DESIGN OF 2-D FIR FILTERS WITH POWER-OF-TWO COEFFICIENTS: A SEMIDEFINITE PROGRAMMING RELAXATION APPROACH

W.-S. Lu

Dept. of Electrical and Computer Engineering
University of Victoria
Victoria, BC, Canada V8W 3P6

ABSTRACT

This paper presents a design method for 2-D FIR digital filters with "sum of power of two" coefficients. Traditionally the design problem is formulated as an integer programming problem which is known to have nonpolynomial-time complexity. The proposed algorithm "relaxes" the design to a semidefinite programming (SDP) problem, which can be solved using efficient interior-point algorithms with polynomial-time complexity.

1. INTRODUCTION

Research interest in digital filters with discrete coefficients, that can be represented as sum of a small number of power-of-two (SP2) terms, has lasted several decades [1]–[7]. Perhaps the primary reason we are interested in digital filters with SP2 coefficients is that these filters admit fast implementations that require no multiplications but simple superposition of shifted versions of the input. When a filter with SP2 coefficients is designed in an optimization setting, the problem is essentially an integer programming (IP) or a mixed integer programming (MIP) problem, which is known to have nonpolynomial complexity [8].

In this paper, we consider the design of 2-D FIR digital filters with SP2 coefficients and tackle the problem by using a semidefinite programming (SDP) relaxation technique. SDP as a relative new class of constrained optimization methods enjoys several desirable features: As a subclass of convex programming problems, a minimizer of an SDP problem is always a global minimizer and SDP problems can be solved using efficient interior-point algorithms with polynomial-time complexity. In a typical weighted least squares design, the relaxation of the MIP problem with binary variables to an SDP problem with continuous variables can be accomplished by neglecting a rank constraint on an augmented matrix. As will be demonstrated in Sec. 3, suboptimal solutions of the design problem in question that are obtained by SDP relaxation (SDPR) are often of excellent quality.

2. A DESIGN METHOD BASED ON SDP RELAXATION

The proposed design is accomplished in three steps, namely, (i) design of a 2-D FIR filter with continuous coefficients that approximates a desired frequency response; (ii) a formulation of the design of a 2-D FIR filter with SP2 coefficients (based on the continuous coefficient filter obtained in Step (i)) as a $\{-1,\ 1\}$-optimization problem; and (iii) an SDPR of the $\{-1,\ 1\}$-optimiza-tion problem formulated in Step (ii).

2.1. Design of Continuous-Coefficient 2-D FIR Filters

Let the transfer function of a 2-D FIR digital filter be denoted by

$$H_c(z_1,\ z_2) = \sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1} h_{ij} z_1^{-i} z_2^{-j} = \mathbf{z}_1^T \hat{\mathbf{H}}_c \mathbf{z}_2 \tag{1}$$

where N_1 and N_2 are odd integers, $\mathbf{z}_1 = [1\ z_1^{-1}\ \cdots\ z_1^{-(N_1-1)}]^T$, $\mathbf{z}_2 = [1\ z_2^{-1}\ \cdots\ z_2^{-(N_2-1)}]^T$, and $\hat{\mathbf{H}} \in R^{N_1\times N_2}$. If the filter has linear phase response, then

$$\hat{\mathbf{H}}_c = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{h}_{12} & \mathbf{H}_{13} \\ \mathbf{h}_{21}^T & h_{22} & \mathbf{h}_{23}^T \\ \mathbf{H}_{31} & \mathbf{h}_{32} & \mathbf{H}_{33} \end{bmatrix}$$

where $\mathbf{H}_{11},\ \mathbf{H}_{13},\ \mathbf{H}_{31},\ \mathbf{H}_{33} \in R^{n_1\times n_2}$, $\mathbf{h}_{12},\ \mathbf{h}_{32} \in R^{n_1\times 1}$, $\mathbf{h}_{21},\ \mathbf{h}_{23} \in R^{n_2\times 1}$, $h_{22} \in R$, $n_1 = (N_1-1)/2$, $n_2 = (N_2-1)/2$, and

$$\begin{aligned} \mathbf{H}_{13} &= \text{flipud }(\mathbf{H}_{33}),\ \mathbf{H}_{31} = \text{fliplr }(\mathbf{H}_{33}), \\ \mathbf{H}_{11} &= \text{flipud }(\text{fliplr }(\mathbf{H}_{33})), \\ \mathbf{h}_{12} &= \text{flipud }(\mathbf{h}_{32}),\ \text{and } \mathbf{h}_{21}^T = \text{fliplr }(\mathbf{h}_{23}^T) \end{aligned}$$

where flipud and fliplr represent the operations of flipping a matrix upside down and from left to right, respectively. Hence the frequency response of the filter is given by

$$H_c(\omega_1,\ \omega_2) = e^{-j(n_1\omega_1+n_2\omega_2)}\mathbf{c}_1^T(\omega_1)\mathbf{H}_c\mathbf{c}_2(\omega_2) \tag{2}$$

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where $\mathbf{c}_i(\omega_i) = [1 \;\; \cos\omega_i \;\; \cdots \;\; \cos n_i\omega_i]^T$ for $i = 1, 2$, and

$$\mathbf{H}_c = \begin{bmatrix} h_{22} & 2\mathbf{h}_{23}^T \\ 2\mathbf{h}_{32} & 4\mathbf{H}_{33} \end{bmatrix} = \{h_{ij}^{(c)}\}$$

Efficient design methods for a variety types of 2-D FIR filters are available. These include the window-function methods, optimization-based methods, and SVD-based methods [9]. For the purpose of illustrating our method, suppose one applies an available technique to design a linear phase FIR filter of odd length (N_1, N_2) that approximates a desirable frequency response $H_d(\omega_1, \omega_2) = \exp[-j(n_1\omega_1 + n_2\omega_2)]A_d(\omega_1+\omega_2)$ such that the weight least squares (WLS) error

$$e = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} W(\omega_1, \omega_2)|H(\omega_1, \omega_2) - H_d(\omega_1, \omega_2)|^2 d\omega_1 d\omega_2 \tag{3}$$

is minimized, and the frequency response of the filter obtained is given by (2).

2.2. A Weighted Least-Squares $\{-1, 1\}$-Optimization

For a given budget of total number of power-of-two (P2) terms M, a certain number of P2 terms, m_{ij}, is allocated to the (i, j)th component, d_{ij}, of matrix $\mathbf{D} = \{d_{ij}\}$ in the discrete-coefficient frequency response

$$H(\omega_1, \omega_2) = e^{-j(n_1\omega_1+n_2\omega_2)}\mathbf{c}_1^T(\omega_1)\mathbf{D}\mathbf{c}_2(\omega_2) \tag{4}$$

with

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} m_{ij} = M \tag{5}$$

Further assume that the P2-terms used in the representation of $\{d_{ij}\}$ and constrained to within a given range, say, between 2^{-U} and 2^{-L} with $L < U$. Thus the discrete coefficient d_{ij} in (4) can be expressed as

$$d_{ij} = \sum_{k=1}^{m_{ij}} b_k^{(ij)} 2^{-q_k^{(ij)}} \tag{6}$$

where $b_k^{(ij)} \in \{-1, 1\}$ and $L \le q_k^{(ij)} \le U$.

For a given continuous-coefficient design $H_c(\omega_1, \omega_2)$, a bit budget $\{m_{ij}\}$, and a range $\{L, U\}$, the *least* SP2 upper bound $\bar{d}_{ij}$ and *largest* SP2 lower bound $\underline{d}_{ij}$ for each continuous coefficient $h_{ij}^{(c)}$ in (2) can be determined such that $\underline{d}_{ij} \le h_{ij}^{(c)} \le \bar{d}_{ij}$ with both $\bar{d}_{ij}$ and $\underline{d}_{ij}$ being of the form (6). Clearly, in each open interval $(\underline{d}_{ij}, \bar{d}_{ij})$, no SP2 numbers of form (6) exist for a given budget $\{m_{ij}\}$.

An easy-to-get suboptimal design of 2-D FIR filter with SP2 coefficients is given by

$$H_s(\omega_1 \, \omega_2) = e^{-j(n_1\omega_1+n_2\omega_2)}\mathbf{c}_1^T(\omega_1)\mathbf{D}_s\mathbf{c}_2(\omega_2) \tag{7a}$$

where $\mathbf{D}_s = \{d_{ij}^{(s)}\}$ with

$$d_{ij}^{(s)} = \begin{cases} \bar{d}_{ij} & \text{if } \bar{d}_{ij} - h_{ij}^{(c)} \le h_{ij}^{(c)} - \underline{d}_{ij} \\ \underline{d}_{ij} & \text{otherwise} \end{cases} \tag{7b}$$

In words, coefficients $\{d_{ij}^{(s)}\}$ are obtained as SP2 bounds of form (6) that are closest to $\{h_{ij}^{(c)}\}$. Unfortunately, as will be demonstrated in Sec. 3, performance of this suboptimal design is often unsatisfactory. Now we denote the midpoint of each interval $[\underline{d}_{ij}, \bar{d}_{ij}]$ as $d_{ij}^{(m)} = (\bar{d}_{ij} + \underline{d}_{ij})/2$ and a half of the interval length as $\delta_{ij} = (\bar{d}_{ij} - \underline{d}_{ij})/2$. Then the SP2 upper and lower bounds $\bar{d}_{ij}$ and $\underline{d}_{ij}$ can be selected as $d_{ij}^{(m)} + b_{ij}\delta_{ij}$ with $b_{ij} = 1$ and $b_{ij} = -1$, respectively. Hence the frequency response of discrete-coefficient FIR transfer function in (4) with

$$d_{ij} = d_{ij}^{(m)} + b_{ij}\delta_{ij} \tag{8}$$

can be expressed as

$$\begin{aligned} H(\omega_1 \, \omega_2) &= e^{-j(n_1\omega_1+n_2\omega_2)}A(\omega_1, \omega_2) \\ A(\omega_1, \omega_2) &= A_m(\omega_1, \omega_2) + \mathbf{c}_1^T(\omega_1)(\mathbf{B} \circ \Delta)\mathbf{c}_2(\omega_2) \end{aligned} \tag{9}$$

where $A_m(\omega_1, \omega_2)$ depends on $\{d_{ij}^{(m)}\}$, and $\mathbf{B} \circ \boldsymbol{\Delta}$ denotes the pointwise product of $\mathbf{B} = \{b_{ij}\}$ and $\boldsymbol{\Delta} = \{\delta_{ij}\}$. Since the second term on the right-hand side of (9) is linear with respect to $\{b_{ij}\}$, we can write

$$A(\omega_1, \omega_2) = A_m(\omega_1, \omega_2) + \mathbf{b}^T\mathbf{v}(\omega_1, \omega_2, \boldsymbol{\Delta}) \tag{10}$$

where $\mathbf{b} = [b_{00} \;\; b_{10} \;\; \ldots \;\; b_{n_1 0} \;\; b_{01} \;\; \ldots \;\; b_{n_1 n_2}]^T$ and $\mathbf{v}$ is a column vector of dimension $N = (n_1 + 1)(n_2 + 1)$ determined by $\mathbf{c}_1(\omega_1)$, $\mathbf{c}_2(\omega_2)$, and $\boldsymbol{\Delta}$. In the light of (10), the WLS objective function in (1) can be written as

$$e = \mathbf{b}^T\mathbf{Q}\mathbf{b} + 2\mathbf{b}^T\mathbf{q} + \text{const} \tag{11}$$

where

$$\mathbf{Q} = \int_0^\pi\int_0^\pi W(\omega_1, \omega_2)\mathbf{v}(\omega_1, \omega_2, \boldsymbol{\Delta})\mathbf{v}^T(\omega_1, \omega_2, \boldsymbol{\Delta})d\omega_1 d\omega_2$$

$$\mathbf{q} = \int_0^\pi\int_0^\pi W(\omega_1, \omega_2)E(\omega_1, \omega_2)\mathbf{v}(\omega_1, \omega_2, \boldsymbol{\Delta})d\omega_1 d\omega_2$$

$$E(\omega_1, \omega_2) = A_m(\omega_1, \omega_2) - A_d(\omega_1, \omega_2)$$

The WLS design of $H(\omega_1, \omega_2)$ with SP2 coefficients can now be formulated as

$$\underset{b_i \in \{-1, 1\}}{\text{Minimize}} \; \mathbf{b}^T\mathbf{Q}\mathbf{b} + 2\mathbf{b}^T\mathbf{q} \tag{12}$$

Since the components of $\mathbf{Q}$ and $\mathbf{q}$ take real values while $\mathbf{b}$ assumes integer entries, (12) is a mixed integer programming (MIP) problem.

2.3. A Semidefinite Programming Relaxation of Problem (12)

Semidefinite programming (SDP) is concerned with a class of constrained optimization problems where a linear objective function is minimized subject to matrix constrains which affinely depend on the variable vector. Typically an SDP problem can be formulated as

$$\begin{aligned} &\text{minimize} && \mathbf{c}^T\mathbf{x} && \text{(13a)}\\ &\text{subject to:} && \mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + \sum_{i=1}^{r} x_i\mathbf{F}_i \succeq \mathbf{0} && \text{(13b)}\end{aligned}$$

where matrices $\mathbf{F}_i$ for $0 \le i \le r$ are symmetric and $\succeq \mathbf{0}$ means positive semidefinite. Efficient polynomial-time interior-point algorithms have been extended to SDP [10][11] and software implementations of the algorithms are available [12]–[14]. The relevance of SDP to the design problem at hand lies in the fact that the MIP problem in (12) can be *relaxed* to an SDP problem so as to obtain an approximate solution which can be solved in polynomial time. Geomans and Williamson [15] was among the first to obtain an SDP relaxation solution of the MAX-CUT problem—a well-known integer quadratic programming problem in graph theory. Following [15], SDP relaxations of various MIP problems have been reported in graph optimization, network management and scheduling [16]. In what follows we present a SDP-relaxation-based solution to our design problem.

We consider the problem in (12) with vector $\mathbf{b}$ replaced by a more conventional notation $\mathbf{x} = [x_1 \ \dots \ x_N]^T$:

$$\begin{aligned} &\text{minimize} && \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{q} && && \text{(14a)}\\ &\text{subject to:} && x_i^2 = 1 && \text{for } 1 \le i \le N && \text{(14b)}\end{aligned}$$

where constraints $x_i \in \{-1, \ 1\}$ have been replaced by the equality constraints in (14b). If we define matrix $\mathbf{X} = \mathbf{x}\mathbf{x}^T$, then the objective function in (14a) can be written as

$$\mathrm{tr}(\mathbf{Q}\mathbf{X}) + 2\mathbf{x}^T\mathbf{q}$$

where $\mathrm{tr}(\cdot)$ denotes trace, and the problem in (14) is equivalent to

$$\begin{aligned} &\text{minimize} && \mathrm{tr}(\mathbf{Q}\mathbf{X}) + 2\mathbf{x}^T\mathbf{q} && && \text{(15a)}\\ &\text{subject to:} && \mathbf{X} = \mathbf{x}\mathbf{x}^T && && \text{(15b)}\\ & && x_i^2 = 1 && \text{for } 1 \le i \le N && \text{(15c)}\end{aligned}$$

Note that the constraints in (15) are equivalent to

$$\mathrm{rank}\begin{bmatrix} \mathbf{X} & \mathbf{x}\\ \mathbf{x}^T & 1\end{bmatrix} = 1, \ X_{ii} = 1, \text{ and } \begin{bmatrix} \mathbf{X} & \mathbf{x}\\ \mathbf{x}^T & 1\end{bmatrix} \succeq \mathbf{0} \quad (16)$$

An SDP relaxation of (15) is obtained by neglecting the rank constraint in (16) while keeping the remaining two constraints, which leads to the minimization problem

$$\begin{aligned} &\text{minimize} && \mathrm{tr}(\mathbf{Q}\mathbf{X}) + 2\mathbf{x}^T\mathbf{q} && && \text{(17a)}\\ &\text{subject to:} && \begin{bmatrix} \mathbf{X} & \mathbf{x}\\ \mathbf{x}^T & 1\end{bmatrix} \succeq \mathbf{0} && && \text{(17b)}\\ & && X_{ii} = 1 && \text{for } 1 \le i \le N && \text{(17c)}\end{aligned}$$

Now let $\hat{\mathbf{X}}$ be the matrix obtained from the symmetric matrix $\mathbf{X}$ with diagonal elements replaced by unity, then the problem in (17) can be simplified to

$$\begin{aligned} &\text{minimize} && \mathrm{tr}(\mathbf{Q}\hat{\mathbf{X}}) + 2\mathbf{x}^T\mathbf{q} && \text{(18a)}\\ &\text{subject to:} && \begin{bmatrix} \hat{\mathbf{X}} & \mathbf{x}\\ \mathbf{x}^T & 1\end{bmatrix} \succeq \mathbf{0} && \text{(18b)}\end{aligned}$$

If we define an augmented variable vector $\hat{\mathbf{x}}$ of dimension $\hat{N} = N(N+1)/2$ that includes the $N(N-1)/2$ independent elements of $\hat{\mathbf{X}}$ and N elements of $\mathbf{x}$, then the objective function in (18a) can be expressed (up to a constant scalar) as $\hat{\mathbf{c}}^T\hat{\mathbf{x}}$ for a constant vector $\hat{\mathbf{c}} \in R^{\hat{N}\times 1}$ and (18) becomes

$$\begin{aligned} &\text{minimize} && \hat{\mathbf{c}}^T\hat{\mathbf{x}} && \text{(19a)}\\ &\text{subject to:} && \begin{bmatrix} \hat{\mathbf{X}} & \mathbf{x}\\ \mathbf{x}^T & 1\end{bmatrix} \succeq \mathbf{0} && \text{(19b)}\end{aligned}$$

Note that the matrix in constraint (19b) is *affine* with respect to $\hat{\mathbf{x}}$, hence (19) is an SDP problem.

Let $(\mathbf{x}^*, \ \hat{\mathbf{X}}^*)$ be a solution of (19), there are two approaches to generating a binary solution for the problem in (14). The first approach is straightforward:

$$\mathbf{x} = \mathrm{sign}(\mathbf{x}^*) \qquad \text{(20a)}$$

where $\mathrm{sign}(\cdot)$ denotes the signum function. The second approach is based on the singular value decomposition of matrix $\mathbf{Z}^* = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^T$ where $\mathbf{Z}^*$ is the $(N+1)\times(N+1)$ matrix in (19b) at $(\mathbf{x}^*, \ \hat{\mathbf{X}}^*)$, and obtains the binary solution $\mathbf{x}$ as

$$\mathbf{x} = \mathrm{sign}[\mathbf{u}_1(N+1)\cdot\mathbf{u}_1(1:N)] \qquad \text{(20b)}$$

where $\mathbf{u}_1$ is the first column vector in $\mathbf{U}$ that corresponds to the largest singular value σ_1 of $\mathbf{Z}^*$. The motivation behind the second approach is that a *perfect* solution $\hat{\mathbf{X}}^*$ satisfying (15b) would imply that matrix $\mathbf{Z}^*$ has rank one, hence $\sigma_1\mathbf{u}_1\mathbf{u}_1^T$ is the best rank-one approximation of $\mathbf{Z}^*$ in the 2-norm sense.

3. DESIGN EXAMPLES

The SDP relaxation (SDPR) method described in Sec. 2 was applied to design linear phase circularly symmetric lowpass 2-D FIR filters of order $(N, \ N)$ with SP2 coefficients based on corresponding WLS FIR prototype filters with continuous coefficients. Seven such filters with normalized passband edge $\omega_p = 0.25$, stopband edge $\omega_a = 0.35$, $W \equiv 1$ in both passband and stopband, $L = 0$, $U = 12$, average number of power-of-two terms per coefficient = 2.0, and filter lengths from 3 to 15 were designed and compared to the suboptimal designs characterized by (7), optimal designs with SP2 coefficients obtained using MIP, and the WLS designs with continuous coefficients in terms of the WLS error e defined by (3). The results are summarized in Table 1 where these four types of designs are labeled as Design (7), SDPR, MIP, and WLS respectively. For SDPR and MIP designs, the number of floating point operations (flops) used are also enclosed as a measure of design efficiency, see the second figures in the SDPR and MIP columns in Table 1.

Table 1: Comparisons of Various Designs

N	Design (7)	SDPR	MIP	WLS
3	0.1431	0.1375/ 5.4e3	0.1375/ 0.8e3	0.1339
5	0.0790	0.0769/89.8e3	0.0769/10.3e4	0.0768
7	0.0292	0.0281/12.8e5	0.0279/37.8e6	0.0272
9	0.0130	0.0095/11.0e6	0.0094/45.3e10	0.0094
11	0.0078	0.0061/65.7e6	0.0059/10.1e16	0.0056
13	0.0039	0.0017/30.0e7	0.0017/17.9e24	0.0016
15	0.0017	0.0011/11.2e8	0.0011/74.8e36	0.0010

4. CONCLUDING REMARKS

From the simulations it is observed that the designs based on (7) show considerably larger approximation error. On the other hand, the SDPR method often yields near-optimal designs (and for $N > 5$) with much reduced computational complexity compared with that required by the MIP designs. It is worthwhile to mention that the SDPR method described in this paper is of illustrative nature as it also applies to a variety of filter designs including the weighted minimax design and WLS designs with constraints in both passbands and stopbands.

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