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Semidefinite Programming: A Versatile Tool for Analysis and Design of Digital Filters

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Abstract

Semidefinite programming (SDP) is a relatively new methodology for constrained optimization of a linear matrix-variable function subject to linear equality and inequality constraints as well as linear positive-semidefinite constraints. The primary purpose of this paper is to demonstrate that many digital-filter analysis and design problems can be formulated as SDP problems and, therefore, they can be solved effectively using powerful SDP solvers.

1 Introduction

Semidefinite programming (SDP) is a relatively new methodology for constrained optimization of linear matrix-variable functions subject to linear equality and inequality constraints as well as linear positive-semidefinite constraints. SDP includes the important linear programming (LP) and convex quadratic programming (QP) problems as its special cases. More importantly, many interior-point optimization algorithms that have proven efficient for LP and QP problems have recently been extended to SDP [1]-[4].

The primary purpose of this paper is to demonstrate that many digital-filter analysis and design problems can be formulated as SDP problems and, therefore, they can be solved effectively using powerful SDP solvers [2][5]. The analysis and design problems to be discussed in this paper include minimum-norm realization of two-dimensional (2-D) digital filter, minimax design of one-dimensional (1-D) and 2-D digital filters, and equiripple-passbands and least-squares-stopbands design of 1-D digital filters.

There are several ways a SDP problem can be formulated, and the one which turns out to be convenient for filter design purposes is given as

$$\text{minimize} \quad c^T x \tag{1a}$$

$$\text{subject to} \quad F(x) \succeq 0 \tag{1b}$$

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \tag{1c}$$

In (1), $x \in R^{n\times 1}$ is the variable, $c \in R^{n\times 1}$, $F_i \in R^{n\times n}$ $(i = 0, 1, \ldots, n)$ are given constant matrices with F_i symmetric, and $F(x) \succeq 0$ denotes that $F(x)$ is positive semidefinite. Note that the constraint matrix $F(x)$ in (1) is *affine* with respect to x. SDP includes both linear and quadratic programming (QP) as special cases, and it represents a broad and important class of convex programming problems. More important, many interior-point methods which have proven efficient for linear programming, have recently been generalized to SDP [2][3].

2 Design of Nonlinear Phase FIR Filters

We consider the problem of designing a nonlinear-phase, FIR digital filter that approximates a desired frequency response (both magnitude and phase responses) in the passbands in the Chebyshev sense, and approximates desired (zero) magnitude response in the stopbands in the least-squares sense. Consideration of such designs has been justified by many, see for example Adams [20]. Design of nonlinear-phase equiripple FIR filters is also considered in the paper. The term "nonlinear-phase design" is referred to a filter design in which the phase response in stopbands and transition bands are *not* required to be linear. As expected, the phase-response relaxation in the stopbands and transition bands from strict linearity has been found useful in enhancing the performance of the filter designed [21][22].

Consider the transfer function of an N-tap FIR filter

$$H(z) = \sum_{k=0}^{N-1} h_k z^{-k} \tag{2}$$

and denote its frequency response by

$$H(\omega) = \sum_{k=0}^{N-1} h_k e^{-jk\omega} = h^T[c(\omega) - js(\omega)] \quad (3)$$

where $h = [h_0 \ \cdots \ h_{N-1}]^T$, $c(\omega) = [1 \ \cos\omega \ \cdots \cos(N-1)\omega]^T$, and $s(\omega) = [0 \ \sin\omega \ \cdots \sin(N-1)\omega]^T$. Here we do *not* assume any symmetry in h. Let $H_d(\omega)$ be the desired frequency response, which is usually complex-valued. In an equiripple design, one seeks to find coefficient vector h that solves the optimization problem

$$\underset{h}{\text{minimize}} \ \underset{\omega\in\Omega}{\text{maximize}} \, W(\omega)|H(\omega) - H_d(\omega)| \quad (4)$$

where Ω is a compact region on $[-\pi, \pi]$. The minimax problem in (4) can be reformulated as

$$\underset{h}{\text{minimize}} \quad \delta \quad (5a)$$

$$\text{subject to } W^2(\omega)|H(\omega) - H_d(\omega)|^2 \le \delta \text{ for } \omega \in \Omega \quad (5b)$$

Now let

$$H_d(\omega) = H_r(\omega) - jH_i(\omega)$$

with $H_r(\omega)$ and $H_i(\omega)$ real, and use (3) to write the left-hand side of the constraint in (5b) as

$$W^2(\omega)|H(\omega) - H_d(\omega)|^2 = \alpha_1^2(\omega) + \alpha_2^2(\omega) \quad (6)$$

where

$$\begin{aligned}
\alpha_1(\omega) &= h^T c_w(\omega) - H_{rw}(\omega)\\
\alpha_2(\omega) &= h^T s_w(\omega) - H_{iw}(\omega)\\
c_w(\omega) &= W(\omega)c(\omega)\\
s_w(\omega) &= W(\omega)s(\omega)\\
H_{rw}(\omega) &= W(\omega)H_r(\omega)\\
H_{iw}(\omega) &= W(\omega)H_i(\omega)
\end{aligned}$$

Constraint (5b) then becomes

$$\delta - \alpha_1^2(\omega) - \alpha_2^2(\omega) \ge 0 \qquad \omega \in \Omega \quad (7)$$

It can be shown that (7) is equivalent to

$$\Delta(\omega) = \begin{bmatrix} \delta & \alpha_1(\omega) & \alpha_2(\omega) \\ \alpha_1(\omega) & 1 & 0 \\ \alpha_2(\omega) & 0 & 1 \end{bmatrix} \succeq 0 \qquad \omega \in \Omega \quad (8)$$

If we denote $x = [\delta \ \ h^T]^T$, then the linear dependence of $\alpha_1(\omega)$ and $\alpha_2(\omega)$ on h implies that matrix $\Delta(\omega)$ in (8) is affine w.r.t. x. Therefore, if $\{\omega_i, \ i = 1, \ \ldots, \ M\} \subseteq \Omega$ is a set of grid points that are sufficiently dense in Ω, then a discretized version of (5) can be described as

$$\text{minimize} \quad c^T x \quad (9a)$$

$$\text{subject to} \quad F(x) \succeq 0 \quad (9b)$$

where $c = [1 \ 0 \ \cdots \ 0]^T$, and $F(x) = \text{diag}\{\Delta(\omega_1), \ \Delta(\omega_2), \ \cdots, \ \Delta(\omega_M)\}$. Obviously, $F(x)$ in (9b) is affine w.r.t. x, hence (9) is a SDP problem. Note that $F(x)$ is a *tridiagonal matrix* of size $3M \times 3M$, which becomes increasingly sparse with M.

In an EPPCLSS type design, one seeks to find h which minimizes the weighted L_2 error function

$$e(h) = \int_\Omega W(\omega)|H(\omega) - H_d(\omega)|^2 \, d\omega \quad (10a)$$

subject to constraints

$$\begin{aligned}
|H(\omega) - H_d(\omega)|^2 &\le \delta_p \qquad \omega \in \Omega_p \quad (10b)\\
|H(\omega)|^2 &\le \delta_a \qquad \omega \in \Omega_a \quad (10c)
\end{aligned}$$

where Ω_p and Ω_a denote the unions of passbands and stopbands, respectively. Simple manipulations of the integral in (10a) yields

$$e(h) = h^T P h - 2h^T q + c_0 \quad (11)$$

where

$$P = \int_\Omega W(\omega)[c(\omega) \ \ s(\omega)]\,[c(\omega) \ \ s(\omega)]^T \, d\omega$$

is positive definite for a compact Ω,

$$q = \int_\Omega W(\omega)[H_r(\omega)c(\omega) + H_i(\omega)s(\omega)] \, d\omega$$

and

$$c_0 = \int_\Omega |H_d(\omega)|^2 \, d\omega$$

Let $P^{1/2}$ be the symmetric square root of P, i.e., $P^{T/2} = P^{1/2}$ and $P^{1/2}P^{1/2} = P$. Then (11) can be written as

$$e(h) = ||P^{1/2}h - P^{-1/2}q||^2 - (||P^{-1/2}q||^2 - c_0)$$

Hence

$$e(h) \le \delta$$

is equivalent to

$$\delta + c_1 - ||P^{1/2}h - P^{-1/2}q||^2 \ge 0 \quad (12)$$

where $c_1 = ||P^{-1/2}q||^2 - c_0$. It can be shown that (12) holds if and only if

$$\Gamma_0 = \begin{bmatrix} \delta + c_1 & h^T P^{1/2} - q^T P^{-1/2} \\ P^{1/2}h - P^{-1/2}q & I_N \end{bmatrix} \succeq 0 \tag{13}$$

where I_N is the $N \times N$ identity matrix. Note that matrix Γ_0 in (13) is *affine* w.r.t. δ and h, and does not depend on ω. Similar to the way we treat constraint (5b), it can be shown that constraint in (10b) is equivalent to

$$\Gamma(\omega) = \begin{bmatrix} \delta_p & \beta_1(\omega) & \beta_2(\omega) \\ \beta_1(\omega) & 1 & 0 \\ \beta_2(\omega) & 0 & 1 \end{bmatrix} \succeq 0 \qquad \omega \in \Omega_p \tag{14}$$

where $\beta_1(\omega) = h^T c(\omega) - H_r(\omega)$ and $\beta_2 = h^T s(\omega) - H_i(\omega)$, and that constraint in (10c) is equivalent to

$$\Phi(\omega) = \begin{bmatrix} \delta_a & \gamma_1(\omega) & \gamma_2(\omega) \\ \gamma_1(\omega) & 1 & 0 \\ \gamma_2(\omega) & 0 & 1 \end{bmatrix} \succeq 0 \qquad \omega \in \Omega_a \tag{15}$$

where $\gamma_1(\omega) = h^T c(\omega)$ and $\gamma_2(\omega) = h^T s(\omega)$. Now if $\{\omega_i^{(p)},\ i = 1, \ldots, M_p\} \subseteq \Omega_p$ $\{\omega_i^{(a)},\ i = 1, \ldots, M_a\} \subseteq \Omega_a$ are the sets of grid points in the passbands and stopbands, respectively, on which the constraints (14) and (15) are imposed, then a discretized version of minimizing (10a) subject to (10b) and (10c) can be formulated as

$$\text{minimize} \quad c^T x \tag{16a}$$

$$\text{subject to} \quad F(x) \succeq 0 \tag{16b}$$

where $x = [\delta\ \delta_p\ \delta_a\ h^T]^T$, $c = [1\ w_p\ w_a\ 0\ \cdots\ 0]^T$ with scalar weights w_p and w_a, and $F(x) = \text{diag}\ \{\Gamma_0,\ \Gamma(\omega_1^{(p)}),\ \cdots,\ \Gamma(\omega_{M_p}^{(p)}),\ \Phi(\omega_1^{(a)}),\ \cdots,\ \Phi(\omega_{M_a}^{(a)})\}$. Clearly, $F(x)$ in (16b) is affine w.r.t. x, therefore (16) is a SDP problem.

Presently, there is only a limited number of software packages available that can be used to solve the SDP problem as formulated in (9). One of them is the LMI Control Toolbox from MathWorks Inc. [5]. The toolbox works with MATLAB and is aimed primarily at solving design problems arising from control engineering by using linear matrix inequality (LMI) techniques [6]. The LMI toolbox includes a command named `mincx` which implements the projective method proposed in [2][7] for solving SDP problems. For the sake of completeness, the next section offers a brief review of several key elements of the projective method.

Figure 1 shows the amplitude response of a 91-tap equiripple FIR filters designed by the proposed method to approximate a lowpass frequency response with normalized $\omega_p = 0.2375$, $\omega_a = 0.2625$, and group delay = 40. The LMI Control Toolbox was used to perform the design with 36 iterations and 190 Kflops.

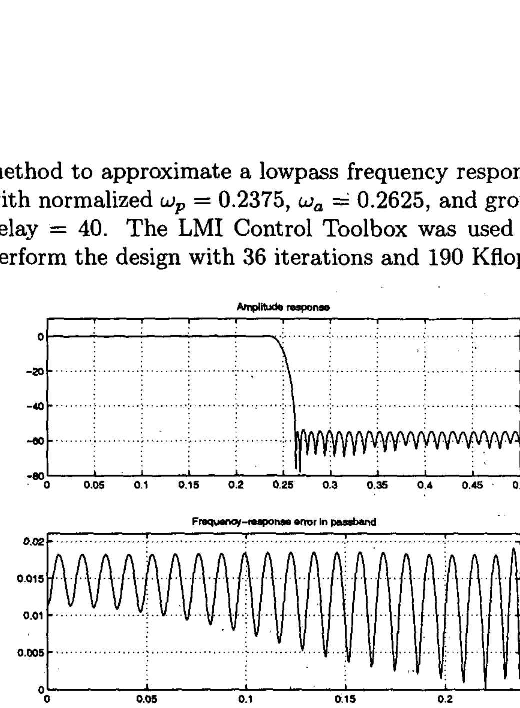

Figure 1

3 Minimum-Norm Realization of 2-D Recursive Filters

Minimum-norm realizations (MNR) of digital filters have been known to possess several desirable properties [8][9]. These include the freedom of overflow limit cycles, low parameter sensitivity, and low roundoff output noise power. For 1-D recursive digital filters, the (spectral) norm of the system matrix of a filter's MNR is equal to the largest magnitude of the filter's poles and obtaining a MNR of a stable recursive filter is a rather simple linear algebraic problem [8]. In the 2-D case, all feasible state-space models, which have been employed for finite-wordlength-effect (FWE) analysis to date, are local models. And the effort to establish an explicit relation of the minimum norm of the system matrix under local state-variable transformations to the system's "poles" is considerably complicated by the fact that the "poles" of a recursive 2-D filter are no longer isolated points but composed of continuous and often unbounded algebraic curves [9][10].

In [11], the problem of obtaining a MNR of a given 2-D recursive filter was formulated as an unconstrained optimization problem, and was tackled using quasi-Newton algorithms [12][13]. A problem with the approach in [11] is its exceedingly high computational complexity even for 2-D filters of moderate sizes. In what follows, we describe an alternative optimization-based method to obtain MNRs for 2-D recursive filters.

The problem at hand is formulated as a *quasiconvex programming* problem, which has recently been intensively studied in the context of convex and semidefinite programming [2][6],

For the 2-D case, let $\Sigma = \{A,\ b,\ c,\ d\}$ be a Roesser's state-space characterization of a given 2-D filter with $A \in \mathcal{R}^{(n_1+n_2)\times(n_1+n_2)}$, $b \in \mathcal{R}^{(n_1+n_2)\times 1}$, $c \in \mathcal{R}^{1\times(n_1+n_2)}$ and $d \in \mathcal{R}$, then $\hat{\Sigma} = \{\hat{A},\ \hat{b},\ \hat{c},\ \hat{d}\} = \{T^{-1}AT,\ T^{-1}b,\ cT,\ d\}$ with nonsingular $T = T_1 \oplus T_2$, $T_1 \in \mathcal{R}^{n_1\times n_1}$, $T_2 \in \mathcal{R}^{n_2\times n_2}$ describes the same 2-D filter. A state-space model $\tilde{\Sigma} = \{\tilde{A},\ \tilde{b},\ \tilde{c},\ \tilde{d}\} = \{\tilde{T}^{-1}A\tilde{T},\ \tilde{T}^{-1}b,\ c\tilde{T},\ d\}$ is said to be a MNR of the filter if

$$\underset{T=T_1\oplus T_2}{\text{minimize}}\ \|T^{-1}AT\| = \|\tilde{A}\|$$

where $\|\cdot\|$ denotes the spectral norm.

By definition, $\|T^{-1}AT\|^2$ is equal to the largest eigenvalue of $T^{-1}ATT^TA^TT^{-T}$. Since the nonzero eigenvalues of YZ and ZY are identical, $\|T^{-1}AT\|^2$ is the largest eigenvalue of $P^{-1}APA^T$ with $P = TT^T = T_1T_1^T \oplus T_2T_2^T$. If λ is an eigenvalue of $P^{-1}APA^T$, then there exists column vector $x \neq 0$ such that

$$P^{-1}APA^Tx = \lambda x$$

i.e.,

$$APA^Tx = \lambda Px$$

Thus λ is also a generalized eigenvalue of matrix pencil $(APA^T,\ P)$ [14]. Therefore, one concludes that $\|T^{-1}AT\|^2$ is the largest generalized eigenvalue of $(APA^T,\ P)$ with $P = TT^T$, and that the problem of finding a MNR of the 2-D filter amounts to solving the minimization problem

$$\underset{P=TT^T}{\text{minimize}}\ \lambda_{max}(P,\ APA^T) \tag{17}$$

where $\lambda_{max}(Y, Z)$ denotes the largest generalized eigenvalue of the pencil $\lambda Y - Z$. Note that matrix P in (17) is positive definite, and the two matrices involved there are *affine* functions of P. Thus the problem in (17) can be reformulated as

$$\begin{aligned} &\text{minimize} && \lambda_{max}(P,\ APA^T) && \text{(18a)}\\ &\text{subject to} && P = P_1 \oplus P_2,\ P_1 > 0,\ P_2 > 0 && \text{(18b)} \end{aligned}$$

Obviously, the constraints in (18b) are convex with respect to parameter matrices P_1 and P_2. In addition, for $P = P_1 \oplus P_2 > 0$, $\hat{P} = \hat{P}_1 \oplus \hat{P}_2 > 0$, and $0 \le \theta \le 1$, it can be shown that [10]

$$\begin{aligned} &\lambda_{max}(\theta P + (1-\theta)\hat{P},\ A[\theta P + (1-\theta)\hat{P}]A^T)\\ &\le \max(\lambda_{max}(P,\ APA^T),\ \lambda_{max}(\hat{P},\ A\hat{P}A^T)) \end{aligned} \tag{19}$$

which means that the objective function in (18a) is *quasiconvex* and the minimization problem (18) is a quasiconvex programming problem [2][6].

We now present a numerical example. Consider a stable 2-D system of order (3, 3), characterized by [16]

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ and } c = [c_1\ \ c_2]$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.38315 & -1.38605 & 1.90670 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.06280 & 0.06190 & 0.00654 \\ -0.02810 & 0.03956 & -0.02248 \\ 1.24452 & -0.57092 & 2.05865 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.00003 & 0.00038 & -0.00053 \\ -0.00001 & 0.00018 & -0.00026 \\ -0.00008 & 0.00023 & -0.00017 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.38238 & -1.38178 & 1.90253 \end{bmatrix}$$

$$\begin{aligned} b_1 &= b_2 = [0\ \ 0\ \ 1]^T \\ c_1 &= [0.01141\ \ -0.00540\ \ 0.01956] \\ c_2 &= [0.01164\ \ -0.00545\ \ 0.01960] \end{aligned}$$

It took the projective algorithm [2] 4364 flops and 1.27 seconds of CPU time on a PentiumPro 200 to generate an MNR of the system as

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix},\ \tilde{b} = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{bmatrix} \text{ and } \tilde{c} = [\tilde{c}_1\ \ \tilde{c}_2]$$

where

$$\tilde{A}_1 = \begin{bmatrix} 0.6683 & 0.3025 & 0.0251 \\ -0.2674 & 0.5781 & 0.3265 \\ 0.0850 & -0.3000 & 0.6603 \end{bmatrix}$$

$$\tilde{A}_2 = \begin{bmatrix} 0.0088 & 0.0075 & 0.0065 \\ -0.0323 & -0.0269 & -0.0230 \\ 0.0344 & 0.0286 & 0.0243 \end{bmatrix}$$

$$\tilde{A}_3 = \begin{bmatrix} 0.0072 & 0.0101 & -0.0190 \\ 0.0077 & -0.0311 & -0.0086 \\ -0.0065 & 0.0316 & 0.0069 \end{bmatrix}$$

$$\tilde{A}_4 = \begin{bmatrix} 0.6419 & 0.2831 & 0.1457 \\ -0.3424 & 0.5865 & 0.2601 \\ -0.3337 & -0.2898 & 0.6742 \end{bmatrix}$$

$$\tilde{b}_1 = [0.0051\ \ -0.0181\ \ 0.0193]^T$$

$$\tilde{b}_2 = [1.5886 \quad -4.4301 \quad 4.6081]^T$$
$$\tilde{c}_1 = [4.5415 \quad 3.5599 \quad 3.1759]$$
$$\tilde{c}_2 = [0.0167 \quad 0.0139 \quad 0.0119].$$

The system norm was reduced from $||A|| = 3.9560$ to $||\tilde{A}|| = 0.7490$. The BGFS based minimization method proposed in [11] also works but requires 16.75×10^6 flops and 13.51 seconds of CPU time. As well, the unweighted mixed L_1/L_2 sensitivity, L_2 sensitivity and the output roundoff-noise power are reduced from 1.7344×10^4, 2.8990×10^5, and 72.8485 for the original system to 8.7248×10^3, 8.7450×10^3, and 10.8377 for the MNR system.

4 A Minimax Design of Nonrecursive 2-D Digital Filters

It is well-known that minimax designs of 2-D FIR filters can be accomplished with various methods as shown, for example, in [16]-[19]. However, computer simulations have demonstrated that SDP can yield the required design with significantly improved computational efficiency.

Let the transfer function of a 2-D FIR digital filter be

$$H(z_1, z_2) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} h_{ij} z^{-i} z^{-j} = z_1^T \hat{H} z_2 \qquad (20)$$

where N_1 and N_2 are odd integers, $z_1 = [1 \;\; z_1^{-1} \;\; \cdots \;\; z_1^{-(N_1-1)}]^T$, $z_2 = [1 \;\; z_2^{-1} \;\; \cdots \;\; z_2^{-(N_2-1)}]^T$, and $\hat{H} \in R^{N_1 \times N_2}$. To derive a compact expression for the transfer function of a linear phase 2-D FIR filter, we partition $\hat{H}$ as

$$\hat{H} = \begin{bmatrix} H_{11} & h_{12} & H_{13} \\ h_{21}^T & h_{22} & h_{23}^T \\ H_{31} & h_{32} & H_{33} \end{bmatrix}$$

where H_{11}, H_{13}, H_{31}, $H_{33} \in R^{n_1 \times n_2}$, h_{12}, $h_{32} \in R^{n_1 \times 1}$, h_{21}, $h_{23} \in R^{n_2 \times 1}$, $h_{22} \in R$, $n_1 = (N_1 - 1)/2$, and $n_2 = (N_2 - 1)/2$. Now if

$$H_{13} = \text{flipud}\,(H_{33}),\; H_{31} = \text{fliplr}\,(H_{33}),$$
$$H_{11} = \text{flipud}\,(\text{fliplr}\,(H_{33})),$$
$$h_{12} = \text{flipud}\,(h_{32}), \text{ and } h_{21}^T = \text{fliplr}\,(h_{23}^T)$$

where flipud and fliplr represent the operations of fliping a matrix upside down and from left to right, respectively, then the filter characterized by $H(z_1, z_2)$ has linear phase response with group delay (n_1T_1, n_2T_2), and the frequency response of the filter is given by

$$H(e^{j\omega_1}, e^{j\omega_2}) = e^{-j(n_1\omega_1 + n_2\omega_2)} c_1^T(\omega_1) H c_2(\omega_2)$$

where $c_i(\omega_i) = [1 \;\; \cos\omega_i \;\; \cdots \;\; \cos n_i\omega_i]^T$ for $i = 1, 2$, and

$$H = \begin{bmatrix} h_{22} & 2h_{23}^T \\ 2h_{32} & 4H_{33} \end{bmatrix}$$

Consequently, the minimax design of a linear-phase 2-D FIR filter is the solution of the optimization problem

$$\underset{H}{\text{minimize}}[\underset{(\omega_1,\omega_2)\in\Omega}{\text{maximize}}\, e(\omega_1, \omega_2, H)] \qquad (21)$$

where

$$e(\omega_1, \omega_2, H) = W(\omega_1, \omega_2)|c_1^T(\omega_1) H c_2(\omega_2) - H_d(\omega_1, \omega_2)|$$

Ω denotes the frequency region of interest, $W(\omega_1, \omega_2) \geq 0$ is a weighting function over region Ω, and $H_d(\omega_1, \omega_2)$ is the desired amplitude response.

Evidently, the optimization problem in (21) is equivalent to

$$\text{minimize} \quad \delta \qquad (22a)$$

$$\text{subject to} \quad e(\omega_1, \omega_2, H) \leq \delta \text{ for } (\omega_1, \omega_2) \in \Omega \qquad (22b)$$

Furthermore, the constraint in (22b) can be expressed as

$$\delta - e(\omega_1, \omega_2, H) \geq 0 \qquad (23)$$

for $(\omega_1, \omega_2) \in \Omega$. The inequality in (23) implies that

$$\delta \geq 0 \qquad \text{for } (\omega_1, \omega_2) \in \Omega \qquad (24)$$

It is known from linear algebra that the inequalities in (23) and (24) hold if and only if

$$\Delta(\omega_1, \omega_2) \succeq 0 \qquad \text{for } (\omega_1, \omega_2) \in \Omega \qquad (25a)$$

where

$$\Delta(\omega_1, \omega_2) = \begin{bmatrix} \delta & e(\omega_1, \omega_2, H) \\ e(\omega_1, \omega_2, H) & 1 \end{bmatrix} \qquad (25b)$$

It is important to note that matrix $\Delta(\omega_1, \omega_2)$ in (25b) is *affine* w.r.t. design matrix H and the scalar auxiliary variable δ. A discretized version of the positive semidefinite condition in (25) is given by

$$D(x) = \text{diag}\,\{\Delta(\omega_1^{(1)}, \omega_2^{(1)}), \ldots, \Delta(\omega_1^{(M)}, \omega_2^{(M)})\} \succeq 0 \qquad (26)$$

where the set $\{(\omega_1^{(i)}, \omega_2^{(i)}),\ 1 \leq i \leq M\}$ forms a sufficiently dense grid in region Ω, and $x = [x_1\, x_2\, \cdots]^T$ is vector of dimension $1 + (n_1+1)(n_2+1)$ where $x_1 = \delta$

and x_2, x_3 $\cdots$ are the entries of H. Taking the above analysis into account, a discretized version of the optimization problem in (22) can be formulated as

$$\text{minimize} \quad c^T x \tag{27a}$$

$$\text{subject to} \quad D(x) \succeq 0 \tag{27b}$$

where $c = [1 \ \ 0 \ \ \cdots \ \ 0]^T$ and $D(x)$ is defined by (26) and (25b). Since matrix $D(x)$ in (27b) is affine w.r.t. x, (27) is an SDP problem.

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