DESIGN OF NONRECURSIVE 2-D DIGITAL FILTERS USING SEMIDEFINITE PROGRAMMING

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ABSTRACT

The design of linear-phase, nonrecursive, two-dimensional (2-D) digital filters in the minimax sense is considered and it is shown that such a design problem can be formulated as a *semidefinite programming* (SDP) problem. An SDP-based algorithm is proposed and is then applied to the design of several nonseparable 2-D FIR filters, including circular symmetric and diamond-shaped filters. Experimental results obtained show that our algorithm compares favorably relative to other existing algorithms in terms of design efficiency.

1. INTRODUCTION

Semidefinite programming (SDP) is a relatively new optimization methodology that has attracted a great deal of research interest in the past several years [1]–[6]. There are several ways an SDP problem can be formulated. For the purpose of filter design, the following formulation turns out to be of convenience:

$$\text{minimize} \quad \mathbf{c}^T\mathbf{x} \tag{1a}$$

$$\text{subject to} \quad \mathbf{F}(\mathbf{x}) \succeq \mathbf{0} \tag{1b}$$

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + \sum_{i=1}^{n} x_i\mathbf{F}_i \tag{1c}$$

In (1), $\mathbf{x} = [x_1 \ \cdots \ x_n]^T$ is the variable vector, $\mathbf{c} \in R^{n\times 1}$, $\mathbf{F}_i \in R^{n\times n}$ for $0 \le i \le n$ are constant matrices with $\mathbf{F}_i$ symmetric, and $\mathbf{F}(\mathbf{x}) \succeq \mathbf{0}$ denotes that $\mathbf{F}(\mathbf{x})$ is positive semidefinite. Note that the constraint matrix $\mathbf{F}(\mathbf{x})$ is *affine* with respect to (w.r.t.) $\mathbf{x}$. SDP includes both linear programming and convex quadratic programming as special cases. More importantly, many interior-point methods, which have proven efficient for linear programming, have recently been extended to SDP [5]-[6].

In this paper, we consider the design of linear-phase, nonrecursive, two-dimensional (2-D) digital filters in the minimax sense and show that such a design problem can be formulated as an SDP problem. It is well-known that minimax designs of 2-D FIR filters can be accomplished with various methods as shown, for example, in [7]-[11]. However, computer simulations have demonstrated that SDP can yield the required design with significantly improved computational efficiency.

For the sake of completeness, a brief review of several key elements of the projective method, which was proposed in [3][11], is included.

2. PROBLEM FORMULATION

Let the transfer function of a 2-D FIR digital filter be

$$H(z_1, z_2) = \sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1} h_{ij}z^{-i}z^{-j} = \mathbf{z}_1^T\hat{\mathbf{H}}\mathbf{z}_2 \tag{2}$$

where N_1 and N_2 are odd integers, $\mathbf{z}_1 = [1 \ \ z_1^{-1} \ \cdots \ z_1^{-(N_1-1)}]^T$, $\mathbf{z}_2 = [1 \ \ z_2^{-1} \ \cdots \ z_2^{-(N_2-1)}]^T$, and $\hat{\mathbf{H}} \in R^{N_1\times N_2}$. To derive a compact expression for the transfer function of a linear phase 2-D FIR filter, we partition $\hat{\mathbf{H}}$ as

$$\hat{\mathbf{H}} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{h}_{12} & \mathbf{H}_{13} \\ \mathbf{h}_{21}^T & h_{22} & \mathbf{h}_{23}^T \\ \mathbf{H}_{31} & \mathbf{h}_{32} & \mathbf{H}_{33} \end{bmatrix}$$

where $\mathbf{H}_{11}, \mathbf{H}_{13}, \mathbf{H}_{31}, \mathbf{H}_{33} \in R^{n_1\times n_2}, \mathbf{h}_{12}, \mathbf{h}_{32} \in R^{n_1\times 1}$, $\mathbf{h}_{21}, \mathbf{h}_{23} \in R^{n_2\times 1}$, $h_{22} \in R$, $n_1 = (N_1 - 1)/2$, and $n_2 = (N_2 - 1)/2$. Now if

$$\begin{aligned} \mathbf{H}_{13} &= \text{flipud}\,(\mathbf{H}_{33}),\ \mathbf{H}_{31} = \text{fliplr}\,(\mathbf{H}_{33}), \\ \mathbf{H}_{11} &= \text{flipud}\,(\text{fliplr}\,(\mathbf{H}_{33})), \\ \mathbf{h}_{12} &= \text{flipud}\,(\mathbf{h}_{32}),\ \text{and}\ \mathbf{h}_{21}^T = \text{fliplr}\,(\mathbf{h}_{23}^T) \end{aligned}$$

where flipud and fliplr represent the operations of fliping a matrix upside down and from left to right, respectively, then the filter characterized by $H(z_1, z_2)$ has linear phase response with group delay (n_1T_1, n_2T_2), and the frequency response of the filter is given by

$$H(e^{j\omega_1}, e^{j\omega_2}) = e^{-j(n_1\omega_1+n_2\omega_2)}\mathbf{c}_1^T(\omega_1)\mathbf{H}\mathbf{c}_2(\omega_2)$$

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where $\mathbf{c}_i(\omega_i) = [1 \;\; \cos\omega_i \;\; \cdots \;\; \cos n_i\omega_i]^T$ for $i = 1,\ 2$, and

$$\mathbf{H} = \begin{bmatrix} h_{22} & 2\mathbf{h}_{23}^T \\ 2\mathbf{h}_{32} & 4\mathbf{H}_{33} \end{bmatrix}$$

Consequently, the minimax design of a linear-phase 2-D FIR filter is the solution of the optimization problem

$$\underset{\mathbf{H}}{\text{minimize}}[\underset{(\omega_1,\,\omega_2)\in\Omega}{\text{maximize}}\, e(\omega_1,\,\omega_2,\,\mathbf{H})] \tag{3}$$

where

$$e(\omega_1,\,\omega_2,\,\mathbf{H}) = W(\omega_1,\,\omega_2)|\mathbf{c}_1^T(\omega_1)\mathbf{H}\mathbf{c}_2(\omega_2) - H_d(\omega_1,\,\omega_2)|$$

Ω denotes the frequency region of interest, $W(\omega_1,\,\omega_2) \geq 0$ is a weighting function over region Ω, and $H_d(\omega_1,\,\omega_2)$ is the desired amplitude response.

Evidently, the optimization problem in (3) is equivalent to

$$\text{minimize} \quad \delta \tag{4a}$$

$$\text{subject to} \quad e(\omega_1,\,\omega_2,\,\mathbf{H}) \leq \delta \text{ for } (\omega_1,\,\omega_2) \in \Omega \tag{4b}$$

Furthermore, the constraint in (4b) can be expressed as

$$\delta - e(\omega_1,\,\omega_2,\,\mathbf{H}) \geq 0 \tag{5}$$

for $(\omega_1,\,\omega_2) \in \Omega$. The inequality in (5) implies that

$$\delta \geq 0 \qquad \text{for } (\omega_1,\,\omega_2) \in \Omega \tag{6}$$

It is known from linear algebra that the inequalities in (5) and (6) hold if and only if

$$\mathbf{\Delta}(\omega_1,\,\omega_2) \succeq \mathbf{0} \qquad \text{for } (\omega_1,\,\omega_2) \in \Omega \tag{7a}$$

where

$$\mathbf{\Delta}(\omega_1,\,\omega_2) = \begin{bmatrix} \delta & e(\omega_1,\,\omega_2,\mathbf{H}) \\ e(\omega_1,\,\omega_2,\mathbf{H}) & 1 \end{bmatrix} \tag{7b}$$

It is important to note that matrix $\mathbf{\Delta}(\omega_1,\,\omega_2)$ in (7b) is *affine* w.r.t. design matrix $\mathbf{H}$ and the scalar auxiliary variable δ. A discretized version of the positive semidefinite condition in (7) is given by

$$\mathbf{D}(\mathbf{x}) = \operatorname{diag}\{\mathbf{\Delta}(\omega_1^{(1)},\,\omega_2^{(1)}),\,\ldots,\,\mathbf{\Delta}(\omega_1^{(M)},\,\omega_2^{(M)})\} \succeq \mathbf{0} \tag{8}$$

where the set $\{(\omega_1^{(i)},\,\omega_2^{(i)}),\ 1 \leq i \leq M\}$ forms a sufficiently dense grid in region Ω, and $\mathbf{x} = [x_1 \; x_2 \; \cdots \;]^T$ is vector of dimension $1 + (n_1+1)(n_2+1)$ where $x_1 = \delta$ and $x_2,\ x_3\ \cdots$ are the entries of $\mathbf{H}$. Taking the above analysis into account, a discretized version of the optimization problem in (4) can be formulated as

$$\text{minimize} \quad \mathbf{c}^T\mathbf{x} \tag{9a}$$

$$\text{subject to} \quad \mathbf{D}(\mathbf{x}) \succeq \mathbf{0} \tag{9b}$$

where $\mathbf{c} = [1 \;\; 0 \;\; \cdots \;\; 0]^T$ and $\mathbf{D}(\mathbf{x})$ is defined by (8) and (7b). Since matrix $\mathbf{D}(\mathbf{x})$ in (9b) is affine w.r.t. $\mathbf{x}$, (9) is an SDP problem.

3. PROJECTIVE METHOD FOR SDP

In this section, we give a brief review of the projective method proposed in [3][11], which has proven effective for SDP problems. To this end we define the set of all positive semidefinite matrices of size $n \times n$ by $\mathcal{K}$ and define the set of all positive definite matrices of size $n \times n$ by $\mathcal{S}$. A set $\mathcal{C}$ is said to be a *cone* if $\mathbf{x} \in \mathcal{C}$ implies $\alpha\mathbf{x} \in \mathcal{C}$ for all $\alpha > 0$. A set $\mathcal{C}$ is said to be a *convex cone* if $\mathcal{C}$ is a cone and is convex. Evidently, both $\mathcal{K}$ and $\mathcal{S}$ are convex cones and $\mathcal{S}$ can be viewed as the interior of $\mathcal{K}$. Given a positive definite matrix $\mathbf{P}$, an inner product can be introduced in $\mathcal{S}$ as

$$< \mathbf{X},\, \mathbf{Y} >_P = \operatorname{tr}(\mathbf{PXPY}) \tag{10}$$

which induces the norm

$$||\mathbf{X}||_P = [\operatorname{tr}(\mathbf{PXPX})]^{1/2}$$

Note that if $\mathbf{P}$ is the identity matrix, then the above norm is reduced to the Frobenius norm

$$||\mathbf{X}||_I = (\operatorname{tr}\mathbf{X}^2)^{1/2} = ||\mathbf{X}||_{Fro}$$

An important concept involved in the development of the projective method is the Dikin ellipsoid [1] which is defined, for a fixed positive definite $\mathbf{X}$, as the set

$$D(\mathbf{X}) = \{\mathbf{Y} : ||\mathbf{Y} - \mathbf{X}||_{X^{-1}} < 1\} \tag{11}$$

It can be shown that the Dikin ellipsoid can be characterized as

$$D(\mathbf{X}) = \{Y : ||\mathbf{X}^{-\frac{1}{2}}\mathbf{Y}\mathbf{X}^{-\frac{1}{2}} - \mathbf{I}||_{Fro} < 1\}$$

and that for a positive definite $\mathbf{X}$, $D(\mathbf{X})$ is always contained in cone $\mathcal{S}$. Therefore, $D(\mathbf{X})$ provides a region around $\mathbf{X}$ in which a search for a better point can be carried but without losing positive definiteness.

With the inner product defined in (10), one can consider the orthogonal projection of a positive definite $\mathbf{X}$ in $\mathcal{S}$ onto a subspace $\mathcal{E}$. In the context of SDP, $\mathcal{E}$ is the range of the linear map $\mathcal{F}$ associated with the LMI constraint in 1(c), i.e.,

$$\mathcal{F}\mathbf{x} = \sum_{i=1}^{n} x_i\mathbf{F}_i \tag{12}$$

and subset $\mathcal{E}$ is characterized by

$$\mathcal{E} = \{\mathbf{Y} : \mathbf{Y} = \mathcal{F}\mathbf{x},\, \mathbf{x} \in R^n\} \tag{13}$$

The orthogonal projection of a given positive definite $\mathbf{X}$ onto subspace $\mathcal{E}$ with respect to the metric $<,>_P$ can be defined as the unique solution of the least-squares problem

$$\underset{\mathbf{Y}\in\mathcal{E}}{\text{minimize}}\, ||\mathbf{Y} - \mathbf{X}||_P = \underset{\mathbf{x}\in R^n}{\text{minimize}}\, ||\mathcal{F}\mathbf{x} - \mathbf{X}||$$

If we denote this orthogonal projection by $\mathbf{X}^+$, then $\mathbf{X}^+$ can be characterized by the optimality condition

$$< \mathbf{X}^+ - \mathbf{X},\ \mathbf{Y} >_P = 0 \qquad \text{for } \mathbf{Y} \in \mathcal{E} \tag{14}$$

which is equivalent to

$$< \mathbf{P}(\mathbf{X}^+ - \mathbf{X})\mathbf{P},\ \mathbf{Y} >_{Fro} = 0 \qquad \text{for } \mathbf{Y} \in \mathcal{E} \tag{15}$$

Like any interior-point optimization method, the projective method starts at a *strictly feasible* initial point $\mathbf{x}_0$ in the sense that matrix $\mathbf{F}(\mathbf{x}_0)$ in (1) is positive definite. For the sake of simplicity, we assume that the SDP problem at hand does have a strictly feasible initial point and that the linear objective function in (1a) has a finite lower bound.

As a preliminary step of the projective method, the problem in (1) is reformulated as the *homogeneous* problem

$$\text{minimize} \quad f(\mathbf{x}) = \frac{\tilde{\mathbf{c}}^T\tilde{\mathbf{x}}}{\tilde{\mathbf{d}}^T\tilde{\mathbf{x}}} \tag{16a}$$

$$\text{subject to:} \quad \tilde{\mathcal{F}}\tilde{\mathbf{x}} \succeq \mathbf{0} \tag{16b}$$

$$\tilde{\mathbf{d}}^T\tilde{\mathbf{x}} \neq 0 \tag{16c}$$

where

$$\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \tau \end{bmatrix}, \tilde{\mathcal{F}}\tilde{\mathbf{x}} = \begin{bmatrix} \mathcal{F}\mathbf{x} + \mathbf{F}_0 & \mathbf{0} \\ \mathbf{0} & \tau \end{bmatrix}, \tilde{\mathbf{c}} = \begin{bmatrix} c \\ 0 \end{bmatrix}, \tilde{\mathbf{d}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The projective method can now be described in terms of the following algorithm.

Projective Algorithm for the SDP Problem in (16)

Step 1 Input a strictly feasible initial point $\tilde{\mathbf{x}}_0$ and tolerance ε and compute $\mathbf{X}_0 = \tilde{\mathcal{F}}\tilde{\mathbf{x}}_0$.
Set $\mathbf{x}_0^* = \tilde{\mathbf{x}}_0$, and evaluate the objective function at $\mathbf{x}_0^*$ as f_0^*.

Step 2 Compute the orthogonal projection $\mathbf{X}_k^+ = \tilde{\mathcal{F}}\tilde{\mathbf{x}}_k$ of $\mathbf{X}_k$ onto $\mathcal{E}$ w.r.t. metric $< \cdot, \cdot >_{X_k^{-1}}$ and check its positive definiteness.
If $\mathbf{X}_k^+ \succ \mathbf{0}$, go to Step 3; otherwise, set $f_k^* = f_{k-1}^*$, $\mathbf{Y}_k = \mathbf{X}_k^+ - \mathbf{X}_k$, and go to Step 4.

Step 3 Reduce $f(\mathbf{x})$ in (16a) until $||\mathbf{X}_k - \mathbf{X}_k^+(f)||_{X_k^{-1}} \geq 0.99$ subject to $\mathbf{X}_k^+(f) \succ \mathbf{0}$, where $\mathbf{X}_k^+(f)$ denotes the orthogonal projection of $\mathbf{X}_k$ onto subspace $\mathcal{E}(f) = \{\mathbf{X} = \tilde{\mathcal{F}}\mathbf{x} :\ (\tilde{\mathbf{c}} - f\tilde{\mathbf{d}})^T\mathbf{x} = 0\}$. Denote the resulting point by $\mathbf{x}_k^*$ and $f_k^* = f(\mathbf{x}_k^*)$, and set $\mathbf{Y}_k = \mathbf{X}_k^+(f_k^*) - \mathbf{X}_k$.

Step 4 If $f_{k-1}^* - f_k^* \leq \varepsilon$, then stop and output $\mathbf{x}_k^*$ as the solution; otherwise generate $\mathbf{X}_{k+1}$ using

$$\mathbf{X}_{k+1}^{-1} = \mathbf{X}_k^{-1} - \gamma_k\mathbf{X}_k^{-1}\mathbf{Y}_k\mathbf{X}_k^{-1}$$

where γ_k is selected such that $\mathbf{X}_{k+1}^{-1} \succ \mathbf{0}$ and $\det(\mathbf{X}_{k+1}^{-1}) \geq \beta \det(\mathbf{X}_k^{-1})$ for some fixed $\beta > 1$.
Repeat from Step 2.

4. ALGORITHM IMPLEMENTATION AND DESIGN EXAMPLES

Presently, there is only a limited number of software packages available that can be used to solve the SDP problem as formulated in (9). One of them is the LMI Control Toolbox from MathWorks Inc. [12]. The toolbox works with MATLAB and is aimed primarily at solving design problems arising in control engineering by using linear matrix inequality (LMI) techniques [4]. The LMI toolbox includes a command named `mincx` which implements the projective method described in Section 3, and can be used to minimize a linear objective function subject to LMI constraints. Evidently, this command is directly applicable to the minimization problem in (9).

The proposed SDP-based algorithm was used to design several nonseparable 2-D FIR digital filters, including circular symmetric and diamond-shaped lowpass, highpass, bandpass, and bandstop filters. As examples, Fig. 1 depicts the amplitude response of a 25×25 linear-phase, circular symmetric, lowpass FIR filter with $\omega_p = 0.425\pi$, $\omega_a = 0.575\pi$, and maximum ripple in both passband and stopband of 0.145. The number of flops required by the algorithm was 299K. Fig. 2 illustrates the amplitude response of a 29×29 linear-phase, diamond-shaped lowpass FIR filter with $\omega_p = 0.8\pi$, $\omega_a = \pi$, and maximum ripple in both passband and stopband of 0.0056. The number of flops used in this case was 461K.

The proposed method was compared with one of the existing methods for minimax design of 2-D linear-phase FIR digital filters, namely, the method described in [10]. Both methods were applied to design several circular symmetric and diamond-shaped lowpass filters. The design results are shown in Tables 1 and 2 for the circular symmetric and diamond-shaped filters, respectively. It is observed from the tables that the proposed method achieve the designs with considerably reduced computational complexity.

Table 1: Comparisons of the proposed method with the method in [10]: Circularly symmetric lowpass filters

N	Maximum ripple	Kflops required by the proposed method	Kflops required by the method in [10]
5	0.3138	198	1,567
9	0.1629	204	3,117
13	0.0841	223	5,901
17	0.0474	229	8,664
21	0.0249	234	12,074
25	0.0145	299	18,195

Table 2: Comparisons of the proposed method with the method in [10]: Diamond-shaped lowpass filters

N	Maximum ripple	Kflops required by the proposed method	Kflops required by the method in [10]
9	0.1633	148	1,044
13	0.0875	188	2,892
17	0.0516	259	7,077
21	0.0292	328	11,220
25	0.0127	391	16,094
29	0.0056	461	23,831

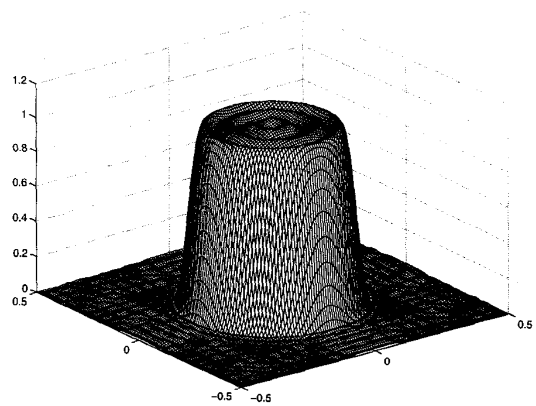

Figure 1: Amplitude response of a 25×25 circular symmetric lowpass FIR filter.

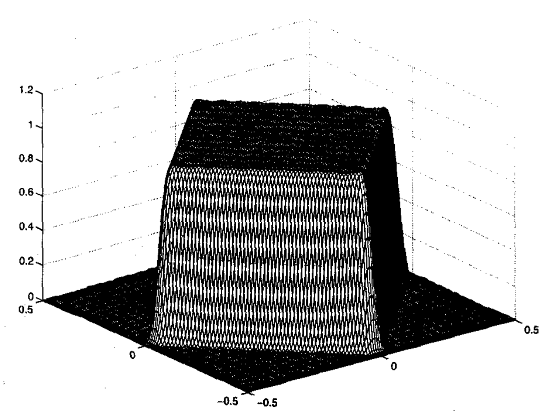

Figure 2: Amplitude response of a 29×29 diamond-shaped lowpass FIR filter.

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