AN ITERATIVE QUADRATIC PROGRAMMING METHOD FOR MULTIRATE FILTER DESIGN

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ABSTRACT
An iterative quadratic programming method for multirate filter design is described. The design problem is formulated as a 4th-order nonlinear optimization problem in which the objective function is a weighted sum of a reconstruction term and an aliasing-error term. Constraints on the filter’s frequency response in the passband and stopband are imposed as a set of linear inequalities. The optimization problem is solved by iteratively minimizing a quadratic function subject to a set of linear constraints. Explicit formulas for evaluating the minimums of these quadratic functions are described, which lead to an efficient and fast algorithm. An example is included to illustrate the design method.

1. INTRODUCTION
Multirate systems can be used for the efficient implementation of some basic signal processing operations such as narrow-band digital filtering [1][2]. A multirate digital filter is a system that consists of two linear time-invariant digital filters together with a down sampler and an up sampler as illustrated in Fig. 1. It can be shown that when the aliasing and imaging effects of the system are negligible, the system acts like a well-behaved digital filter [1][3].

\[ x(n) \rightarrow H(z) \rightarrow y(n) \rightarrow H_{1}(z) \rightarrow y'(n) \]

Figure 1: Multirate system for filtering.

The advantages of using multirate implementation for narrow-band digital filters over conventional FIR filters with a fixed sampling rate include reduction in the amount of computation as well as reduction in finite wordlength effects [4]. Unfortunately, the analysis and design of multirate filters are complicated due to the fact that multirate systems are periodically time-varying systems which are more difficult to deal with than their time-invariant counterparts [1][5].

In [3], the design of multirate filters was formulated as a model-matching problem [6] where a linear periodically time-varying system approximates a desired linear time-invariant system. A detailed analysis on the basis of a relative \( L^2 \) error measure was carried out and the design problem was solved using nonsmooth optimization techniques that deal with the class of optimization problems with nondifferentiable objective functions.

In this paper, we propose an alternative approach to the design of multirate lowpass filters. The design problem is formulated in the frequency domain as a 4th-order nonlinear optimization problem in which the objective function is a weighted sum of a reconstruction term and an aliasing-error term. Constraints on the filter’s frequency response in the passband and stopband are imposed as a set of linear inequalities. The optimization problem is solved by iteratively minimizing a quadratic function subject to a set of inequality constraints. Explicit formulas for evaluating the objective function are derived, which lead to an efficient and fast algorithm.

2. PROBLEM FORMULATION AND DESIGN METHOD

2.1. Objective Function
Let the filters represented by \( H(z) \) and \( H_{1}(z) \) in Fig. 1 be FIR filters with impulse responses \( h(n) \) and \( h_{1}(n) \), respectively, and assume that \( h_{1}(n) = h(N - n - 1) \) for \( 0 \leq n \leq N - 1 \) where \( N \) is the length of the filters. Then

\[ H_{1}(z) = z^{-(N-1)}H(z^{-1}) \]

The input-output relation in frequency domain can be expressed as

\[ Y(z) = \left[ \frac{1}{M}z^{-(N-1)}H(z^{-1})H(z) \right] X(z) + \left[ \frac{1}{M} \sum_{i=1}^{M-1} z^{-(N-1)}H(z^{-1})H(z e^{2\pi i/M}) \right] X(z e^{2\pi i/M}) \]

The first term on the right-hand side is the \( z \)-transform of the lowpass filtered signal while the second term is an aliasing-error term. In order for the multirate system in Fig. 1 to approximate an ideal lowpass filter, we seek an \( H(z) \) such that

\[ \frac{1}{M}H(z^{-1})H(z) \approx \begin{cases} 1 & \text{for } \omega \in [0, \omega_p] \\ 0 & \text{for } \omega \in [\omega_s, \pi] \end{cases} \]

and
\[
\frac{1}{M} \sum_{i=1}^{M-1} |H(\omega^{-1})H(\omega^{2\pi i/M})|^2 \approx 0
\]  
(2)

where \( \omega_p \) and \( \omega_s \) are the required passband and stopband edges, respectively. Equation (1) specifies the lowpass filter and (2) represents aliasing-error minimization. To facilitate the design, a weighted objective function is constructed as

\[
E = E_1 + \alpha_1 E_2 + \alpha_2 E_3
\]  
(3)

with

\[
E_1 = \int_{\omega_p}^{\omega_s} \left| \frac{1}{M} H(e^{j\omega})H(e^{-j\omega}) - 1 \right|^2 d\omega
\]

\[
E_2 = \int_{\omega_s}^{\pi} |H(e^{j\omega})|^2 d\omega
\]

\[
E_3 = \sum_{i=1}^{M-1} \int_{\omega_p}^{\omega_s} \left| H(e^{j\omega})H(e^{-j\omega}) \right|^2 d\omega
\]

\[
\mu = \omega + 2\pi i/M
\]

Coefficients are sought for \( H(z) \) to minimize the objective function \( E \) in (3), which is a 4th-order polynomial in terms of the impulse response \( h(n) \), for \( 0 \leq n \leq N - 1 \).

2.2. Constraints

The filter's performance is assured by minimizing \( E \) in (3) subject to constraints imposed on the passband and stopband. Based on the design requirements in (1) and (2), we impose

\[
\frac{1}{M} H(e^{j\omega})H(e^{-j\omega}) - 1 \leq \delta_p \quad \text{for} \quad \omega \in [0, \omega_p]
\]  
(4)

\[
\frac{1}{M} H(e^{j\omega})H(e^{-j\omega}) \leq \delta_s \quad \text{for} \quad \omega \in [\omega_s, \pi]
\]  
(5)

where \( \delta_p \) and \( \delta_s \) are the maximum errors allowed in the passband and stopband, respectively.

2.3. Closed-Form Formulas for Objective Function

The design problem at hand is to find \( H_k(z) \) such that the objective function \( E \) in (3) is minimized. This turns out to be a typical 4th-order optimization problem as will now be demonstrated.

In (3), the fourth-order terms come from \( H(e^{j\omega})H(e^{-j\omega}) \) and \( H(e^{j\omega})H(e^{-j\omega}) \), By fixing one frequency response, say \( H(e^{j\omega}) \) to be \( F(e^{-j\omega}) \), we can write

\[
E_1 = \frac{1}{M^2} E_{11} - \frac{2}{M} \text{real} E_{12} + \omega_p
\]

and

\[
E_3 = \sum_{i=1}^{M-1} \int_{\omega_p}^{\omega_s} \left| H(e^{j\omega})F(e^{-j\omega}) \right|^2 d\omega
\]

where

\[
E_{11} = \int_{\omega_p}^{\omega_s} \left| H(e^{j\omega})F(e^{-j\omega}) \right|^2 d\omega
\]

\[
E_{12} = \int_{\omega_p}^{\omega_s} H(e^{j\omega})F(e^{-j\omega})d\omega
\]

By expressing the transfer function \( H(z) \) as

\[
H(z) = h^Tz
\]

with

\[
h = [h_0 \ h_1 \ \cdots \ h_{N-1}]^T
\]

\[
z = [1 \ z^{-1} \ \cdots \ z^{-(N-1)}]^T
\]

we obtain

\[
E_{11} = h^T P_1(f) h
\]  
(6)

\[
E_{12} = h^T b_1 f
\]  
(7)

\[
E_2 = h_i^T b_i f
\]  
(8)

\[
E_5 = h_i^T P_2(f) h
\]  
(9)

where \( P_1(f), b_1, b_2, P_2(f), \) and \( f \) are given by

\[
P_1(f) = \int_{\omega_p}^{\omega_s} C(\omega)|F(e^{-j\omega})|^2 d\omega
\]

\[
b_1 = \int_{\omega_p}^{\omega_s} C(\omega) d\omega
\]

\[
b_2 = \int_{\omega_p}^{\omega_s} C(\omega) d\omega
\]

\[
P_2(f) = \sum_{i=1}^{M-1} \int_{0}^{\omega_s} C(\mu)|F(e^{-j\omega})|^2 d\omega
\]

\[
f = [f_0 \ f_1 \ \cdots \ f_{N-1}]^T
\]

\[
C(\omega) = \begin{bmatrix}
1 & \cos \omega & \cdots & \cos(N-1)\omega \\
\cos \omega & 1 & \cdots & \cos(N-2)\omega \\
\vdots & \vdots & \ddots & \vdots \\
\cos(N-1)\omega & \cos(N-2)\omega & \cdots & 1
\end{bmatrix}
\]

From (6)-(9), with a fixed \( f \) the objective function \( E \) in (3) can be expressed as a quadratic function

\[
E(h) = \frac{1}{M^2} E_{11} - \frac{2}{M} \text{real} E_{12} + \omega_p
\]

2.4. Closed-Form Formulas for Constraints

In the proposed method, the objective function \( E \) in (3) is reformulated by changing \( H(e^{j\omega}) \) to \( F(e^{-j\omega}) \) in the fourth-order terms where \( F(e^{-j\omega}) \) is assumed to be known from the preceding iteration. Similarly, the constraints in (4) and (5) are modified to a set of inequalities which are linear with respect to the coefficients of \( H(z) \). These inequalities can be put together as

\[
A_f h \leq b
\]  
(11)

where vector \( h \) contains the coefficients of \( H(z) \), and

\[
A_f = [A_{f1} \ A_{f2} \ A_{f3}]^T
\]

\[
b = [b_1 \ b_2 \ b_3]^T
\]

with
2.5. Iterative Algorithm

The algorithm starts with an initial point \( f^{(0)} \) that corresponds to the impulse response of a conventional lowpass filter. Quadratic programming \([7]\) is applied to minimize \( E(h) \) in (10) subject to the constraints in (11). The result is \( h^{(0)} \). Vectors \( f^{(0)} \) and \( h^{(0)} \) are then used to form vector

\[
f^{(i+1)} = \tau h^{(i)} + (1 - \tau)f^{(i)}
\]

where the smoothing factor \( \tau \) usually takes a value between 0.3 and 0.7. Next, vector \( f^{(0)} \) is used to evaluate the quadratic objective function as well as \( A_f \) and \( b \) in (11), and the iteration continues until \( \| h^{(k)} - f^{(k)} \| \) is less than a prescribed tolerance \( \epsilon \). A step-by-step description of the algorithm is given below.

Algorithm

**Step 1** Input sampling rate \( M \), filter length \( N \), passband edge \( \omega_p \), stopband edge \( \omega_s \), and tolerance \( \epsilon \). Design a lowpass FIR filter to form initial vector \( f^{(0)} \). Set \( n = 0 \).

**Step 2** Use \( f^{(0)} \) to evaluate \( E(h) \) in (10) and \( A_f \) and \( b \) in (11).

**Step 3** Minimize \( E(h) \) subject to constraint (11) using a quadratic programming algorithm. Denote the minimizer obtained as \( h^{(0)} \).

**Step 4** If \( \| h^{(i)} - f^{(i)} \| < \epsilon \), output \( h = h^{(i)} \) and stop. Otherwise, compute \( f^{(i+1)} \) using (12), set \( i = i + 1 \) and repeat from Step 2.

3. DESIGN EXAMPLE AND APPLICATION

The proposed algorithm was used to design a narrow-band lowpass filter with the following specifications: \( M = 5 \), \( N = 50 \), \( \alpha_1 = 1000 \), \( \alpha_2 = 0 \), \( \epsilon = 5 \times 10^{-6} \), \( \omega_p = 0.09 \), \( \omega_s = 0.103 \), \( \delta_p = 0.078 \), and \( \delta_s = 0.011 \). The initial coefficient vector was obtained by designing a linear-phase FIR filter of length \( N = 50 \) using the window method with a Hamming window \([8]\). The algorithm converged after 43 iterations. The filter coefficients are listed in Table 1.

### Table 1: Coefficients of the filter designed

<table>
<thead>
<tr>
<th>( i )</th>
<th>( h_i )</th>
<th>( i )</th>
<th>( h_i )</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0.010474239775775</td>
<td>25</td>
<td>-0.03156857751461</td>
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<td>1</td>
<td>0.00993353303465</td>
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<td>0.001255852140017</td>
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<td>-0.0019910119906</td>
<td>29</td>
<td>0.02109869297343</td>
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<tr>
<td>6</td>
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<td>31</td>
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<td>0.038594594223</td>
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<td>0.00347189668134</td>
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<td>0.01012489719028</td>
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<td>10</td>
<td>0.01372327929470</td>
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<td>11</td>
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<td>0.15781672768872</td>
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<td>18</td>
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<td>0.02462139321393</td>
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<td>22</td>
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<td>0.02275494385314</td>
</tr>
</tbody>
</table>

The magnitude responses of the filter represented by the optimal \( H(z) \) and the filter designed in [3] are shown in Fig. 2. It is observed in Fig. 2 that the lowpass filter designed has an improved stopband magnitude response relative to that of the filter reported in [3] although the passband responses are similar.

The narrow-band filter in [3] was designed using a different optimization method. The performance of the filter was evaluated using an error measure known as the normalized frequency response. This error measure is defined as

\[
N(f) = \frac{1}{\sqrt{M}} \| T(\omega) - T_{\text{ideal}}(\omega) \|_2
\]

where \( T(\omega) \) is the outer product of the polyphase components of \( H(z) \) and \( H_1(z) \), i.e.,

\[
T(z) = \left[ \begin{array}{c}
H_1(z) \\
\vdots \\
H_{M-1}(z)
\end{array} \right]
\]

evaluated at \( z = e^{j\omega} \) and \( T_{\text{ideal}}(\omega) \) is the ideal frequency response of the filter. The normalized frequency responses for the FIR filters designed by the method of [3] and the proposed method are depicted in Fig. 3. From Fig. 3, it is observed that the error measure \( N(f) \) for our design is less than that for the filter obtained in [3] in most of the passband. This means that our design will perform better in applications where the signal energy is of importance and noise needs to be removed.
problem is formulated as a 4th-order constrained nonlinear optimization problem which is solved using an iterative quadratic programming approach. Explicit formulas for evaluating the integrals involved have been derived for fast and precise implementation. The algorithm was illustrated by designing an example which was used for the elimination of noise.

Table 2: Mean-square Error of Test Signals Using the Proposed System and the System Reported in [3]

<table>
<thead>
<tr>
<th>Test Signal</th>
<th>SNR (dB)</th>
<th>MSE of the System in [3]</th>
<th>MSE of the Proposed System</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14.8256</td>
<td>0.0219</td>
<td>0.0218</td>
</tr>
<tr>
<td>2</td>
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<td>0.0218</td>
</tr>
<tr>
<td>3</td>
<td>9.4365</td>
<td>0.0222</td>
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</tr>
<tr>
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<td>0.0238</td>
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<tr>
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</tr>
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</table>

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5. REFERENCES


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