DESIGN OF DIGITAL DIFFERENTIATORS SATISFYING PRESCRIBED SPECIFICATIONS USING OPTIMIZATION TECHNIQUES

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ABSTRACT

A procedure is described which can be used to design digital differentiators satisfying prescribed specifications. The procedure is based on formulating the error between the practical digital differentiator and the ideal digital differentiator as a quadratic function. Standard optimization methods are used to minimize the error function.

I. INTRODUCTION

A signal $x(t)$, whose values are known at $t = nT$ for $n = 0, 1, 2, \ldots,$ where T is the sampling period, can be differentiated by using the many available numerical differentiation formulas such as the Gregory-Newton forward- and backward-difference formulas or the Bessel, Everett and Stirling central-difference formulas. Alternatively, $x(t)$ can be differentiated by using nonrecursive digital differentiators.

Nonrecursive digital differentiators can be designed by using the equiripple method advanced by Rabiner, McClellan, and Parks [1] or by using the Fourier-series method in conjunction with the Kaiser window function [2]. In the first approach, the passband error is minimized using the Remez exchange algorithm so as to achieve an equiripple weighted error. In the second approach, the passband error is minimized with respect to a single independent parameter α which is inherent in the Kaiser window function. Digital differentiators satisfying prescribed specifications can be designed by using the methods in [1] or [2] in conjunction with the prediction technique for differentiator order described in [3].

This contribution describes an alternative procedure for the design of digital differentiators. The new procedure, which entails two distinct but similar methods, involves formulating the error between the practical and ideal digital differentiators as a quadratic function which can be readily minimized using standard optimization algorithms or routines. The new methods are simpler to apply than the method in [1] and lead to better designs than the methods in [2] and [3].

II. PROBLEM FORMULATION

An Nth-order nonrecursive filter can be represented by the transfer function [4]

$$H(z) = \sum_{n=-K}^{K} h(nT) z^{-n} \tag{1}$$

where

$$K = \begin{cases} (N-1)/2 & \text{for odd N} \\ N/2 & \text{for even N} \end{cases}$$

If the impulse response of the filter is assumed to be antisymmetrical such that

$$h(0) = 0$$

$$h(nT) = -h(-nT)$$

then

$$\begin{aligned} H(e^{j\omega T}) &= M(\omega)\, e^{j\theta(\omega)} \\ &= e^{j\pi/2} \sum_{n=1}^{K} c_n \sin(\omega n T) \end{aligned} \tag{2}$$

where

$$M(\omega) = |\, H(e^{j\omega T})\,| = |\sum_{n=1}^{K} c_n \sin(\omega n T)\,|$$

$$\theta(\omega) = \arg H(e^{j\omega T}) = \pi/2$$

and

$$c_n = 2h(nT)$$

An ideal digital differentiator has a frequency response

$$\begin{aligned} H_I(e^{j\omega T}) &= M_I(\omega)\, e^{j\theta_I(\omega)} \\ &= j\omega \quad \text{for} \quad 0 \le |\,\omega\,| \le \omega_s/2 \end{aligned} \tag{3}$$

where

$$M_I(\omega) = |\,\omega\,|$$

$$\theta_I(\omega) = \pi/2$$

and $\omega_s = 2\pi/T$ is the sampling frequency in rad/s. On comparing eqns. (2) and (3), we note that an antisymmetrical impulse response yields a phase shift of $\pi/2$, as required in the case of a differentiator. If coefficients c_n in eqn. (2) are chosen such that the error function

$$e(\mathbf{C}) = \int_0^{\omega_s/2} [W(\omega)E(\omega)]^2\, d\omega \tag{4}$$

is minimized, where $W(\omega)$ is a weighting function,

$$E(\omega) = M(\omega) - M_I(\omega)$$

and

$$\mathbf{C} = [c_1\ c_2\ \ldots\ c_K]^T$$

then a digital differentiator can be designed.

In practice, a digital differntiator is required to have a specific degree of accuracy over a specific range of frequencies, depending on the application and the spectrum of the signal to be processed. Hence the main specifications of a differentiator are its bandwidth ω_p and the maximum passband error δ. The weighting function $W(\omega)$ can be used to emphasize the error function in one or more frequency bands so as to achieve lower error in these bands. In order to minimize the relative error defined as $E(\omega)/M_I(\omega)$, the weighting function [5]

$$W(\omega) = 1/\omega \qquad 0 \le \omega \le \omega_p \tag{5}$$

can be used in eqn. (4).

Two design methods are possible depending on whether an approximate or exact evaluation of $e(\mathbf{C})$ is utilized.

A. Method I

The error function $e(\mathbf{C})$ can be minimized by minimizing the function

$$e_1(\mathbf{C}) = \sum_{k=0}^{m} [W(\omega_k)E(\omega_k)]^2 \tag{6}$$

where $0 \le \omega_k \le \omega_p$ and m is the number of points at which the error is sampled. For $\omega_k > 0$

$$E(\omega_k) = M(\omega_k) - \omega_k \tag{7}$$

and if the weighting function in eqn. (5) is used, eqns. (6) and (7) yield

$$e_1(\mathbf{C}) = \sum_{k=0}^{m} [\frac{M(\omega_k)}{\omega_k} - 1]^2$$

Using matrix notation, the error function may be written as

$$e_1(\mathbf{C}) = \sum_{k=0}^{m} [\frac{\mathbf{C^T P}_k}{\omega_k} - 1]^2$$

where $\mathbf{P}_k$ is the vector given by

$$\mathbf{P}_k = \begin{pmatrix} sin(\omega_k T) \\ sin(2\omega_k T) \\ \vdots \\ sin(K\omega_k T) \end{pmatrix}$$

Thus

$$e_1(\mathbf{C}) = \sum_{k=0}^{m} (\mathbf{C}^T\mathbf{S}_k\mathbf{S}_k^T\mathbf{C} - 2\mathbf{C}^T\mathbf{S}_k) + \lambda \tag{8}$$

where $\mathbf{S}_k = \mathbf{P}_k/\omega_k$, and λ is a constant independent of $\mathbf{C}$. Evidently, $e_1(\mathbf{C})$ can be minimized by minimizing the modified error function

$$e_1(\mathbf{C}) = \sum_{k=0}^{m} (\mathbf{C}^T\mathbf{S}_k\mathbf{S}_k^T\mathbf{C} - 2\mathbf{C}^T\mathbf{S}_k) \tag{9}$$

It can readily be seen that the function in eqn. (9) represents a quadratic function of the form [6]

$$f(\mathbf{X}) = \frac{1}{2}\mathbf{X}^T\mathbf{Q}\mathbf{X} - \mathbf{X}^T\mathbf{b} \tag{10}$$

where

$$\mathbf{Q} = 2\sum_{k=0}^{m} \mathbf{S}_k\mathbf{S}_k^T$$

$$\mathbf{b} = 2\sum_{k=0}^{m} \mathbf{S}_k$$

and

$$\mathbf{X} = \mathbf{C}$$

If N is small, $e_1(\mathbf{C})$ can be minimized exactly by obtaining the inverse of $\mathbf{Q}$ and then multiplying it by vector $\mathbf{b}$. On the other hand, if N is large $e_1(\mathbf{C})$ can be minimized in at most $N/2$ iterations by using the well-known conjugate gradient unconstraint optimization method [6]. These two algorithms are as follows:

Algorithm 1

1. Evaluate matrix $\mathbf{Q}$ in eqn. 10
2. Compute $\mathbf{Q}^{-1}$
3. Obtain the solution as $\mathbf{C} = \mathbf{Q}^{-1}\mathbf{b}$

Algorithm 2

1. Pick an initial point $\mathbf{C}_0 \in \mathbf{E}^{N/2}$, compute $\mathbf{g}_0 = \mathbf{QC}_0 - \mathbf{b}$ and set $\mathbf{r}_0 = -\mathbf{g}_0$
2. Compute
$$\alpha_k = -\frac{\mathbf{g}_k^T\mathbf{r}_k}{\mathbf{r}_k^T\mathbf{Q}\mathbf{r}_k}$$
$$\mathbf{C}_{k+1} = \mathbf{C}_k + \alpha_k\mathbf{r}_k$$
3. Compute $\mathbf{g}_{k+1} = \mathbf{QC}_{k+1} - \mathbf{b}$
4. Compute
$$\beta_k = \frac{\mathbf{g}_{k+1}^T\mathbf{Q}\mathbf{r}_k}{\mathbf{r}_k^T\mathbf{Q}\mathbf{r}_k}$$
and
$$\mathbf{r}_{k+1} = -\mathbf{g}_{k+1} + \beta_k\mathbf{r}_k$$
5. If
$$\|\mathbf{g}_{k+1}\| > \epsilon$$
where $\epsilon = 10^{-8}$ (say) replace k by $k+1$ and return to step (2); otherwise output
$$\mathbf{C}_{k+1}$$
as the required solution.

B. Method II

In the second method, the error function $e(\mathbf{C})$ can be minimized by minimizing the function

$$e_2(\mathbf{C}) = \int_0^{\omega_p} [M(\omega T) - \omega]^2 \; d\omega \tag{11}$$

Integration was considered to be an alternative for the reason that the functions involved are trignometric and not complex. Thus integration gives exact results over a range of frequencies in the passband and stopband. Eqn. (11) can be expressed as

$$e_2(\mathbf{C}) = \int_0^{\omega_p} [\mathbf{C}^T\mathbf{S}\mathbf{S}^T\mathbf{C} - 2\mathbf{C}^T\mathbf{S}\omega] \; d\omega + \xi \tag{12}$$

where ξ is a constant independent of $\mathbf{C}$. Evidently, $e_2(\mathbf{C})$ can be minimized by minimizing the modified error function

$$e_2(\mathbf{C}) = \mathbf{C}^T[\int_0^{\omega_p} \mathbf{S}\mathbf{S}^T \; d\omega]\mathbf{C} - 2\mathbf{C}^T[\int_0^{\omega_p} \mathbf{S}\omega \; d\omega] \tag{13}$$

On comparing eqn. (13) with eqn. (10), we have

$$\mathbf{Q} = 2\int_0^{\omega_p} \mathbf{S}\mathbf{S}^T \; d\omega \tag{14}$$

$$\mathbf{b} = 2\int_0^{\omega_p} \mathbf{S}\omega \; d\omega \tag{15}$$

where the integration is carried out for each term in matrix $\mathbf{Q}$ and for each term in vector $\mathbf{b}$.

In this method, the weighting function ω^{-1} is not used as the integration becomes very difficult. Instead piecewise weighting is used. The integration is broken into several parts between the limits 0 and ω_p and weights are assigned to each integral to emphasize a particular region in the passband. For example, if the integration is broken into three parts, eqns. (14) and (15) give

$$\mathbf{Q} = 2[W_1 \int_0^{\omega_{p_1}} \mathbf{S}\mathbf{S}^T\, d\omega + W_2 \int_{\omega_{p_1}}^{\omega_{p_2}} \mathbf{S}\mathbf{S}^T\, d\omega + W_3 \int_{\omega_{p_2}}^{\omega_{p_3}} \mathbf{S}\mathbf{S}^T\, d\omega]$$

$$\mathbf{b} = 2[W_1 \int_0^{\omega_{p_1}} \mathbf{S}\omega\, d\omega + W_2 \int_{\omega_{p_1}}^{\omega_{p_2}} \mathbf{S}\omega\, d\omega + W_3 \int_{\omega_{p_2}}^{\omega_{p_3}} \mathbf{S}\omega\, d\omega]$$

where W_1, W_2, and W_3 are appropriate weights and $\omega_{p_3} = \omega_p$.

The minimization of the error function can be accomplished by using Algorithms 1 and 2 described earlier.

III. DESIGN EXAMPLES

As in [3], the plot of log (δ) against N was found to be almost linear. Thus given the inband error, the order of the filter can be estimated and vice versa by using the technique described in [3]. Fig. 1 shows a plot of the variation of the error with the order of the filter for the case of odd N using Method I.

The above methods were used to design three digital differentiators assuming a prescribed bandwidth $\omega_p = 2.0$ rad/s and prescribed maximum inband errors $\delta = 0.01, 0.005, 0.001$. The sampling frequency was assumed to be $\omega_s = 2\pi$ rad/s. In Method I, the value of m was taken to be 100. In Method II, three weights were used namely $W_1 = 1$, $W_2 = \frac{2}{3}$, and $W_3 = \frac{1}{3}$. The optimum values of N for the two methods are given in Table 1. The variation of $E(\omega)$ for $\omega_p = 2.0$ rad/s with weighting for the case of even N using Method I is shown in Fig. 2.

The same procedure was then used to design three digital differentiators assuming a prescribed maximum inband error $\delta = 0.01$ and prescribed bandwidths $\omega_p = 2.25, 2.5, 2.75$ rad/s. The optimum values of N for the two methods are given Table 2. The variation of $E(\omega)$ for $\omega_p = 2.25$, 2.50, and 2.75 rad/s with $\delta = 0.01$ with weighting for the case of odd N using Method II are shown in Fig. 3.

The procedure was also used to design a wideband differentiator having a prescribed bandwidth of $\omega_p = 3.0$ rad/s and a prescribed maximum inband error $\delta = 0.01$. The optimum values of N for the two methods are given in Table 3. The variation of $E(\omega)$ without weighting for the case of even N using Method II is shown in Fig. 4.

In Method I, the weighting functions $\omega^{-1/2}, \omega^{-1/3}$ were also tried but did not yield better results than ω^{-1}.

As can be observed, the design of digital differentiators using even N is better and yields lower-order differentiators than the case of odd N . The reason for this is that $M(\omega) = 0$ for $\omega = \pi$ for odd N and $M(\omega) \neq 0$ for $\omega = \pi$ for even N. Thus the error introduced in the case of odd N is larger near the passband edges, particularly in the design of wideband digital differentiators. Consequently, an odd N yields a larger order for the differentiators. It should be noted that in the case of odd N, values of the impulse response occur at instants nT while for the case of even N samples occur at $(n + \frac{1}{2})T$. This can lead to problems in some applications.

IV. CONCLUSIONS

Two methods for the design of nonrecursive digital differentiators based on standard optimization methods have been described. The methods were used in conjunction with the prediction technique for the differentiator order described in [3] to design several differentiators assuming a prescribed bandwidth and various prescribed maximum inband errors. The methods were then used to design several differentiators assuming a prescribed maximum inband error and various prescribed bandwidths. They were then used to design a wideband differentiator. The designs obtained satisfy the prescribed specifications in all cases.

The results obtained show that the even-order digital differentiators require lower order for the same prescribed specifications. It has also been found that the error function with weighting yields better results in terms of relative error, as may expected. Method II yields better results and is more efficient than Method I since the integrals involved provide an exact closed-form expression for the error function which can be evaluated with less computational effort.

The new methods are simpler to apply than the method in [1] and lead to reduced differentiator error relative to the methods in [2]-[3]. The amount of computation involved by different methods is currently being investigated.

References

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Table 1: Optimum N for Design Examples ($\omega_p = 2.0$ rad/s)

δ	Odd N without weighting	Odd N with weighting	Even N without weighting	Even N with weighting
Method I				
0.010	9	11	6	6
0.005	11	11	6	6
0.001	15	15	8	8
Method II				
0.010	9	11	6	6
0.005	11	11	6	6
0.001	13	15	8	8

Table 2: Optimum N for Design Examples ($\delta = 0.01$)

ω_p	Odd N without weighting	Odd N with weighting	Even N without weighting	Even N with weighting
Method I				
2.25	13	13	6	6
2.50	19	19	8	8
2.75	31	33	10	12
Method II				
2.25	13	13	6	6
2.50	17	19	8	8
2.75	31	31	10	10

Table 3: Optimum N for Wideband Differentiator

Odd N without weighting	Odd N with weighting	Even N without weighting	Even N with weighting
Method I			
103	109	22	24
Method II			
93	95	20	20

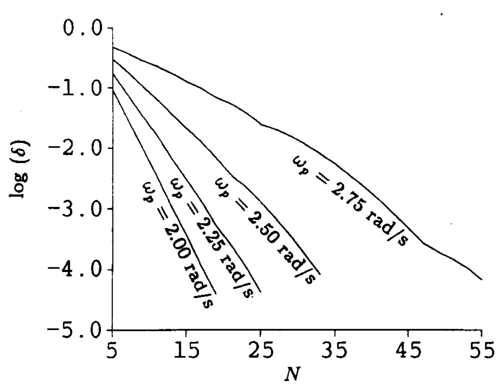

Fig. 1. Variation of log (δ) with N

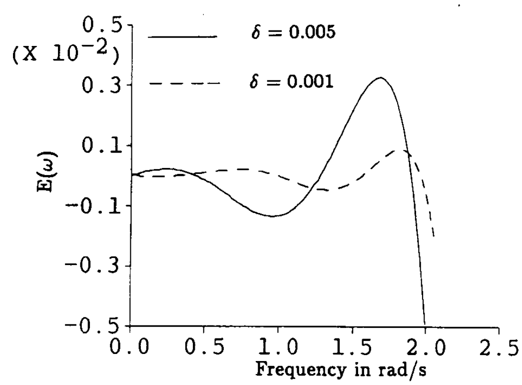

Fig. 2. Variation of $E(\omega)$ with ω

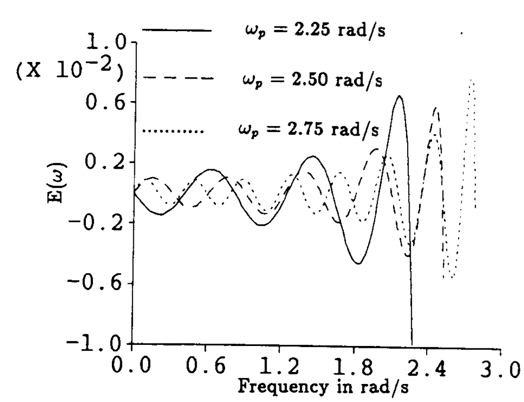

Fig. 3. Variation of $E(\omega)$ with ω

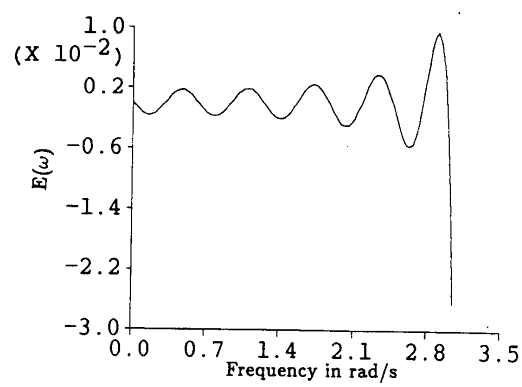

Fig. 4. Variation of $E(\omega)$ with ω for a wideband differentiator

(217) 333-4789 Further information can be obtained from Dr. W. Kenneth Jenkins.