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Order Reduction of 2-D Discrete Systems Via Bounded Real Balancing*

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Abstract A new order reduction method for 2-D discrete systems based on bounded real (BR) balancing is presented. The concept of BR balancing was first introduced by Opdenacker and Jonckheere to reduce the order of 1-D continuous-time systems. It is shown in this paper that if the 2-D discrete system considered satisfies the 2-D Lyapunov condition, the BR balanced approximation then yields a stable lower-order 2-D system with a reasonably small approximation error.

I. INTRODUCTION

The problem of reducing the order of two-dimensional (2-D) discrete systems has been a subject for study during the last several years [1]-[7]. While the balanced approximation method [3] appears to be a satisfactory scheme [6], its application to certain 2-D systems with very small stability margin might yield unstable reduction [6].

In this paper, a new order reduction method for 2-D discrete systems based on bounded real (BR) balancing is presented. The concept of BR balancing was first introduced by Opdenacker and Jonckheere [8] to reduce the order of 1-D continuous-time systems. It is shown that if the 2-D discrete system considered satisfies the 2-D Lyapunov condition described below, the BR balanced approximation then yields a stable lower-order 2-D system with a reasonably small approximation error.

II. THE MAIN RESULTS

To describe the reduction procedure, let $\Sigma = (A, B, C)$ with $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}_{(n_1+n_2)\times(n_1+n_2)}$, $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}_{(n_1+n_2)\times m}$, $C = (C_1 \quad C_2)_{p\times(n_1+n_2)}$

be the Roesser's state-space representation of the original 2-D system. It is assumed throughout that system Σ is L-stable meaning that there exists a block-diagonal positive-definite matrix $W = W_1 \oplus W_2$ such that

$$W - A^T W A > 0 \tag{1}$$

The transfer function of system Σ can be written as

$$H(z_1, z_2) = (c_1 \quad c_2)\begin{pmatrix} z_1 I - A_1 & -A_2 \\ -A_3 & z_2 I - A_4 \end{pmatrix}^{-1}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$
$$= C(z_1)(z_2 I - A(z_1))^{-1} B(z_1) + D(z_1)$$

where

$$\begin{aligned} A(z_1) &= A_3(z_1 I - A_1)^{-1} A_2 + A_4 \\ B(z_1) &= A_3(z_1 I - A_1)^{-1} B_1 + B_2 \\ C(z_1) &= C_1(z_1 I - A_1)^{-1} A_2 + C_2 \\ D(z_1) &= C_1(z_1 I - A_1)^{-1} B_1 \end{aligned} \tag{2}$$

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In order to meet a technical condition regarding the real boundness of a relevant transfer matrix described below, the proposed reduction procedure shall be carried out for a "scaled" transfer function $H_\alpha(z_1, z_2)$ where

$$H_\alpha(z_1, z_2) = \alpha^2 H(z_1, z_2)$$
$$= (\alpha C_1 \quad \alpha C_2)\begin{pmatrix} z_1 I - A_1 & -A_2 \\ -A_3 & z_2 I - A_4 \end{pmatrix}^{-1}\begin{pmatrix} \alpha B_1 \\ \alpha B_2 \end{pmatrix}$$
$$= C_\alpha(z_1)(z_2 I - A(z_1))^{-1} B_\alpha(z_1) + D_\alpha(z_1) \tag{3}$$

where $\alpha > 0$ is a scaling factor to be chosen,

$$\begin{aligned} B_\alpha(z_1) &= A_3(z_1 I - A_1)^{-1} \alpha B_1 + \alpha B_2 \\ C_\alpha(z_1) &= \alpha C_1(z_1 I - A_1)^{-1} A_2 + \alpha C_2 \\ D_\alpha(z_1) &= \alpha C_1(z_1 I - A_1)^{-1} \alpha B_1 \end{aligned}$$

When the order of $H_\alpha(z_1,z_2)$ is reduced, the desired approximation of $H(z_1,z_2)$ can be obtained immediately by a re-scaling.

Notice that due to the L-stability of system Σ, $A(z_1)$ defined by (2) (up to a similarity transformation) can be assumed to be strictly bounded real (SBR) without loss of generality [9], i.e.

$$\|A(z_1)\|_\infty < 1 \tag{4}$$

Further notice that in order to reduce the order of system Σ, one may seek a lower-order 2-D system, say $\hat{\Sigma}$, for which the corresponding transfer function

$$\hat{H}_\alpha(z_1, z_2) = \hat{C}_\alpha(z_1)(z_2 I - \hat{A}(z_1))^{-1} \hat{B}_\alpha(z_1) + \hat{D}_\alpha(z_1)$$

is close to $H_\alpha(z_1, z_2)$ in the sense that $\hat{A}(z_1)$, $\hat{B}_\alpha(z_1)$, $\hat{C}_\alpha(z_1)$, and $\hat{D}_\alpha(z_1)$ are *simultaneously* close to $A(z_1)$, $B_\alpha(z_1)$, $C_\alpha(z_1)$, and $D_\alpha(z_1)$ in L_∞ norm, respectively.

One way to accomplish such a simultaneous approximation is to reduce the order of the transfer function matrix $\Phi_\alpha(z_1)$ defined by

$$\Phi_\alpha(z_1) = \begin{pmatrix} A(z_1) & B_\alpha(z_1) \\ C_\alpha(z_1) & D_\alpha(z_1) \end{pmatrix} = H_\alpha(z_1 I - A_1)^{-1} G_\alpha + J_\alpha \tag{5a}$$

where

$$G_\alpha = (A_2 \quad \alpha B_1),\ H_\alpha = \begin{pmatrix} A_3 \\ \alpha C \end{pmatrix},\ J_\alpha = \begin{pmatrix} A_4 & \alpha B_2 \\ \alpha C_2 & 0 \end{pmatrix} \tag{5b}$$

Since can be written as

$$\Phi_\alpha(z_1) = \begin{pmatrix} A(z_1) & \alpha B(z_1) \\ \alpha C(z_1) & \alpha^2 D(z_1) \end{pmatrix}$$

condition (4) implies that their exists > 0 such that

$$\|\Phi_\alpha(z_1)\|_\infty < 1 \tag{6}$$

We are now in a position to begin our reduction procedure with Φ_α defined by (5a) satisfying SBR condition (6).

Step 1 Compute the two extremal positive-definite (PD) solutions P_{max} and P_{min} of the discrete algebraic Riccati equation(DARE) for system $\Sigma_\alpha =$

$(A_1, G_\alpha, H_\alpha, J_\alpha)$:

$$-P + A_1^T P A_1 + H_\alpha^T H_\alpha +$$
$$(A_1^T P G_\alpha + H_\alpha^T J_\alpha)(I - J_\alpha^T J_\alpha - G_\alpha^T P G_\alpha)(A_1^T P G_\alpha + H_\alpha^T J_\alpha) = 0$$

Note that the two extremal PD solutions Q_{max} and Q_{min} of the DARE for dual system $\Sigma_\alpha^T = (A_1^T, H_\alpha^T, G_\alpha^T, J_\alpha^T)$:

$$-Q + A_1 Q A_1^T + G_\alpha G_\alpha^T +$$
$$(A_1 Q H_\alpha^T + G_\alpha J_\alpha^T)(I - J_\alpha J_\alpha^T - H_\alpha Q H_\alpha^T)(A_1 Q H_\alpha^T + G_\alpha J_\alpha^T) = 0$$

are related to P_{max} and P_{min} by $Q_{max} = P_{min}^{-1}$ and $Q_{min} = P_{max}^{-1}$. It can readily be verified that with any similarity transformation T_1, the associated PD solutions P_{max}, P_{min}, Q_{max}, Q_{min} of the equivalent system

$$\Sigma_\alpha = (T_1 A_1 T_1^{-1}, T_1 G_\alpha, H_\alpha T_1^{-1}, J_\alpha)$$

and its dual system are related to P_{max}, P_{min}, Q_{max} and Q_{min} by $P_{max}=T_1^{-T}P_{max}T_1^{-1}$, $P_{min}=T_1^{-T}P_{min}T_1^{-1}$, $Q_{max}=T_1Q_{max}T_1^T$, and $Q_{min}=T_1Q_{min}T_1^T$.

In other words, the eigenvalues of $P_{min}Q_{min}$ (i.e. $P_{min}P_{max}^{-1}$), for instance, are real, positive and invariant under similarity transformations. This enables us to proceed with the next step.

Step 2 Find a nonsingular T_1 such that

$$P_{min} = P_{max}^{-1} = diag\{\xi_i\} \quad with \quad \xi_1 \ge \xi_2 \ge \ldots \ge \xi_{n_1} > 0$$

This can be accomplished by using a standard procedure [10].

Step 3 (order reduction along the horizontal direction)

If $\xi_{r_1} >> \xi_{r_1+1}$, *then partition* Σ_α *as*

$$A_1 = T_1 A_1 T_1^{-1} = \begin{pmatrix} A_{1r_1} & * \\ * & * \end{pmatrix}, G_\alpha = T_1 G_\alpha = \begin{pmatrix} A_{2r_1} & \alpha B_{1r_1} \\ * & * \end{pmatrix}$$

$$H_\alpha = H_\alpha T_1^{-1} = \begin{pmatrix} A_{3r_1} & * \\ \alpha C_{1r_1} & * \end{pmatrix}, J_\alpha = J_\alpha$$

where $A_{1r_1} \in R^{r_1 \times r_1}, A_{2r_1} \in R^{r_1 \times n_2}, A_{3r_1} \in R^{n_2 \times r_1}$,

$$B_{1r_1} \in R^{r_1 \times m}, C_{1r_1} \in R^{p \times r_1},$$

and form the horizontally reduced 2-D system

$$\Sigma_{a1} = (A_{r1}, B_{a1}, C_{a1}),$$

with
$$A_{r_1} = \begin{pmatrix} A_{1r_1} & A_{2r_1} \\ A_{3r_1} & A_4 \end{pmatrix}, B_{r_1} = \begin{pmatrix} B_{1r_1} \\ B_1 \end{pmatrix}, C_{r1} = \begin{pmatrix} C_{1r_1} & C_2 \end{pmatrix}$$

It can be shown that A_{r1} is 2-D L-stable and the reduction error is given in terms of ξ_i, $r_1+1 < i < n_1$.

To reduce the order of system Σ along the vertical direction, we form the transfer function matrix $\Psi_\beta(z_2)$ defined by

$$\Psi_\beta(z_2) = \begin{pmatrix} F(z_2) & B_\beta(z_2) \\ C_\beta(z_2) & D_\beta(z_2) \end{pmatrix} = H_\beta(z_2 I - A_4)^{-1} G_\beta + J_\beta$$

where $\beta > 0$ is a scaling factor to be chosen, and

$$F(z_2) = A_{2r_1}(z_2 I - A_4)^{-1} A_{3r_1} + A_{1r_1},$$

$$B_\beta(z_2) = A_{2r_1}(z_2 I - A_4)^{-1} \beta B_2 + \beta B_{1r_1},$$

$$C_\beta(z_2) = \beta C_2(z_2 I - A_4)^{-1} A_{3r_1} + \beta C_{1r_1},$$

$$D_\beta(z_2) = \beta C_2(z_2 I - A_4)^{-1} \beta B_2.$$

$$G_\beta = (A_{3r_1} \quad \beta B_2), H_\beta = \begin{pmatrix} A_{2r_1} \\ \beta C_2 \end{pmatrix}, J_\beta = \begin{pmatrix} A_{1r_1} & \beta B_{1r_1} \\ \beta C_{1r_1} & 0 \end{pmatrix}$$

Steps analogous to step 1 - step 3 can then be proceeded to finally obtain an L-stable reduced 2-D system of order (r_1, r_2).

References

[1] E.B.Lee, W.-S.Lu, and N.E.Wu, "Approximation of 2-D systems", Proc. Johns Hopkins Conf. Information Sciences, March 1985.

[2] E.I.Jury and K.Premaratne, "Model reduction of 2-D discrete systems", IEEE Trans. Circuits Syst., vol.CAS-33, pp. 558-562, May 1986.

[3] W.-S. Lu, E.B. Lee, and Q.-T. Zhang, "Balanced approximation of two-dimensional and delay-differential systems", Proc. 25th. IEEE Conf. Decision and Control, pp. 917-922, Dec. 1986.

[4] P.N.Paraskevopoulos, "Pade type order reduction of two-dimensional systems", IEEE Trans. Circuits Syst., vol.CAS-27, pp. 413-416, May 1980.

[5] B.Lashgari, L.M.Silverman, and J.F.Abramatic, "Approximation of 2D separable in denominator filters", IEEE Trans. Circuits Syst., vol.CAS-30 pp. 107-121, Feb. 1983.

[6] K.Premaratne and E.I.Jury, "Model reduction of two-dimensional discrete systems via balanced realization",Proc. 21st. Asilomar Conf., Nov. 1987.

[7] A.Kumar, F.W.Fairman, and J.R.Sveinsson, "Separately balanced realization and model reduction of 2D separable denominator transfer functions from input output data", IEEE Trans. Circuits Syst.,vol.CAS-24, pp.233-239,March, 1987.

[8] P.C.Opdenacker and E.A.Jonckheere, "A contraction mapping preserving balanced reduction scheme and its infinity norm error bounds", IEEE Trans. Circuits Syst., vol.CAS-35, pp. 184-189, Feb. 1988.

[9] B.D.O.Anderson, P.Agathoklis, E.I.Jury, and M.Mansour, "Stability and the matrix Lyapunov equation for discrete 2-dimensional systems", IEEE Trans. Circuits Syst., vol.CAS-33, pp. 261-266, March 1986.

[10]A.J.Laub, M.T.Heath, C.C.Paige, and R.C.Ward, "Computation of system balancing transformations and other applications of simultaneous diagonalization algorithm", IEEE Trans. Automat. Contr., vol.AC-32, pp. 115-122, Feb. 1987.