_i(z_1) g_i(z_2) \\ &= \mathbf{z}_1^T \mathbf{C} \mathbf{z}_2^T \end{aligned} \qquad (10)$$

where $\mathbf{C}$ is an $N \times N$ coefficient matrix, and

$$\begin{aligned} \mathbf{z_1} &= [1\, z_1^{-1} \cdots z_1^{1(N-1)}]^T \\ \mathbf{z_2} &= [1\, z_2^{-1} \cdots z_2^{1(N-1)}]^T \end{aligned}$$

Step3 Perform the LU decomposition of matrix $\mathbf{C}$

$$\mathbf{C} = \mathbf{L}_c \mathbf{U}_c$$

and write the 2-D transfer function as

$$\begin{aligned} H(z_1, z_2) &= \mathbf{z}_1^T \mathbf{L}_c \mathbf{U}_c \mathbf{z}_2 \\ &= \sum_{i=1}^{r_c} L_{ci}(z_1) U_{ci}(z_2) \end{aligned} \qquad (11)$$

where $\mathbf{L_c}$ and $\mathbf{U_c}$ are lower and upper triangular matrices, respectively, and r_c is the rank of $\mathbf{C}$.

The significance of (11) in the realization of transfer function $H(z_1, z_2)$ in (10) is that in practice we often have $r_c \ll r$ and, therefore, (11) represents a better realization as compared to (10). Furthermore, since both $\mathbf{L_c}$ and $\mathbf{U_c}$ contain a large number of zero entries, extra savings can be obtained by the realization based on (11).

VI. Conclusions

The SVD design method has been surveyed and several important recent developments of the method have been addressed. The method can be applied in conjunction with 1-D design techniques for the design of causal, FIR or IIR, 2-D digital filters with arbitrary amplitude and phase responses. Furthermore, it is relatively simple to apply and leads to a parallel arrangement of pairs of cascaded 1-D filter sections. Structures of this type allow a large amount of concurrent processing and can be implemented in terms of systolic arrays. Therefore, they are amenable to VLSI implementation.

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