AN ITERATIVE QUADRATIC PROGRAMMING METHOD FOR MULTIRATE FILTER DESIGN

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ABSTRACT

An iterative quadratic programming method for multirate filter design is described. The design problem is formulated as a 4th-order nonlinear optimization problem in which the objective function is a weighted sum of a reconstruction term and an aliasing-error term. Constraints on the filter's frequency response in the passband and stopband are imposed as a set of linear inequalities. The optimization problem is solved by iteratively minimizing a quadratic function subject to a set of linear constraints. Explicit formulas for evaluating the minimums of these quadratic functions are described, which lead to an efficient and fast algorithm. An example is included to illustrate the design method.

1. INTRODUCTION

Multirate systems can be used for the efficient implementation of some basic signal processing operations such as narrow-band digital filtering [1][2]. A multirate digital filter is a system that consists of two linear time-invariant digital filters together with a downsampler and an upsampler as illustrated in Fig. 1. It can be shown that when the aliasing and imaging effects of the system are negligible, the system acts like a well-behaved digital filter [1]-[3].

Figure 1: Multirate system for filtering.

The advantages of using multirate implementation for narrow-band digital filters over conventional FIR filters with a fixed sampling rate include reduction in the amount of computation as well as reduction in finite wordlength effects [4]. Unfortunately, the analysis and design of multirate filters are complicated due to the fact that multirate systems are periodically time-varying systems which are more difficult to deal with than their time-invariant counterparts [1]-[5].

In [3], the design of multirate filters was formulated as a model-matching problem [6] where a linear periodically time-varying system approximates a desired linear time-invariant system. A detailed analysis on the basis of a relative l^2 error measure was carried out and the design problem was solved using nonsmooth optimization techniques that deal with the class of optimization problems with nondifferentiable objective functions.

In this paper, we propose an alternative approach to the design of multirate lowpass filters. The design problem is formulated in the frequency domain as a 4th-order nonlinear optimization problem in which the objective function is a weighted sum of a reconstruction term and an aliasing-error term. Constraints on the filter's frequency response in the passband and stopband are imposed as a set of linear inequalities. The optimization problem is solved by iteratively minimizing a quadratic function subject to a set of inequality constraints. Explicit formulas for evaluating the objective function are derived, which lead to an efficient and fast algorithm.

2. PROBLEM FORMULATION AND DESIGN METHOD

2.1. Objective Function

Let the filters represented by $H(z)$ and $H_1(z)$ in Fig. 1 be FIR filters with impluse responses $h(n)$ and $h_1(n)$, respectively, and assume that $h_1(n) = h(N-n-1)$ for $0 \le n \le N-1$ where N is the length of the filters. Then

$$H_1(z) = z^{-(N-1)}H(z^{-1})$$

The input-output relation in frequency domain can be expressed as

$$Y(z) = \left[\frac{1}{M}z^{-(N-1)}H(z^{-1})H(z)\right]X(z) + \left[\frac{1}{M}\sum_{l=1}^{M-1} z^{-(N-1)}H(z^{-1})H(ze^{2\pi jl/M})\right]X(ze^{2\pi jl/M})$$

The first term on the right-hand side is the z transform of the lowpass filtered signal while the second term is an aliasing-error term. In order for the multirate system in Fig. 1 to approximate an ideal lowpass filter, we seek an $H(z)$ such that

$$\frac{1}{M}H(z^{-1})H(z) \approx \begin{cases} 1 & \text{for } \omega \in [0, \omega_p] \\ 0 & \text{for } \omega \in [\omega_a, \pi] \end{cases} \qquad (1)$$

and

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$$\frac{1}{M}\sum_{l=1}^{M-1}\left|H(z^{-1})H(ze^{2j\pi l/M})\right|^2 \approx 0 \tag{2}$$

where ω_p and ω_a are the required passband and stopband edges, respectively. Equation (1) specifies the lowpass filter and (2) represents aliasing-error minimization. To facilitate the design, a weighted objective function is constructed as

$$E = E_1 + \alpha_1 E_2 + \alpha_2 E_3 \tag{3}$$

with

$$\begin{aligned} E_1 &= \int_0^{\omega_p}\left[\frac{1}{M}H(e^{j\omega})H(e^{-j\omega})-1\right]^2 d\omega \\ E_2 &= \int_{\omega_a}^{\pi}\left|H(e^{j\omega})\right|^2 d\omega \\ E_3 &= \sum_{l=1}^{M-1}\int_0^{\pi}\left|H(e^{j\mu})H(e^{-j\omega})\right|^2 d\omega \\ \mu &= \omega + 2\pi l/M \end{aligned}$$

Coefficients are sought for $H(z)$ to minimize the objective function E in (3), which is a 4th-order polynomial in terms of the impulse response $h(n)$, for $0 \le n \le N-1$.

2.2. Constraints

The filter's performance is assured by minimizing E in (3) subject to constraints imposed on the passband and stopband. Based on the design requirements in (1) and (2), we impose

$$\left|\frac{1}{M}H(e^{j\omega})H(e^{-j\omega})-1\right| \le \delta_p \quad \text{for } \omega \in [0,\omega_p] \tag{4}$$

$$\left|\frac{1}{M}H(e^{j\omega})H(e^{-j\omega})\right| \le \delta_s \quad \text{for } \omega \in [\omega_a,\pi] \tag{5}$$

where δ_p and δ_s are the maximum errors allowed in the passband and stopband, respectively.

2.3. Closed-Form Formulas for Objective Function

The design problem at hand is to find $H_k(z)$ such that the objective function E in (3) is minimized. This turns out to be a typical 4th-order optimization problem as will now be demonstrated.

In (3), the fourth-order terms come from $H(e^{j\omega})H(e^{-j\omega})$ and $H(e^{j\mu})H(e^{-j\omega})$. By fixing one frequency response, say $H(e^{-j\omega})$ to be $F(e^{-j\omega})$, we can write

$$E_1 = \frac{1}{M^2}E_{11} - \frac{2}{M}\,\text{real}\ E_{12} + \omega_p$$

and

$$E_3 = \sum_{l=1}^{M-1}\int_0^{\pi}\left|H(e^{j\mu})F(e^{-j\omega})\right|^2 d\omega$$

where

$$\begin{aligned} E_{11} &= \int_0^{\omega_p}\left|H(e^{j\omega})F(e^{-j\omega})\right|^2 d\omega \\ E_{12} &= \int_0^{\omega_p} H(e^{j\omega})F(e^{-j\omega})d\omega \end{aligned}$$

By expressing the transfer function $H(z)$ as

$$H(z) = \mathbf{h}^T\mathbf{z}$$

with

$$\begin{aligned} \mathbf{h} &= [\, h_0 \ \ h_1 \ \ \cdots \ \ h_{N-1}\,]^T \\ \mathbf{z} &= [\, 1 \ \ z^{-1} \ \ \cdots \ \ z^{-(N-1)}\,]^T \end{aligned}$$

we obtain

$$E_{11} = \mathbf{h}^T\mathbf{P}_1(\mathbf{f})\,\mathbf{h} \tag{6}$$
$$E_{12} = \mathbf{h}^T\mathbf{b}_1\mathbf{f} \tag{7}$$
$$E_2 = \mathbf{h}^T\mathbf{b}_2\mathbf{f} \tag{8}$$
$$E_3 = \mathbf{h}^T\mathbf{P}_2(\mathbf{f})\mathbf{h} \tag{9}$$

where $\mathbf{P}_1(\mathbf{f})$, $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{P}_2(\mathbf{f})$, and $\mathbf{f}$ are given by

$$\begin{aligned} \mathbf{P}_1(\mathbf{f}) &= \int_0^{\omega_p}\mathbf{C}(\omega)|F(e^{-j\omega})|^2 d\omega \\ \mathbf{b}_1 &= \int_0^{\omega_p}\mathbf{C}(\omega)d\omega \\ \mathbf{b}_2 &= \int_{\omega_c}^{\pi}\mathbf{C}(\omega)d\omega \\ \mathbf{P}_2(\mathbf{f}) &= \sum_{l=1}^{M-1}\int_0^{\pi}\mathbf{C}(\mu)|F(e^{-j\omega})|^2 d\omega \\ \mathbf{f} &= [\, f_0 \ \ f_1 \ \ \cdots \ \ f_{N-1}\,]^T \end{aligned}$$

$$\mathbf{C}(\omega) = \begin{bmatrix} 1 & \cos\omega & \cdots & \cos(N-1)\omega \\ \cos\omega & 1 & \cdots & \cos(N-2)\omega \\ \vdots & \vdots & \ddots & \vdots \\ \cos(N-1)\omega & \cos(N-2)\omega & \cdots & 1 \end{bmatrix}$$

From (6)-(9), with a fixed $\mathbf{f}$ the objective function E in (3) can be expressed as a *quadratic* function

$E(\mathbf{h}) =$

$$\mathbf{h}^T\left[\frac{\mathbf{P}_1(\mathbf{f})}{M^2}+\alpha_1\mathbf{b}_2+\alpha_2\mathbf{P}_2(\mathbf{f})\right]\mathbf{h} - \mathbf{h}^T\frac{2}{M}\mathbf{b}_1\mathbf{f}+\omega_p \tag{10}$$

2.4. Closed-Form Formulas for Constraints

In the proposed method, the objective function E in (3) is reformulated by changing $H(e^{-j\omega})$ to $F(e^{-j\omega})$ in the fourth-order terms where $F(e^{-j\omega})$ is assumed to be known from the preceding iteration. Similarly, the constraints in (4) and (5) are modified to a set of inequalities which are linear with respect to the coefficients of $H(z)$. These inequalities can be put together as

$$\mathbf{A}_f\mathbf{h} \le \mathbf{b} \tag{11}$$

where vector $\mathbf{h}$ contains the coefficients of $H(z)$, and

$$\begin{aligned} \mathbf{A}_f &= [\ \mathbf{A}_{1f} \ \ \mathbf{A}_{2f} \ \ \mathbf{A}_{3f}\ ]^T \\ \mathbf{b} &= [\ \mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \mathbf{b}_3\ ]^T \end{aligned}$$

with

$$
\begin{aligned}
\mathbf{A}_{1f} &= \mathbf{c}\mathbf{T}_f \\
\mathbf{A}_{2f} &= -\mathbf{c}\mathbf{T}_f \\
\mathbf{A}_{3f} &= \mathbf{c}\mathbf{T}_f \\
\mathbf{b}_1 &= \delta_p + \mathbf{M} \\
\mathbf{b}_2 &= \delta_p - \mathbf{M} \\
\mathbf{b}_3 &= \delta_s^2
\end{aligned}
$$

$$\mathbf{c} = [\ \cos(N-1)\omega\ \cdots\ \cos\omega\ \ 1\ \ \cos\omega\ \cdots\ cos(N-1)\omega\ ]$$

$$\mathbf{T}_f = \begin{bmatrix} f_{N-1} & 0 & \cdots & 0 \\ f_{N-2} & f_{N-1} & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ f_0 & \cdots & \cdots & f_{N-1} \\ 0 & f_0 & \cdots & f_{N-2} \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & f_0 \end{bmatrix}$$

2.5. Iterative Algorithm

The algorithm starts with an initial point $\mathbf{f}^{(0)}$ that corresponds to the impluse response of a conventional lowpass filter. Quadratic programming [7] is applied to minimize $E(\mathbf{h})$ in (10) subject to the constraints in (11). The result is $\mathbf{h}^{(0)}$. Vectors $\mathbf{f}^{(0)}$ and $\mathbf{h}^{(0)}$ are then used to form vector

$$\mathbf{f}^{(i+1)} = \tau\mathbf{h}^{(i)} + (1-\tau)\mathbf{f}^{(i)} \qquad (12)$$

where the smoothing factor τ usually takes a value between 0.3 and 0.7. Next, vector $\mathbf{f}^{(i)}$ is used to evaluate the quadratic objective function as well as $\mathbf{A}_f$ and $\mathbf{b}$ in (11), and the iteration continues until $||\mathbf{h}^{(k)} - \mathbf{f}^{(k)}||$ is less than a prescribed tolerance ϵ. A step-by-step description of the algorithm is given below.

Algorithm

Step 1 Input sampling rate M, filter length N, passband edge ω_p, stopband edge ω_a, and tolerance ϵ. Design a lowpass FIR filter to form initial vector $\mathbf{f}^{(0)}$. Set $i = 0$.

Step 2 Use $\mathbf{f}^{(i)}$ to evaluate $E(\mathbf{h})$ in (10) and A_f and $\mathbf{b}$ in (11).

Step 3 Minimize $E(\mathbf{h})$ subject to constraint (11) using a quadratic programming algorithm. Denote the minimizer obtained as $\mathbf{h}^{(i)}$.

Step 4 If $||\ \mathbf{h}^{(i)} - \mathbf{f}^{(i)}\ || < \epsilon$, output $\mathbf{h} = \mathbf{h}^{(i)}$ and stop. Otherwise, compute $\mathbf{f}^{(i+1)}$ using (12), set $i = i + 1$ and repeat from Step 2.

3. DESIGN EXAMPLE AND APPLICATION

The proposed algorithm was used to design a narrow-band lowpass filter with the following specifications: $M = 5$, $N = 50$, $\alpha_1 = 1000$, $\alpha_2 = 0$, $\epsilon = 5 \times 10^{-6}$, $\omega_p = 0.09$, $\omega_a = 0.103$, $\delta_p = 0.078$, and $\delta_s = 0.011$. The initial coefficient vector was obtained by designing a linear-phase FIR filter of length $N = 50$ using the window method with a Hamming window [8]. The algorithm converged after 43 iterations. The filter coefficients are listed in Table 1.

Table 1: Coefficients of the filter designed

i	h_i	i	h_i
0	0.01047423977757	25	−0.03156887751461
1	0.00999353303465	26	−0.03385947172852
2	0.00425852140017	27	−0.02362737810830
3	−0.00323225082729	28	−0.00321267782883
4	−0.00919910119906	29	0.02109869297343
5	−0.01149216426680	30	0.04092284633083
6	−0.00945823127795	31	0.04850686255175
7	−0.00385949549223	32	0.03948609599111
8	0.00347189668134	33	0.01474292111087
9	0.01012489719028	34	−0.01933778272510
10	0.01372237928470	35	−0.05243627589050
11	0.01263327694875	36	−0.07329358994544
12	0.00664620881249	37	−0.07320510279297
13	−0.00266965109989	38	−0.04884539931035
14	−0.01220475670053	39	−0.00344634014256
15	−0.01825065975258	40	0.05407835225987
16	−0.01790914853197	41	0.11173078435832
17	−0.01042995755215	42	0.15781672768872
18	0.00214779740099	43	0.18397591741493
19	0.01545245713994	44	0.18694722531621
20	0.02428210997529	45	0.16871605178309
21	0.02462139212339	46	0.13528828699650
22	0.01537153198667	47	0.09471683044832
23	−0.00093713417576	48	0.05506077709626
24	−0.01879829607868	49	0.02275494385314

The magnitude responses of the filter represented by the optimal $H(z)$ and the filter designed in [3] are shown in Fig. 2. It is observed in Fig. 2 that the lowpass filter designed has an improved stopband magnitude response relative to that of the filter reported in [3] although the passband responses are similar.

The narrow-band filter in [3] was designed using a different optimization method. The performance of the filter was evaluated using an error measure known as the normed frequency response. This error measure is defined as

$$N(f) = \frac{1}{\sqrt{M}}||\mathbf{T}(\omega) - \mathbf{T}_{\text{ideal}}(\omega)||_2$$

where $\mathbf{T}(\omega)$ is the outer product of the polyphase components of $H(z)$ and $H_1(z)$, i.e.,

$$\mathbf{T}(z) = \begin{bmatrix} H_{1,0}(z) \\ \vdots \\ H_{1,M-1}(z) \end{bmatrix} [\ H_{0,0}(z)\ \cdots\ z^{-(M-1)}H_{0,M-1}(z)\ ]$$

evaluated at $z = e^{j\omega}$ and $\mathbf{T}_{\text{ideal}}(\omega)$ is the ideal frequency response of the filter. The normed frequency responses for the FIR filters designed by the method of [3] and the proposed method are depicted in Fig. 3. From Fig. 3, we note that error measure $N(f)$ for our design is less than that for the filter obtained in [3] in most of the passband. This means that our design will perform better in applications where the signal energy is of importance and noise needs to be removed.

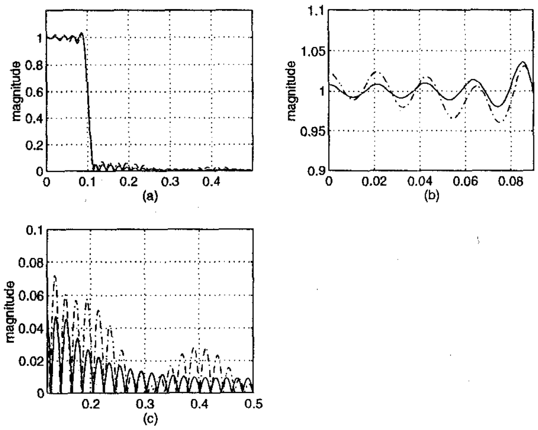

Figure 2: Magnitude responses of the filters designed in [3] (dash-dot line) and by the proposed method (solid line): (a) entire frequency band, (b) passband, (c) stopband.

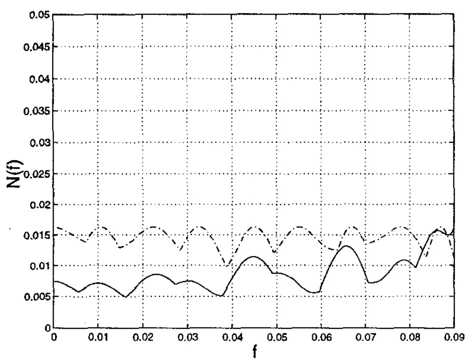

Figure 3: Normed frequency responses of the filters designed in [3] (dash-dot line) and by the proposed method (solid line).

The system designed above and the system reported in [3] were applied to a set of ten noise-corrupted test signals

$$x_n(t) = x(t) + n(t) \text{ with } x(t) = \sum_{k=1}^{n} a_k \cos(2\pi\omega_k t)$$

where ω_k for $1 \leq k \leq n$ are normalized frequencies that are uniformly distributed in the range [0, 0.09], coefficients a_i are chosen randomly, and $n(t)$ is the additive random noise obtained by highpass filtering a gaussian white noise sequence. The signal-to-noise ratio of the test signals varies from 4.8017 dB to 14.8256 dB. The system performance was measured by comparing the mean-square error (MSE) of the filtered signal with the noise-free signal $x(t)$. The simulation results are listed in Table 2.

4. CONCLUSION

We have proposed an alternative approach to the design of multirate lowpass filters. In this approach, the design problem is formulated as a 4th-order constrained nonlinear optimization problem which is solved using an iterative quadratic programming approach. Explicit formulas for evaluating the integrals involved have been derived for fast and precise implementation. The algorithm was illustrated by designing an example which was used for the elimination of noise.

Table 2: Mean-square Error of Test Signals Using the Proposed System and the System Reported in [3]

Test Signal	SNR (dB)	MSE of the System in [3]	MSE of the Proposed System
1	14.8256	0.0219	0.0218
2	10.8350	0.0220	0.0218
3	9.4365	0.0222	0.0220
4	8.9514	0.0238	0.0236
5	8.4362	0.0236	0.0234
6	7.9127	0.0230	0.0228
7	7.2313	0.0237	0.0236
8	6.5382	0.0209	0.0207
9	5.7809	0.0238	0.0237
10	4.8017	0.0233	0.0232

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5. REFERENCES

[1] R. E. Crochiere and L. R. Rabiner, *Multirate Digital Signal Processing,* Englewood Cliffs, NJ: Prentice-Hall, 1983.

[2] R. G. Shenoy, D. Burnside, and T. W. Parks, "Linear periodic systems and multirate filter design, " *IEEE Trans. Signal Processing,* vol. 42, pp. 2242-2256, Sept. 1994.

[3] W. M. Campbell and T. W. Parks, "Optimal design of partial-band time-varying systems," *IEEE Trans. Circuits Syst.,* vol. 44, pp. 274-283, April 1997.

[4] R. A. Meyer and C. S. Burrus, "A unified analysis of multirate and periodically time-varying digital filters," *IEEE Trans. Circuits Syst.,* vol. 22, pp. 162-168, Mar. 1975.

[5] J. S. Prater and C. M. Loeffler, "Analysis and design of periodically time-varying IIR filters, with applications to transmultiplexing, " *IEEE Trans. Signal Processing,* vol. 40, pp. 2715-2725, Nov. 1992.

[6] B. A. Francis, *A Course in H_∞ Control Theory,* New York: Springer-Verlag, 1987.

[7] R. Fletcher, *Practical Optimization Methods,* 2nd ed., New York: Wiley, 1987.

[8] A. Antoniou, *Digital Filters: Analysis, Design, and Applications,* 2nd ed., New York: McGraw-Hill, 1993.