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DESIGN OF STABLE MINIMAX IIR DIGITAL FILTERS USING SEMIDEFINITE PROGRAMMING

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ABSTRACT

Semidefinite programming (SDP) has recently been found useful in designing various types of FIR digital filters. This paper describes a SDP-based method for the design of stable minimax IIR filters. A single linear matrix inequality (LMI) constraint assures the filter's stability and fits nicely into the SDP-based design setting. Design efficiency and performance of the proposed method are illustrated by simulations and comparisons to a well-known design by Deczky.

1. INTRODUCTION

Recursive (also known as infinite-impulse-response (IIR)) digital filters offer improved selectivity, computation efficiency and reduced system delay compared to what can be achieved by nonrecursive (also known as finite-impulse-response (FIR)) digital filters of comparable approximation accuracy [1]–[6]. The major drawbacks of IIR designs are that linear phase response can be achieved only approximately and the design must handle the stability problem which does not exist in the FIR case. Several recent methods formulate an IIR design as a constrained optimization problem [6]–[10]. In [7], the filter's stability is ensured by imposing a family of linear inequalities, in which there is a frequency parameter varying over the entire domain $[-\pi, \pi]$. The design methods developed in [8][9] take the advantage of the above constraint's linearity to formulate least-squares IIR designs as iterative quadratic programming (QP) problems. However, the design practice reported in [6][8][9][10] indicated that the stability constraint proposed in [7] is highly restrictive, excluding many excellent deign candidates. Moreover, unless the designer is patient to carry out a design by trial and error, an arbitrary discrete implementation of the stability constraint often yields either a unnecessarily large number of discretized constraints leading to numerically expensive QP problems or an insufficient number of discretized constraints that fail to assure the stability. In [6][10], a stability constraint based on Rouché's theorem (RT) [11] is proposed. Again, it is a sufficient constraint for stability, but the simulation results described in [6][10] indicated that the constraint is considerably less restrictive than that of [7]. It is noted that the RT-based constraint contains a parameter from 0 to 2π. To avoid a discrete implementation, the algorithm in [6][10] calls for a one-dimensional (1-D) optimization to search the maximum of a constraint function. In addition, in its original form the RT-based constraint is not a linear one.

Semidefinite programming (SDP) is a relatively new interior-point optimization methodology that has attracted a great deal of research interest in the past several years [12]–[18]. Methods for the design of 1-D and 2-D digital filters using SDP have been reported [19]–[21]. In this paper, we describe an SDP-based method for the design of stable minimax IIR filters. Unlike the existing design methods, the stability is ensured by imposing a *single* linear matrix inequality (LMI) constraint derived from the well-known Lyapunov theory [22]. The LMI contains no continuous parameters and hence eliminates the need of discrete implementation. More important, it fits nicely into the SDP-based design setting. To illustrate the proposed method, a design example is presented and compared to a well-known design by Deczky [5].

2. DESIGN UTILITIES

2.1. SDP

There are several ways a SDP problem can be formulated. For the purpose of filter design, the following formulation turns out to be of convenience:

$$\text{minimize} \quad \mathbf{c}^T\mathbf{x} \tag{1a}$$

$$\text{subject to} \quad \mathbf{F}(\mathbf{x}) \succeq \mathbf{0} \tag{1b}$$

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + \sum_{i=1}^{n} x_i\mathbf{F}_i \tag{1c}$$

In (1), $\mathbf{x} = [x_1 \ \ldots \ x_n]^T$ is the variable vector, $\mathbf{c} \in R^{n\times 1}$, $\mathbf{F}_i \in R^{n\times n}$ for $0 \le i \le n$ are constant matrices with $\mathbf{F}_i$ symmetric, and $\mathbf{F}(\mathbf{x}) \succeq \mathbf{0}$ denotes that the constraint matrix $\mathbf{F}(\mathbf{x})$ is positive semidefinite. Note that the constraint matrix is *affine* with respect to $\mathbf{x}$. SDP includes both linear and convex quadratic programming as special cases. More importantly, many interior-point methods, which have proven efficient for linear programming, have recently been extended to SDP [13][14]. Efficient and user-friendly software implementations of SDP algorithms are available. In particular, we mention the LMI Control Toolbox, which works with MATLAB, as an excellent implementation tool [23].

2.2. An LMI Constraint for Stability

An IIR digital filter is stable if its poles are located strictly inside the unit circle. Let $d(z) = z^r + d_1z^{r-1} + \cdots + d_r$ be the denominator polynomial of an IIR filter. It is well known that the zeros

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of $d(z)$ are the eigenvalues of the canonical matrix

$$\mathbf{D} = \begin{bmatrix} -d_1 & -d_2 & \cdots & -d_r \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} \tag{2}$$

A matrix $\mathbf{D}$ is said to be stable if the modulus of every eigenvalue of $\mathbf{D}$ is strictly less than 1. Hence an irreducible transfer function $b(z)/d(z)$ represents a stable IIR filter if and only if $\mathbf{D}$ in (2) is a stable matrix. The well known Lyapunov theory [22] states that $\mathbf{D}$ is stable if and only if there exists a positive matrix $\mathbf{P}$ such that $\mathbf{P} - \mathbf{D}^T\mathbf{P}\mathbf{D}$ is positive definite, i.e.,

$$\mathbf{P} - \mathbf{D}^T\mathbf{P}\mathbf{D} \succ \mathbf{0} \tag{3}$$

In other words, $\mathbf{D}$ is stable if and only if the feasible set $\{\mathbf{P}\}$ of constraints

$$\begin{aligned} \mathbf{P} &\succ \mathbf{0} && \text{(4a)} \\ \mathbf{P} - \mathbf{D}^T\mathbf{P}\mathbf{D} &\succ \mathbf{0} && \text{(4b)} \end{aligned}$$

is nonempty. Using simple linear algebraic manipulations, it can be verified that the feasible set of the constraints in (4) can be characterized as the feasible set $\{\mathbf{P}\}$ of the LMI constraint

$$\begin{bmatrix} \mathbf{P}^{-1} & \mathbf{D} \\ \mathbf{D}^T & \mathbf{P} \end{bmatrix} \succ \mathbf{0} \tag{5}$$

Note that, unlike constraint (4), matrix $\mathbf{D}$ (hence the coefficients of $d(z)$) in (5) appears *affinely*.

3. A SDP FORMULATION AND DESIGN ALGORITHM

3.1. The Design Problem

We seek to find a stable transfer function

$$H(z) = \frac{b(z)}{\hat{d}(z)} = \frac{b(z)}{z^{n-r}d(z)} \tag{6}$$

where

$$b(z) = \sum_{i=0}^{n} b_i z^{n-i} \tag{7}$$

$$d(z) = \sum_{i=0}^{r} d_i z^{r-i} \quad \text{with } d_0 = 1 \tag{8}$$

and r is an integer between 0 and n, to best approximate the desired frequency response $H_d(\omega)$ in the minimax sense:

$$\begin{aligned} &\text{minimize} && \underset{-\pi \le \omega \le \pi}{\text{maximize}}\, W(\omega)|H(\omega) - H_d(\omega)| && \text{(9a)} \\ &\text{subject to} && d(z) \neq 0 \quad \text{for } |z| \geq 1 && \text{(9b)} \end{aligned}$$

where $W(\omega) \geq 0$ is a weighting function. The form of $\hat{d}(z)$ in (6) is convenient to reflect the fact that assigning a certain number of poles at the origin might be beneficial for the design of several types of digital filters as was observed in [6]. Note that the frequency response of the filter can be written as

$$H(\omega) = \frac{b(\omega)}{d(\omega)} \tag{10}$$

where

$$b(\omega) = \sum_{i=0}^{n} b_i e^{-ji\omega},\ d(\omega) = \sum_{i=0}^{r} d_i e^{-ji\omega} \tag{11}$$

In the rest of the paper, the following notation will be adopted:

$$\begin{aligned} \mathbf{b} &= [b_0\ b_1\ \cdots\ b_n]^T,\ \mathbf{d} = [d_1\ d_2\ \cdots\ d_r]^T \\ \mathbf{c}(\omega) &= [1\ \cos\omega\ \cdots\ \cos n\omega]^T,\ \mathbf{s}(\omega) = [0\ \sin\omega\ \cdots\ \sin n\omega]^T \\ \mathbf{c}_1(\omega) &= [\cos\omega\ \cdots\ \cos r\omega]^T,\ \mathbf{s}_1(\omega) = [\sin\omega\ \cdots\ \sin r\omega]^T \\ H_d(\omega) &= H_r(\omega) - jH_i(\omega) \end{aligned}$$

where $H_r(\omega)$ and $-H_i(\omega)$ are the real and imaginary parts of $H_d(\omega)$, respectively.

3.2. A SDP Formulation

The minimization problem in (9) can be re-formulated as

$$\begin{aligned} &\text{minimize} && \delta && \text{(12a)} \\ &\text{subject to} && W^2(\omega)|H(\omega) - H_r(\omega)|^2 \leq \delta && \text{(12b)} \\ & && d(z) \neq 0 \quad \text{for } |z| \geq 1 && \text{(12c)} \end{aligned}$$

where the upper bound δ is treated as an auxiliary variable in the design. Next we write

$$W^2(\omega)|H(\omega) - H_d(\omega)|^2 = \frac{W^2(\omega)}{|d(\omega)|^2}|b(\omega) - d(\omega)H_d(\omega)|^2$$

which suggests the following iterative scheme: At the kth iteration, one seeks to find $b_k(z)$ and $d_k(z)$ that solve the constrained minimization problem

$$\begin{aligned} &\text{minimize} && \delta && \text{(13a)} \\ &\text{subject to} && \frac{W^2(\omega)}{|d_{k-1}(\omega)|^2}|b(\omega) - d(\omega)H_d(\omega)|^2 \leq \delta && \text{(13b)} \\ & && d(z) \neq 0 \quad \text{for } |z| \geq 1 && \text{(13c)} \end{aligned}$$

By straightforward manipulations, it can be shown that the constraint in (13b) is equivalent to

$$\mathbf{\Gamma}(\mathbf{x},\, \omega) \succeq 0 \tag{14}$$

where

$$\mathbf{\Gamma}(\mathbf{x}, \omega) = \begin{bmatrix} \delta & a_1 & a_2 \\ a_1 & 1 & 0 \\ a_2 & 0 & 1 \end{bmatrix} \tag{15}$$

with

$$\begin{aligned} a_1 &= \mathbf{x}^T\mathbf{c}_k - H_{rw},\ a_2 = \mathbf{x}^T\mathbf{s}_k - H_{iw} \\ \mathbf{x} &= \begin{bmatrix} \mathbf{b} \\ \mathbf{d} \end{bmatrix},\ \mathbf{c}_k = \begin{bmatrix} \mathbf{c}_w \\ \mathbf{u}_w \end{bmatrix},\ \mathbf{s}_k = \begin{bmatrix} \mathbf{s}_w \\ -\mathbf{v}_w \end{bmatrix} \end{aligned} \tag{16}$$

$$\begin{aligned} w_k &= W(\omega)/|d_{k-1}(\omega)| \\ \mathbf{c}_w &= w_k\mathbf{c}(\omega),\ \mathbf{s}_w = w_k\mathbf{s}(\omega) \\ H_{rw} &= w_kH_r(\omega),\ H_{iw} = w_kH_i(\omega) \\ \mathbf{u}_w &= w_k[H_i(\omega)\mathbf{s}_1(\omega) - H_r(\omega)\mathbf{c}_1(\omega)] \\ \mathbf{v}_w &= w_k[H_i(\omega)\mathbf{c}_1(\omega) + H_r(\omega)\mathbf{s}_1(\omega)] \end{aligned}$$

The term $d_{k-1}(\omega)$ in (13b) is the denominator polynomial obtained from the preceding i.e. the $(k-1)$th iteration. Because of the stability constraint in (13c), $d_{k-1}(\omega)$ is nonzero for

$-\pi \leq \omega \leq \pi$, hence the modified weight w_k involved in matrix $\mathbf{\Gamma}$ is well defined. It should be pointed out that the iterative scheme set forth in (13) is a close resemblance to the Steiglitz-McBride (SM) scheme which has found useful in identification and adaptive filtering [24]. The difference between (13) and the SM scheme is that the SM scheme iterates a least-squares objective function while (13b) is an iterative constraint in a minimax problem.

Concerning the stability constraint in (13c), from Sec. 2.2 it follows that for a stable $d_{k-1}(z)$ there exists a $\mathbf{P}_{k-1} \succ \mathbf{0}$ that solves the Lyapunov equation

$$\mathbf{P}_{k-1} - \mathbf{D}_{k-1}^T \mathbf{P}_{k-1} \mathbf{D}_{k-1} = \mathbf{I} \tag{17}$$

where $\mathbf{I}$ is the $r \times r$ identify matrix and $\mathbf{D}_{k-1}$ is the $r \times r$ canonical matrix with $-\mathbf{d}_{k-1}$ in its first row. From Sec. 2.2, a natural stability constraint of $d(z)$ is

$$\begin{bmatrix} \mathbf{P}_{k-1}^{-1} & \mathbf{D} \\ \mathbf{D}^T & \mathbf{P}_{k-1} \end{bmatrix} \succ \mathbf{0}$$

or

$$\mathbf{Q}_k(\mathbf{x}) = \begin{bmatrix} \mathbf{P}_{k-1}^{-1} - \tau \mathbf{I} & \mathbf{D} \\ \mathbf{D}^T & \mathbf{P}_{k-1} - \tau \mathbf{I} \end{bmatrix} \succeq \mathbf{0} \tag{18}$$

where $\mathbf{D}$ is defined by (2) and $\tau > 0$ is a scalar that can be used to control the stability margin of the filter. Two remarks on constraint (18): (a) $\mathbf{Q}_k(\mathbf{x})$ in (18) depends on $\mathbf{D}$ (hence $\mathbf{x}$) affinely, therefore (18) is an LMI; (b) because of (17), the positive definite matrix $\mathbf{P}_{k-1}$ in (18) is "constrained". Consequently (17) is a sufficient (but not necessary) constraint for the stability. However, if the iterative algorithm converges, then the matrix sequence $\{\mathbf{D}_k\}$ converges as well. Since the existence of a $\mathbf{P}_{k-1} \succ \mathbf{0}$ in (17) is a necessary and sufficient condition for the stability of $d_{k-1}(z)$, the LMI constraint in (18) gets less and less restrictive as the algorithm proceeds. We consider this a desirable feature of the proposed algorithm.

The constrained optimization problem involved in the kth iteration can now be formulated as

$$\text{minimize} \quad \hat{\mathbf{c}}^T \hat{\mathbf{x}} \tag{19a}$$

$$\text{subject to} \quad \begin{bmatrix} \mathbf{\Gamma}_k(\mathbf{x}) & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_k(\mathbf{x}) \end{bmatrix} \succeq \mathbf{0} \tag{19b}$$

where $\hat{\mathbf{x}}$ is the augmented variable vector defined by

$$\hat{\mathbf{x}} = \begin{bmatrix} \delta \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \delta \\ \mathbf{b} \\ \mathbf{d} \end{bmatrix}, \; \hat{\mathbf{c}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{19c}$$

and $\mathbf{\Gamma}_k(\mathbf{x}) \succeq \mathbf{0}$ is a discrete implementation of (14) on a set of frequencies $\{\omega_i, \; 1 \leq i \leq m\}$ on the region of interest:

$$\mathbf{\Gamma}_k(\mathbf{x}) = \text{diag}\,\{\mathbf{\Gamma}(\mathbf{x}, \omega_1), \cdots, \mathbf{\Gamma}(\mathbf{x}, \omega_m)\} \tag{19d}$$

Note that the stability constraint $\mathbf{Q}_k(\mathbf{x}) \succeq \mathbf{0}$ needs no discretization. Since both $\mathbf{\Gamma}_k(\mathbf{x})$ and $\mathbf{Q}_k(\mathbf{x})$ depend on $\mathbf{x}$ affinely, (19) is a SDP problem as defined in (1). Moreover, since an SDP problem is a convex programming problem, any solution of (19) (if exists) is a global solution.

3.3. An Iterative SDP Algorithm

Given a desired frequency response $H_d(\omega)$, a weighting function $W(\omega)$, and filter order (n, r), one chooses a convenient starting point $\mathbf{x}_0 = [\mathbf{b}_0^T \; \mathbf{d}_0^T]$ with $\mathbf{d}_0 = \mathbf{0}$ and $\mathbf{b}_0$ obtained by designing an FIR filter using a routine design algorithm. One then solves the SDP problem in (19) for $k = 1$ and compare the solution $\mathbf{x}_k$ with $\mathbf{x}_{k-1}$. If $||\mathbf{x}_k - \mathbf{x}_{k-1}||$ is less than a prescribed tolerance ε then $\mathbf{x}_k$ is deemed as a solution of the design problem. Otherwise the algorithm proceeds by solving (19) for $k = 2$, etc. This algorithm is illustrated in the next section by presenting a design example and some comparison results.

4. A DESIGN EXAMPLE

A stable lowpass IIR filter of order 12, which is optimal in the minimax sense as specified in (9), was designed to meet the following specifications: passband edge $\omega_p = 0.25$, stopband edge $\omega_a = 0.3$, maximum passband error ≤ 0.02, minimum stopband attenuation $\geq$ 34 dB, group delay in passband = 9 samples with maximum ripple less than 1 sample. With $n = 12$, $r = 6$, $W(\omega) \equiv 1$, $\tau = 10^{-4}$, and $\varepsilon = 0.005$, it took the proposed algorithm 50 iterations to reach a solution. The poles and zeros of the filter obtained are given in Table I, and $b_0 = 0.00789947$. The largest magnitude of the poles is 0.9440. The performance of the filter designed is summarized in Table II and Fig. 1. In the design, the constraint $\Gamma_0(x, \omega) \succeq 0$ was discretized over a set of 240 equally-spaced frequencies on $[0, \omega_p] \bigcup [\omega_a, 0.5]$.

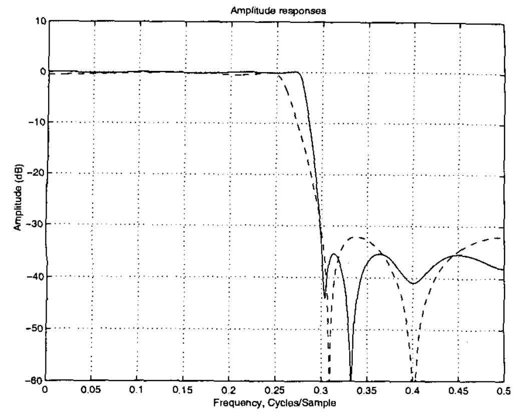
Table I Zeros and poles of the transfer function

Zeros	Poles
-2.12347973	
-1.22600378	$-0.15960464 \pm 0.93037014j$
$1.49482238 \pm 0.55741991j$	$-0.03719150 \pm 0.55679505j$
$0.75350472 \pm 1.37716837j$	$0.24717453 \pm 0.18656749j$
$-0.89300316 \pm 0.65496710j$	Plus another 6 poles
$-0.32277491 \pm 0.93626367j$	at the origin
$-0.49091195 \pm 0.86511412j$	

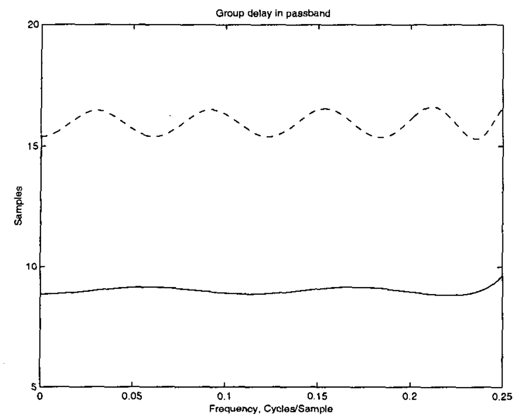
Table II Performance comparisons

Filter	Proposed IIR	Deczky[5]
Maximum passband error in magnitude	0.0171	0.0549
Minimum stopband attenuation (dB)	34.7763	31.5034
Maximum ripple of group delay in passband (samples)	0.8320	1.3219

The above design was compared to a well-known design by Deczky as Example 1 in [5]. The Deczky filter has the same order n, ω_p, and ω_a as our filter. Its performance is illustrated in Fig. 1 and Table II. It is observed that our filter offers improved performance with a lower and flatter group delay profile in the passband. It is also noted that our filter has only 6 non-origin poles which leads to reduced computational complexity for implementation.



(a)



(b)

Figure 1: Frequency responses of the IIR filter (solid lines) and Deczky's filter (dashed lines): (a) Amplitude responses, (b) Group delay in passband.

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