ON STABILITY ROBUSTNESS OF DISCRETE-TIME SYSTEMS: THE COMPLEX-VARIABLE APPROACH OF MASTORAKIS

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ABSTRACT

The key element of Mastorakis' approach is the well-known theorem of Rouché: Suppose $f(z)$ and $d(z)$ are analytic in domain D and on its simple closed boundary ∂D, and suppose that $|d(z)| < |f(z)|$ on ∂D. Then $f(z)$ and $f(z) + d(z)$ have the same number of zeros in D. By taking the nominal and perturbed denominator polynomials of a stable discrete-time transfer function as $f(z)$ and $f(z) + d(z)$, respectively, and taking the unit disk as D, Rouché's theorem is directly connected to a stability-robustness study. This paper proposes an enhanced complex-variable approach initiated by Mastorakis to derive several improved bounds for stable coefficient perturbations of a nominal Schur polynomial.

1. INTRODUCTION

Stability and stability robustness of dynamic systems have been a subject of research in the past several decades with many striking results [1]–[8]. In a recent paper [9], Mastorakis proposed a complex-variable approach to investigating stability robustness of discrete-time systems. The key element of the approach is the well-known theorem of Rouché: Suppose $f(z)$ and $d(z)$ are analytic in domain D and on its simple closed boundary ∂D, and suppose that

$$|d(z)| < |f(z)| \qquad \text{on} \quad \partial D \tag{1}$$

then $f(z)$ and $f(z) + d(z)$ have the same number of zeros in D [10]. Consider a nominal stable transfer function whose denominator polynomial is denoted by $f(z)$, and let $\tilde{f}(z)$ be a perturbed denominator polynomial. Mastorakis [9] describes a novel application of Rouché's theorem to studying the stability of $\tilde{f}(z)$ by taking $d(z) = \tilde{f}(z) - f(z)$ and $D =$ the open unit disk, leading to several bounds for stable coefficient perturbations of the nominal Schur polynomial $f(z)$.

This paper proposes an enhanced complex-variable approach to derive several improved bounds for stable coefficient perturbations of a nominal Schur polynomial. Examples are included to demonstrate that the improvement achieved can be substantial. In the second part of the paper, the complex-variable approach is extended to investigate robust stability of two-dimensional discrete (2-D) systems.

2. BOUNDS FOR STABLE COEFFICIENT PERTURBATIONS

2.1. Notation, Definitions, and the Results of Mastorakis

Consider an nth-order polynomial in one complex variable z:

$$f(z) = \sum_{k=0}^{n} a_k z^k \qquad a_k \in R \tag{2}$$

$f(z)$ is said to be a *Schur polynomial or Schur stable* if it's zeros are strictly inside the unit circle $T = \{z : |z| = 1\}$. A time-invariant, discrete-time, dynamic system is asymptotically stable if and only if the denominator of its transfer function is a Schur polynomial [1]. Let a perturbed polynomial of $f(z)$ be denoted by

$$\tilde{f}(z) = \sum_{k=0}^{n} \tilde{a}_k z^k \tag{3}$$

where coefficients $\{\tilde{a}_k, \ k = 0, \ \ldots, \ n\}$ vary in a hyper-rectangular region centered at $\{a_k, \ k = 0, \ldots, \ n\}$:

$$|\tilde{a}_k - a_k| \le r_k \qquad 0 \le k \le n \tag{4}$$

Denoting

$$r = \max\{r_k, \ 0 \le k \le n\} \tag{5a}$$

one can write

$$r_k = \lambda_k r \qquad \text{with some} \qquad \lambda_k \in [0, \ 1] \tag{5b}$$

One of the problems to be studied in this paper can be stated as

Hyberrectangular Type Robust Stability (HRS) Problem:

Given an nth-order (nominal) Schur polynomial $f(z)$ and the λ_k's in (5b), find a bound ρ for r in (5a) such that $\tilde{f}(z)$ in (3) remains Schur stable if $r < \rho$.

The main result of Mastorakis [9] as related to the HRS problem is that $\tilde{f}(z)$ remains Schur stable if $r < \rho_{hm}$ where

$$\rho_{hm} = \min_{|z|=1} |f(z)| / \textstyle\sum_{k=0}^{n} \lambda_k \tag{6}$$

In the case $\lambda_k \equiv 1$, then the bound ρ_{hm} becomes

$$\bar{\rho}_{hm} = \min_{|z|=1} |f(z)|/(n+1) \tag{7}$$

Another problem to be considered in this paper is the robust stability of $f(z)$ with its coefficients varying in a ball centered at $\{a_k, \ k = 0, \ \ldots, \ n\}$ in the $(n+1)$-dimensional parameter space. This problem will be referred to as

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Ball Type Robust Stability (BRS) Problem:
Given an nth-order (nominal) Schur polynomial $f(z)$ and assume that the perturbed coefficients $\{\tilde{a}_k\}$ are restricted to be in the ball

$$\left[\sum_{k=0}^{n}(\tilde{a}_k - a_k)^2\right]^{1/2} \leq r \tag{8}$$

find a bound ρ for r in (8) such that $\tilde{f}(z)$ in (3) remains Schur stable if $r < \rho$.

2.2. New Bounds

2.2.1. *Rouché's Theorem as Related to Robust Stability*

Let D be the open unit disk and $d(z) = \tilde{f}(z) - f(z)$. Rouché's theorem implies that all the zeros of $\tilde{f}(z)$ are in D (hence the stability of $\tilde{f}(z)$) if

$$|\tilde{f}(z) - f(z)| < |f(z)| \qquad \text{on} \quad T = \{z : |z| = 1\} \tag{9}$$

With the notation in Sec. 2.1, (9) becomes

$$\left|\sum_{k=0}^{n}(\tilde{a}_k - a_k)e^{jk\omega}\right| < |f(e^{j\omega})| \quad \text{for} \quad \omega \in [0, \ \pi] \tag{10}$$

In (10), we have taken the advantage of the fact that coefficients of $f(z)$ and $\tilde{f}(z)$ are real, therefore parameter ω needs only to vary from 0 to π.

2.2.2. *A New Bound for the HRS Problem*

Considering the HRS problem, one can write

$$\tilde{a}_k = a_k + \tilde{\lambda}_k r \tag{11}$$

with $|\tilde{\lambda}_k| \leq \lambda_k$. This leads (10) to

$$r\left|\sum_{k=0}^{n}\tilde{\lambda}_k e^{jk\omega}\right| < |f(e^{j\omega})| \qquad \omega \in [0, \ \pi] \tag{12}$$

A key step in developing improved bounds is to treat $|\sum_{k=0}^{n}\tilde{\lambda}_k e^{jk\omega}|$ as the Euclidean norm of the vector

$$\begin{bmatrix} c^T(\omega)\tilde{\lambda} \\ s^T(\omega)\tilde{\lambda} \end{bmatrix}$$

with

$$\begin{aligned} c(\omega) &= [1 \ \cos\omega \ \cdots \ \cos n\omega]^T \\ s(\omega) &= [0 \ \sin\omega \ \cdots \ \sin n\omega]^T \end{aligned}$$

and

$$\tilde{\lambda} = [\tilde{\lambda}_0 \ \tilde{\lambda}_1 \ \cdots \ \tilde{\lambda}_n]^T \in \Lambda = \{\tilde{\lambda} : |\tilde{\lambda}_k| < \lambda_k, \text{ for } 0 \leq k \leq N\}$$

where Λ is referred to the uncertainty polyhedron of polynomial $\tilde{f}(z)$. Define the *frequency-dependent* $2 \times (n+1)$ matrix

$$H(\omega) = \begin{bmatrix} c^T(\omega) \\ s^T(\omega) \end{bmatrix} \tag{13}$$

then we have

$$\left|\sum_{k=0}^{n}\tilde{\lambda}_k e^{jk\omega}\right| = ||H(\omega)\tilde{\lambda}|| \tag{14}$$

where $\lambda = [\lambda_0 \ \lambda_1 \ \cdots \ \lambda_n]^T$ is a *known* vector.

By (12) and (14), one concludes that $\tilde{f}(z)$ remains Schur stable if $r < \rho_h$ where

$$\rho_h = \min_{\substack{\omega\in[0,\pi] \\ \tilde{\lambda}\in\Lambda}} \frac{|f(e^{j\omega})|}{||H(\omega)\tilde{\lambda}||} \tag{15}$$

The bound ρ_h is the global minimizer of $(n+2)-$variable function $|f(e^{j\omega})|/||H(\omega)\tilde{\lambda}||$ in the region $(\omega, \ \tilde{\lambda}) \in [0, \ \pi] \times \Lambda$. Two easy-to-use bounds can be derived from (15) as follows.

Note that

$$\min_{\substack{\omega\in[0,\pi] \\ \tilde{\lambda}\in\Lambda}} \frac{|f(e^{j\omega})|}{||H(\omega)\tilde{\lambda}||} = \min_{\omega\in[0,\pi]} \frac{|f(e^{j\omega})|}{\max\limits_{\tilde{\lambda}\in\Lambda}||H(\omega)\tilde{\lambda}||} \tag{16}$$

For a fixed ω, $||H(\omega)\tilde{\lambda}||$ is convex with respect to $\tilde{\lambda}$, so the maximum of $||H(\omega)\tilde{\lambda}||$ is achieved at one of the verteices whose ith entry is either λ_i or $-\lambda_i$. A total of $2^{(n+1)}$ function evaluations are required to compute

$$h(\omega) = \max_{\tilde{\lambda}\in\Lambda}||H(\omega)\tilde{\lambda}|| = \max_{\tilde{\lambda}_i=\pm\lambda_i}||H(\omega)\tilde{\lambda}|| \tag{17}$$

Once $h(\omega)$ is obtained, the bound in (15) becomes

$$\rho_h = \min_{\omega\in[0,\pi]} \frac{|f(e^{j\omega})|}{h(\omega)} \tag{18}$$

To avoid the combinatorial optimization (17), we estimate

$$\max_{\tilde{\lambda}\in\Lambda}||H(\omega)\tilde{\lambda}|| \leq ||H(\omega)|| \max_{\tilde{\lambda}\in\Lambda}||\tilde{\lambda}|| = ||H(\omega)|| \cdot ||\lambda||$$

where $||H(\omega)||$ denotes the spectral norm of $H(\omega)$, and $||\lambda||$ is the Euclidean norm of λ. Since

$$\min_{\substack{\omega\in[0,\pi] \\ \tilde{\lambda}\in\Lambda}} \frac{|f(e^{j\omega})|}{||H(\omega)\tilde{\lambda}||} \geq \frac{1}{||\lambda||} \cdot \min_{\omega\in[0,\pi]} \frac{|f(e^{j\omega})|}{||H(\omega)||}$$

one concludes that $\tilde{f}(z)$ remains Schur stable if $r < \rho_{h_1}$ where

$$\rho_{h_1} = \frac{1}{||\lambda||} \min_{\omega\in[0,\pi]} f_h(\omega) \tag{19a}$$

with

$$f_h(\omega) = \frac{|f(e^{j\omega})|}{||H(\omega)||} \tag{19b}$$

If $\lambda_k \equiv 1$, then bound ρ_{h_1} becomes

$$\bar{\rho}_h = \frac{1}{\sqrt{n+1}} \min_{\omega\in[0,\pi]} f_h(\omega) \tag{20}$$

From (13), it follows that [11]

$$1 \leq ||H(\omega)|| \leq \sqrt{n+1} \tag{21}$$

with its upper bound reached at $\omega = 0$ and $\omega = \pi$. By (19) and (21), we see that

$$\bar{\rho}_h \geq \min_{\omega\in[0,\pi]} |f(e^{j\omega})|/(n+1) = \bar{\rho}_{h_m} \tag{22}$$

where $\bar{\rho}_{h_m}$ is the Mastorakis bound in (7). Therefore, (20) offers an improved bound over the Mastorakis bound. The examples given in Sec. 4 will demonstrate that the improvement can be substantial.

The chief reason why the improvement can be significant is that the frequencies, at which $||H(\omega)||$ is far less than $\sqrt{n+1}$, consist of a large part of interval $[0,\ \pi]$. In fact, the curve $||H(\omega)||$ versus ω on $[0,\ \pi]$ has a U-shape, and as order n increases, the frequency subset corresponding to the bottom part of the U-shape curve quickly dominates the $[0,\ \pi]$ interval. See Fig. 1 for two plots of $||H(\omega)||$ with $n = 10$ and $n = 50$.

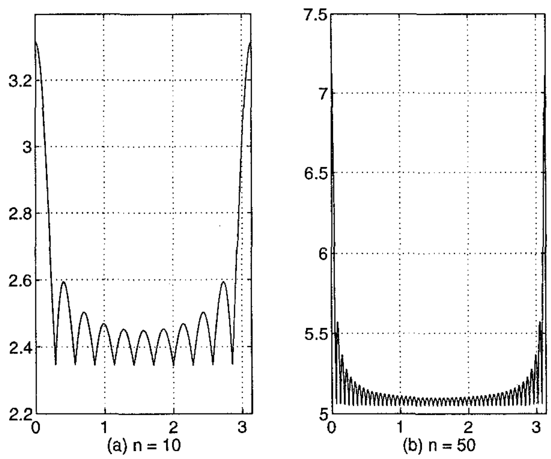

Figure 1. $||H(\omega)||$ **on** $[0,\ \pi]$**: (a)** $n = 10$**; (b)** $n = 50$.

2.2.3. A Bound for the BRS Problem

Consider the perturbed coefficients

$$\tilde{a}_k = a_k + \tilde{\lambda}_k r \tag{23a}$$

where $\tilde{\lambda} = [\tilde{\lambda}_0\ \cdots\ \tilde{\lambda}_n]^T$ varies in the unit ball:

$$\tilde{\lambda} \in \mathcal{B} = \{\tilde{\lambda} :\ ||\tilde{\lambda}|| \le 1\} \tag{23b}$$

By an argument similar to that in Sec. 2.2.2, one concludes that $\tilde{f}(z)$ remains Schur stable if $r < \rho_b$ where

$$\rho_b = \min_{\substack{\omega\in[0,\pi] \\ \tilde{\lambda}\in\mathcal{B}}} \frac{|f(e^{j\omega})|}{||H(\omega)\tilde{\lambda}||} \tag{24}$$

Since

$$\min_{\substack{\omega\in[0,\pi] \\ \tilde{\lambda}\in\mathcal{B}}} \frac{|f(e^{j\omega})|}{||H(\omega)\tilde{\lambda}||} = \min_{\omega\in[0,\pi]} \frac{|f(e^{j\omega})|}{\max\limits_{\tilde{\lambda}\in\mathcal{B}}||H(\omega)\tilde{\lambda}||} = \min_{\omega\in[0,\pi]} \frac{|f(e^{j\omega})|}{||H(\omega)||}$$

The bound ρ_b in (24) can be expressed as

$$\rho_b = \min_{\omega\in[0,\pi]} f_h(\omega) \tag{25}$$

with $f_h(\omega)$ given by (19b).

2.2.4. Computational Complexity

First we remark that as long as the order of $f(z)$ is known $||H(\omega)||$ can be evaluated *prior* to the stability test. Likewise, for a given uncertainty polyhedron Λ, function $h(\omega)$ defined in (17) can be computed and used for testing robust stability of any nth-order polynomial as long as its uncertainty polyhedron is contained in Λ.

It is well known that function $||H(\omega)||$ is continuous but not differentiable, see for example Fig. 1. In addition, for a fixed n $||H(\omega)||$ exhibits on $[0,\ \pi]$ exactly n local minimums and $n+1$ local maximums. Consequently, $f_h(\omega)$ is continuous but not differentiable, and in general has several local minimums. In [12], a global optimization algorithm for one-variable continuous functions is proposed. For minimizing a continuous function, the algorithm in [12] finds the global minimizer by generating a sequence of *lower-bounding* functions obtained from sample values of the objective function. Therefore, the minimum value obtained from the algorithm serves as a very tight lower bound of the true minimum value of the objective function. Obviously, this lower-bound (rather than upper-bound) nature makes the algorithm especially suitable for estimating bounds ρ_{h_1} and $\bar{\rho}_h$.

3. BOUNDS FOR 2-D DISCRETE SYSTEMS

Let

$$f(z_1,\ z_2) = \sum_{l=0}^{n}\sum_{k=0}^{m} a_{lk} z_1^l z_2^k \tag{26}$$

be the denominator polynomial of the transfer function of a 2-D discrete system, and let a perturbed polynomial of $f(z_1,\ z_2)$ be given by

$$\tilde{f}(z_1,\ z_2) = \sum_{l=0}^{n}\sum_{k=0}^{m} \tilde{a}_{lk} z_1^l z_2^k \tag{27}$$

It is well-known that the system associated with $f(z_1,\ z_2)$ in (26) is bounded-Input-bounded-Output (BIBO) stable if and only if [13, Theorem 5.6]

(a) $f_1(z) = \sum_{l=0}^{m} a_{lm} z^l$ is stable and (28a)

(b) $f_2(\omega_1,\ z_2) = \sum_{k=0}^{m}\big(\sum_{l=0}^{n} a_{lk} e^{jl\omega_1}\big) z_2^k$ is stable for each $\omega_1 \in [-\pi,\ \pi]$. (28b)

Let us consider the HRS problem for 2-D discrete systems: Given an $(n,\ m)$th-order (nominal) BIBO stable polynomial $f(z_1,\ z_2)$ and assume the coefficients of the perturbed polynomial $\tilde{f}(z_1,\ z_2)$ in (27) vary in a hypertectangular region centered at $\{a_{l,k},\ l = 1,\ \ldots,\ n;\ k = 1,\ \ldots,\ m\}$

$$|\tilde{a}_{lk} - a_{lk}| \le \lambda_{lk} r \text{ with } \lambda_{l,k} \in [0, 1] \tag{29}$$

find a bound ρ_2 such that $\tilde{f}(z_1,\ z_2)$ remains BIBO stable if r in (29) is strictly less than ρ_{h_2}. Denote

$$\tilde{f}_1(z) = \sum_{l=0}^{n} \tilde{a}_{lm} z^l \tag{30a}$$

$$\tilde{f}_2(\omega_1,\ z_2) = \sum_{k=0}^{m}\left(\sum_{l=0}^{n} \tilde{a}_{lk} e^{jl\omega_1}\right) z_2^k \tag{30b}$$

$$\tilde{a}_{lk} = a_{lk} + \tilde{\lambda}_{lk} r \quad \text{with } |\tilde{\lambda}_{lk}| \le \lambda_{lk} \qquad (30c)$$

$$\lambda_2^{(0)} = [\lambda_{0m} \ \cdots \ \lambda_{nm}]^T$$

$$\lambda_2 = [\lambda_{00} \ \cdots \ \lambda_{0m} \ \lambda_{10} \ \cdots \ \lambda_{nm}]^T$$

The results obtained in Sec. 2.2.2 are directly applicable to $\tilde{f}_1(z)$ in (30a) to conclude that $\tilde{f}_1(z)$ remains stable if $r < \rho_2^{(0)}$ where

$$\rho_2^{(0)} = \frac{1}{||\lambda_2^{(0)}||} \min_{\omega \in [0,\pi]} h_2^{(0)}(\omega) \qquad (31a)$$

with

$$h_2^{(0)}(\omega) = \frac{|f_1(e^{j\omega})|}{||H(\omega)||} \qquad (31b)$$

Based on Rouché's theorem, an argument similar to that used in Sec. 2.2.2 can be adopted to claim that $\tilde{f}_2(\omega_1, z_2)$ is stable for each $\omega_1 \in [-\pi, \pi]$ if

$$\left| r \sum_{k=0}^{m} \sum_{l=0}^{n} \tilde{\lambda}_{lk} e^{j(l\omega_1 + k\omega_2)} \right| < \left| \sum_{k=0}^{m} \sum_{l=0}^{n} a_{lk} e^{j(l\omega_1 + k\omega_2)} \right| \qquad (32)$$

and Eq. (32) holds if $r < \rho_2$ where

$$\rho_2 = \frac{1}{||\lambda_2||} \min_{(\omega_1, \omega_2) \in S} h_2(\omega_1, \omega_2) \qquad (33a)$$

with $S = [-\pi, \pi] \times \pi$,

$$h_2(\omega_1, \omega_2) = \frac{|f(e^{j\omega_1}, e^{j\omega_2})|}{||H_2(\omega_1, \omega_2)||} \qquad (33b)$$

and

$$H_2(\omega_1, \omega_2) = \begin{bmatrix} 1 & \cos\omega_1 & \cdots & \cos n\omega_1 & \cos(\omega_2) & \cdots & \cos(n\omega_1 + m\omega_2) \\ 0 & \sin\omega_1 & \cdots & \sin n\omega_1 & \sin(\omega_2) & \cdots & \sin(n\omega_1 + m\omega_2) \end{bmatrix}$$

Using the stability test in (28), one now concludes that $\tilde{f}(z_1, z_2)$ remains BIBO stable if $r < \rho_{h_2}$ where

$$\rho_{h_2} = \min(\rho_2^{(0)}, \rho_2) \qquad (34)$$

with $\rho_2^{(0)}$ and ρ_2 determined by (31) and (33), respectively. If all λ_{lk} are equal to 1, then the bound becomes

$$\bar{\rho}_{h_2} = \min(\bar{\rho}_2^{(0)}, \bar{\rho}_2) \qquad (35a)$$

with

$$\bar{\rho}_2^{(0)} = \frac{1}{\sqrt{n+1}} \min_{\omega \in [0,\pi]} h_2^{(0)}(\omega) \qquad (35b)$$

and

$$\bar{\rho}_2 = \frac{1}{\sqrt{(n+1)(m+1)}} \min_{(\omega_1, \omega_2) \in S} h_2(\omega_1, \omega_2) \qquad (35c)$$

4. EXAMPLES

Consider the Schur polynomial of order 4 that was used in [9] to illustrate the method of Mastorakis:

$$P_4(z) = 12z^4 + 2z^3 - 2z^2 - 3z + 1$$

and the Schur polynomial of order 8:

$$P_8(z) = 148z^8 + 48z^7 - 44z^6 - 80z^5 + 12z^4 + 16z^3 + 5z^2 - 6z - 2$$

Assume $\lambda_k \equiv 1$, the bounds $\bar{\rho}_{hm}$, ρ_h, and $\bar{\rho}_h$ for $P_4(z)$ and $P_8(z)$ are listed in Table 1.

Table 1. Bounds for $P_2(z)$ and $P_4(z)$

Polynomial	$\bar{\rho}_{hm}$	ρ_h	$\bar{\rho}_h$
$P_4(z)$	1.5131	2.0000	1.9525
$P_8(z)$	6.8913	10.0719	9.2538

Now consider the BIBO stable 2-D polynomial $f(z_1, z_2) = Z_1^T C Z_2$ with $Z_1 = [z_1^2 \ z_1 \ 1]^T$, $Z_2 = [z_2^2 \ z_2 \ 1]^T$, and

$$C = \begin{bmatrix} 40 & -28 & 7 \\ -30 & 25 & -5 \\ 6 & -4 & 1 \end{bmatrix}$$

and assume $\lambda_{lk} \equiv 1$. The $\bar{\rho}_2^{(0)}$ and $\bar{\rho}_{h_2}$ are found to be 5.3333 and 1.2005, respectively, which lead to the bound $\bar{\rho}_2 = 1.2005$.

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