A NEW METHOD FOR THE DESIGN OF STABLE IIR 2-D DIGITAL FILTERS USING SEQUENTIAL SEMIDEFINITE PROGRAMMING

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ABSTRACT

A method for the design of quadrantally symmetric 2-D IIR digital filters using sequential semidefinite programming (SDP) is developed. The design problem is solved by formulating the objective function in the SDP framework using a linear approximation of the transfer function. The issue of filter stability is addressed by converting the stability constraint into a set of linear matrix inequalities based on the well-known Lyapunov stability theory.

1. INTRODUCTION

A great deal of research on the design of one- and two-dimensional (1- and 2-D) digital filters has been carried out in the past [1]-[5]. The design of 2-D IIR filters presents two additional challenges over and above the design of 2-D FIR filters. First, while 2-D FIR filters are inherently stable, the stability of 2-D IIR filters is fairly difficult to formulate in a manner suitable for constrained optimization. Second, since the transfer functions of IIR filters are rational functions, the degree of nonlinearity involved in their design by optimization is considerably higher than that for FIR filters.

In this paper, we develop a method for the design of IIR quadrantally symmetric 2-D filters based on semidefinite programming (SDP). The method is essentially a sequential minimization algorithm and can be described as follows. At the kth iteration a linear approximation for the transfer function is obtained using its Taylor expansion. In this way, a weighted least-squares objective function can be deduced which is a convex quadratic function of the design variables. The objective function is then minimized subject to several constraints. These constraints include a stability constraint and possibly other constraints on the filter's maximum passband ripple and minimum stopband attenuation. The stability constraint is converted into a linear matrix inequality (LMI) and it is shown that each iteration of the design can be performed using SDP. A design example is included to illustrate the proposed method.

2. LINEAR APPROXIMATION OF TRANSFER FUNCTION

Consider a quadrantally symmetric 2-D IIR digital filter whose transfer function is given by

$$H(z_1,\, z_2) = \frac{B(z_1,\, z_2)}{A(z_1,\, z_2)} \tag{1}$$

where

$$B(z_1,\, z_2) = \sum_{i=0}^{n_1}\sum_{k=0}^{n_2} b_{ik} z_1^{-i} z_2^{-k}$$

and

$$\begin{aligned} A(z_1,\, z_2) &= A_1(z_1)A_2(z_2) \\ A_1(z_1) &= \sum_{i=0}^{r_1} a_{1i} z_1^{-i}, \qquad a_{10} = 1 \\ A_2(z_2) &= \sum_{i=0}^{r_2} a_{2i} z_2^{-i}, \qquad a_{20} = 1 \end{aligned}$$

The design variables are $\{b_{ik},\ 0 \le i \le n_1,\ 0 \le k \le n_2\}$, $\{a_{1i},\ 1 \le i \le r_1\}$, and $\{a_{2i},\ 1 \le i \le r_2\}$, which form a $[r_1 + r_2 + (n_1+1)(n_2+1)]$-dimensional vector

$$\mathbf{x} = [a_{11}\ \cdots\ a_{1r_1}\ a_{21}\ \cdots\ a_{2r_2}\ b_{00}\ \cdots\ b_{n_1,n_2}]^T$$

Denote vector $\mathbf{x}$ in the kth iteration as $\mathbf{x}_k$ and the frequency response of the filter for $\mathbf{x} = \mathbf{x}_k$ as $H(e^{j\omega_1},\, e^{j\omega_2},\, \mathbf{x}_k)$. In the vicinity of $\mathbf{x}_k$, the design variable can be expressed as

$$\mathbf{x} = \mathbf{x}_k + \boldsymbol{\delta}$$

As in the approach used in [6] for the 1-D case, the transfer function can be approximated in terms of a linear function of $\boldsymbol{\delta}$ as

$$H(e^{j\omega_1},\, e^{j\omega_2},\, \mathbf{x}) \approx H(e^{j\omega_1},\, e^{j\omega_2},\, \mathbf{x}_k) + \mathbf{g}_k^T\boldsymbol{\delta} \tag{2}$$

where $\mathbf{g}_k$ is the gradient of $H(e^{j\omega_1},\, e^{j\omega_2},\, \mathbf{x})$ for $\mathbf{x} = \mathbf{x}_k$, i.e.,

$$\mathbf{g}_k = \nabla H(e^{j\omega_1},\, e^{j\omega_2},\, \mathbf{x}_k)$$

From (1), the gradient vector can be evaluated as

$$\frac{\partial H}{\partial x_i} = \begin{cases} \frac{-B}{A_1^2 A_2} e^{-ji\omega_1} & \text{for } 1 \le i \le r_1 \\ \frac{-B}{A_1 A_2^2} e^{-ji\omega_2} & \text{for } r_1 + 1 \le i \le r_1 + r_2 \\ \frac{1}{A_1 A_2} e^{-j(l\omega_1 + m\omega_2)} & \text{otherwise} \end{cases}$$

where B, A_1, A_2 denote the values of $B(e^{j\omega_1}, e^{j\omega_2})$, $A_1(e^{j\omega_1})$, and $A_2(e^{j\omega_2})$ evaluated at $\mathbf{x} = \mathbf{x}_k$, respectively.

For a weighted least-squares design, the objective function is given by

$$e(\mathbf{x}) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W(\omega_1, \omega_2) |E(\omega_1, \omega_2, \mathbf{x})|^2 \, d\omega_1 \omega_2 \quad (3)$$

where $E(\omega_1, \omega_2, \mathbf{x}) = H(e^{j\omega_1}, e^{j\omega_2}, \mathbf{x}) - H_d(\omega_1, \omega_2)$ and $H_d(\omega_1, \omega_2)$ is the desired frequency response. By using the linear approximation of $H(e^{u\omega_1}, e^{j\omega_2}, \mathbf{x})$ in (2), the objective function $e(\mathbf{x})$ in a small neighborhood of $\mathbf{x}_k$ can be expressed as

$$e(x) \approx \boldsymbol{\delta}^T \mathbf{Q}_k \boldsymbol{\delta} - 2\mathbf{q}_k^T \boldsymbol{\delta} + c_0 \quad (4a)$$

$$\mathbf{Q}_k = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W(\omega_1, \omega_2) \mathcal{R}e(\bar{\mathbf{g}}_k \mathbf{g}_k^T) \, d\omega_1 d\omega_2 \quad (4b)$$

where

$$\mathbf{q}_k = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W(\omega_1, \omega_2) \mathcal{R}e[E_k(\omega_1, \omega_2) \bar{\mathbf{g}}_k] d\omega_1 \, d\omega_2 \quad (4c)$$

$$c_0 = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W(\omega_1, \omega_2) |E_k(\omega_1, \omega_2)^2| \, d\omega_1 d\omega_2 \quad (4d)$$

$$E_k(\omega_1, \omega_2) = E(\omega_1, \omega_2, \mathbf{x}_k) \quad (4e)$$

Therefore, in the kth iteration vector $\mathbf{x}_k$ can be updated as

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \boldsymbol{\delta}_k$$

where α_k is a scalar between 0 and 1 and $\boldsymbol{\delta}_k$ solves the minimization problem

$$\text{minimize} \quad \boldsymbol{\delta}^T \mathbf{Q}_k \boldsymbol{\delta} - 2\mathbf{q}_k^T \boldsymbol{\delta} + c_0 \quad (5a)$$

subject to: $\mathbf{x}_k + \boldsymbol{\delta}$ corresponds to a stable IIR filter (5b)

3. STABILITY CONSTRAINTS

The stability constraint in (5b) needs appropriate treatment to make it tractable for numerical optimization. Denote the vectors formed from the first r_1 components and the next r_2 components of $\mathbf{x}_k + \boldsymbol{\delta}$ by $\mathbf{a}_{1k} + \boldsymbol{\delta}_1$ and $\mathbf{a}_{2k} + \boldsymbol{\delta}_2$, respectively. Since the denominator of $H(z_1, z_2)$ is separable, it can be shown [7] that the IIR filter with coefficient vector $\mathbf{x}_k + \boldsymbol{\delta}$ is stable if and only if the magnitudes of the eigenvalues of matrices

$$\mathbf{D}_{1k} = \begin{bmatrix} -(\mathbf{a}_{1k} + \boldsymbol{\delta}_1)^T \\ \hat{\mathbf{I}}_{r_1} \end{bmatrix}$$

and

$$\mathbf{D}_{2k} = \begin{bmatrix} -(\mathbf{a}_{2k} + \boldsymbol{\delta}_2)^T \\ \hat{\mathbf{I}}_{r_2} \end{bmatrix}$$

are all strictly less than one, where $\hat{\mathbf{I}}_r$ denotes a matrix of size $r \times (r+1)$ obtained by augmenting the identity matrix with a zero column on the right. In such a case, $\mathbf{D}_{1k}$ and $\mathbf{D}_{2k}$ are said to be stable matrices. Applying the well-known Lyapunov theory [7], one concludes that matrices $\mathbf{D}_{1k}$ and $\mathbf{D}_{2k}$ are stable if and only if there exist positive definite matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ such that

$$\mathbf{P}_1 - \mathbf{D}_{1k}^T \mathbf{P}_1 \mathbf{D}_{1k} \succ \mathbf{0} \quad (6)$$

$$\mathbf{P}_2 - \mathbf{D}_{2k}^T \mathbf{P}_2 \mathbf{D}_{2k} \succ \mathbf{0} \quad (7)$$

where $\mathbf{M} \succ \mathbf{0}$ denotes that matrix $\mathbf{M}$ is positive definite. It can be readily verified that the matrix inequalities in (6) and (7) hold if and only if

$$\begin{bmatrix} \mathbf{P}_1^{-1} & \mathbf{D}_{1k} \\ \mathbf{D}_{1k}^T & \mathbf{P}_1 \end{bmatrix} \succ \mathbf{0} \text{ and } \begin{bmatrix} \mathbf{P}_2^{-1} & \mathbf{D}_{2k} \\ \mathbf{D}_{2k}^T & \mathbf{P}_2 \end{bmatrix} \succ \mathbf{0} \quad (8)$$

To assure a stability margin for the IIR filter, the constraints in (8) are modified as

$$\mathbf{Y}_{1k} = \begin{bmatrix} \mathbf{P}_1^{-1} - \tau \mathbf{I}_{r_1} & \mathbf{D}_{1k} \\ \mathbf{D}_{1k}^T & \mathbf{P}_1 - \tau \mathbf{I}_{r_1} \end{bmatrix} \succeq \mathbf{0} \quad (9a)$$

$$\mathbf{Y}_{2k} = \begin{bmatrix} \mathbf{P}_2^{-1} - \tau \mathbf{I}_{r_2} & \mathbf{D}_{2k} \\ \mathbf{D}_{2k}^T & \mathbf{P}_2 - \tau \mathbf{I}_{r_2} \end{bmatrix} \succeq \mathbf{0} \quad (9b)$$

where $\mathbf{Y} \succeq \mathbf{0}$ denotes that $\mathbf{Y}$ is positive semidefinite and τ is a positive scalar that controls the stability margin of the filter.

4. AN SDP DESIGN FORMULATION

4.1. Semidefinite Programming

An SDP problem can be formulated in several ways. For the purpose of filter design, the following formulation turns out to be convenient:

$$\text{minimize} \quad \mathbf{c}^T \mathbf{x} \quad (10a)$$

$$\text{subject to} \quad \mathbf{F}(\mathbf{x}) \succeq \mathbf{0} \quad (10b)$$

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + \sum_{i=1}^{n} x_i \mathbf{F}_i \quad (10c)$$

In (10), $\mathbf{x} = [x_1 \ \cdots \ x_n]^T$ is the variable vector, $\mathbf{c} \in R^{n \times 1}$, $\mathbf{F}_i \in R^{n \times n}$ for $0 \le i \le n$ are constant matrices with $\mathbf{F}_i$ symmetric. Note that the constraint matrix is *affine* with respect to $\mathbf{x}$, and that SDP includes both linear and convex quadratic programming as special cases. More importantly, many interior-point methods that have proven efficient for linear programming have recently been extended to SDP

[8][9]. Efficient and user-friendly software implementations of various SDP algorithms are available, for example, the LMI Control Toolbox [10], which works with MATLAB, and freeware SDPT 3 [11] and SeDuMi [12].

4.2. Design Formulation

With the constraint in (5b) replaced by those in (9), the minimization problem in the kth iteration can be described as

$$\text{minimize} \quad \boldsymbol{\delta}^T \mathbf{Q}_k \boldsymbol{\delta} - 2\mathbf{q}_k^T \boldsymbol{\delta} + c_0 \tag{11a}$$

$$\text{subject to:} \quad \mathbf{Y}_k = \begin{bmatrix} \mathbf{Y}_{k_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_{k_2} \end{bmatrix} \succeq \mathbf{0} \tag{11b}$$

If we denote the upper bound of the objective function in (11a) as μ, then minimizing the objective function is equivalent to minimizing its upper bound and the problem in (11) can be converted into

$$\text{minimize} \quad \mu \tag{12a}$$

$$\text{subject to:} \quad \boldsymbol{\delta}^T \mathbf{Q}_k \boldsymbol{\delta} - 2\mathbf{q}_k^T \boldsymbol{\delta} + c_0 \le \mu \tag{12b}$$

$$\mathbf{Y}_k \succeq \mathbf{0} \tag{12c}$$

where μ is treated as an *additional design variable*. To convert the constraint in (12b) to an LMI, we rewrite (12b) as

$$(\mu + 2\mathbf{q}_k^T \boldsymbol{\delta} - c_0) - (\mathbf{Q}_k^{\frac{1}{2}} \boldsymbol{\delta})^T (\mathbf{Q}_k^{\frac{1}{2}} \boldsymbol{\delta}) \ge 0 \tag{13}$$

where $\mathbf{Q}_k^{1/2}$ is the symmetric square root of $\mathbf{Q}_k$. It can be readily shown that (13) holds if and only if

$$\mathbf{S}_k = \begin{bmatrix} \mathbf{I} & \mathbf{Q}_k^{1/2} \boldsymbol{\delta} \\ \boldsymbol{\delta}^T \mathbf{Q}_k^{1/2} & \mu + 2\mathbf{q}_k^T \boldsymbol{\delta} - c_0 \end{bmatrix} \succeq \mathbf{0} \tag{14}$$

The problem in (12) can now be converted to

$$\text{minimize} \quad \hat{\mathbf{c}}^T \hat{\mathbf{x}} \tag{15a}$$

$$\text{subject to:} \quad \begin{bmatrix} \mathbf{S}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_k \end{bmatrix} \succeq \mathbf{0} \tag{15b}$$

where $\hat{\mathbf{x}}$ denotes the augmented vector $\hat{\mathbf{x}} = [\mu \ \boldsymbol{\delta}^T]^T$ and $\hat{\mathbf{c}} = [1 \ 0 \ \cdots \ 0]^T$. From (9) and (14), we see that matrices $\mathbf{S}_k$ and $\mathbf{Y}_k$ depend on vector $\hat{\mathbf{x}}$ *affinely*. The matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ in (9) are *not* considered as design variables. Rather, these positive definite matrices are fixed in each iteration and can be obtained, for example, by solving the Lyapunov equations

$$\mathbf{P}_1 - \hat{\mathbf{D}}_{1k}^T \mathbf{P}_1 \hat{\mathbf{D}}_{1k} = \mathbf{I} \tag{16a}$$

and

$$\mathbf{P}_2 - \hat{\mathbf{D}}_{2k} \mathbf{P}_2 \hat{\mathbf{D}}_{2k} = \mathbf{I} \tag{16b}$$

respectively, where

$$\hat{\mathbf{D}}_{1k} = \begin{bmatrix} -(\mathbf{a}_{1k})^T \\ \hat{\mathbf{I}}_{r_1} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{D}}_{2k} = \begin{bmatrix} -(\mathbf{a}_{2k})^T \\ \hat{\mathbf{I}}_{r_2} \end{bmatrix}$$

It is well-known [7] that if the IIR filter with coefficient vector $\mathbf{x}_k$ is stable, then the solutions of the equations in (16) are unique and positive definite.

With $\mathbf{P}_1$ and $\mathbf{P}_2$ fixed in $\mathbf{Y}_k$, the minimization problem in (15) is an SDP problem of size $1+r_1+r_2+(n_1+1)(n_2+1)$.

4.3. Design Steps

Given the order of the IIR filter $(n_1, \ n_2, \ r_1, \ r_2)$ and the desired frequency response $H_d(\omega_1, \ \omega_2)$, the proposed design method starts with an initial point $\mathbf{x}_0$ that corresponds to a stable design obtained using a conventional method. For example, one can design an FIR filter of order $(n_1, \ n_2)$ to approximate $H_d(\omega_1, \ \omega_2)$ and simply set $A_1(z_1) \equiv 1$ and $A_2(z_2) \equiv 1$ as the initial design. With this $\mathbf{x}_0$, positive definite matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ can be obtained by solving the Lyapunov equations in (16), and quantities $\mathbf{Q}_k$, $\mathbf{q}_k$, and c_0 can be evaluated by using (4b)–(4e). Next we solve the SDP problem in (15) whose solution $\hat{\mathbf{x}}^* = [\mu^* \ \ \boldsymbol{\delta}^{*T}]^T$ can be used to update $\mathbf{x}_0$ to $\mathbf{x}_1 = \mathbf{x}_0 + \boldsymbol{\delta}^*$. The iteration continues until $||\boldsymbol{\delta}^*||$ is less than a prescribed tolerance ε.

5. A DESIGN EXAMPLE

The proposed method was applied to design several types of 2-D IIR filters, including circularly symmetric lowpass, highpass, bandpass filters, and diamond-shaped lowpass filters. The design example presented here is a circularly symmetric lowpass filter of order $(n_1, \ n_2, \ r_1, \ r_2) = (12, \ 12, \ 8, \ 8)$ with normalized passband and stopband edges $\omega_p = 0.5\pi$ and $\omega_a = 0.7\pi$. The desired frequency response was assumed to be

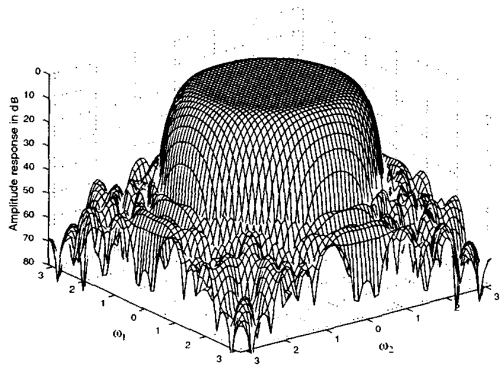
$$H_d(\omega_1, \ \omega_2) = \begin{cases} e^{-j7.5(\omega_1+\omega_2)} & \text{for } \sqrt{\omega_1^2 + \omega_2^2} \le \omega_c \\ 0 & \text{elsewhere} \end{cases}$$

where $\omega_c = 0.5(\omega_p + \omega_a)$. The weighting function used was given by

$$W(\omega_1, \ \omega_2) = \begin{cases} 1 & \text{for } \sqrt{\omega_1^2 + \omega_2^2} \le \omega_p \text{ or} \\ & \quad \sqrt{\omega_1^2 + \omega_2^2} \ge \omega_a \\ 0.1 & \text{elsewhere} \end{cases}$$

The initial design was obtained by designing a 2-D FIR filter of order (12, 12) using the singular-value decomposition method [13] and setting $A_1(z_1) \equiv 1$ and $A_2(z_2) \equiv 1$. With $\varepsilon = 3 \times 10^{-4}$ and $\alpha_k \equiv 1$, it took the algorithm 19 iterations to converge to a solution. The amplitude response and the passband group-delay characteristic are depicted in Fig. 1. The weighted least-squares error defined by (3) for the filter designed was 0.0023, and the maximum modulus of the poles was 0.7210. For the sake of comparison, another circularly symmetric lowpass IIR filter of order (12,

12) satisfying the same design specifications was designed using the method presented in [14], and the weighted least-squares error achieved was 0.0087.



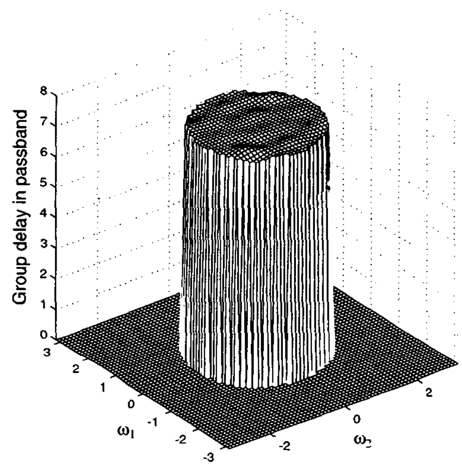


Figure 1: (a) Amplitude response and (b) passband group-delay characteristic.

6. CONCLUSION

A method for the design of IIR 2-D digital filters has been presented. The method is based on a sequential application of SDP, where in each iteration an SDP algorithm is applied to solve the optimization problem. It has been demonstrated that the algorithm converges quickly to a solution with satisfactory performance and guaranteed stability.

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