ON STABILITY ROBUSTNESS OF DISCRETE-TIME SYSTEMS:
THE COMPLEX-VARIABLE APPROACH OF MASTORAKIS

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ABSTRACT
The key element of Mastorakis’ approach is the well-known theorem of Rouché: Suppose $f(z)$ and $d(z)$ are analytic in domain $D$ and on its simple closed boundary $\partial D$, and suppose that $|d(z)| < |f(z)|$ on $\partial D$. Then $f(z)$ and $f(z) + d(z)$ have the same number of zeros in $D$. By taking the nominal and perturbed denominator polynomials of a stable discrete-time transfer function as $f(z)$ and $f(z) + d(z)$, respectively, and taking the unit disk as $D$, Rouché’s theorem is directly connected to a stability-robustness study. This paper proposes an enhanced complex-variable approach initiated by Mastorakis to derive several improved bounds for stable coefficient perturbations of a nominal Schur polynomial.

1. INTRODUCTION
Stability and stability robustness of dynamic systems have been a subject of research in the past several decades with many striking results [1]–[8]. In a recent paper [9], Mastorakis proposed a complex-variable approach to investigating stability robustness of discrete-time systems. The key element of the approach is the well-known theorem of Rouché: Suppose $f(z)$ and $d(z)$ are analytic in domain $D$ and on its simple closed boundary $\partial D$, and suppose that

$$|d(z)| < |f(z)| \quad \text{on} \quad \partial D$$

then $f(z)$ and $f(z) + d(z)$ have the same number of zeros in $D$ [10]. Consider a nominal stable transfer function whose denominator polynomial is denoted by $f(z)$, and let $\tilde{f}(z)$ be a perturbed denominator polynomial. Mastorakis [9] describes a novel application of Rouché’s theorem to studying the stability of $\tilde{f}(z)$ by taking $d(z) = \tilde{f}(z) - f(z)$ and $D$ = the open unit disk, leading to several bounds for stable coefficient perturbations of the nominal Schur polynomial $f(z)$.

This paper proposes an enhanced complex-variable approach to derive several improved bounds for stable coefficient perturbations of a nominal Schur polynomial. Examples are included to demonstrate that the improvement achieved can be substantial. In the second part of the paper, the complex-variable approach is extended to investigate robust stability of two-dimensional discrete (2-D) systems.

2. BOUNDS FOR STABLE COEFFICIENT PERTURBATIONS
2.1. Notation, Definitions, and the Results of Mastorakis

Consider an $n$th-order polynomial in one complex variable $z$:

$$f(z) = \sum_{k=0}^{n} a_k z^k \quad a_k \in \mathbb{R}$$

(2)

$f(z)$ is said to be a Schur polynomial or Schur stable if its zeros are strictly inside the unit circle $T = \{ z : |z| = 1 \}$. A time-invariant, discrete-time, dynamic system is asymptotically stable if and only if the denominator of its transfer function is a Schur polynomial [1]. Let a perturbed polynomial of $f(z)$ be denoted by

$$\tilde{f}(z) = \sum_{k=0}^{n} \tilde{a}_k z^k$$

(3)

where coefficients $\{ \tilde{a}_k, k = 0, \ldots, n \}$ vary in a hyper-rectangular region centered at $\{ a_k, k = 0, \ldots, n \}$:

$$|\tilde{a}_k - a_k| \leq r_k \quad 0 \leq k \leq n$$

(4)

Denoting

$$r = \max\{ r_k, 0 \leq k \leq n \}$$

(5a)

one can write

$$r_k = \lambda_k r \quad \text{with some} \quad \lambda_k \in [0, 1]$$

(5b)

One of the problems to be studied in this paper can be stated as

Hyper-Rectangular Type Robust Stability (HRS) Problem:

Given an $n$th-order (nominal) Schur polynomial $f(z)$ and the $\lambda_k$’s in (5b), find a bound $\rho$ for $r$ in (5a) such that $\tilde{f}(z)$ in (3) remains Schur stable if $r < \rho$.

The main result of Mastorakis [9] as related to the HRS problem is that $\tilde{f}(z)$ remains Schur stable if $r < \rho_h$, where

$$\rho_h = \min_{\{ |z| = 1 \}} |f(z)| / \sum_{k=0}^{n} \lambda_k$$

(6)

In the case $\lambda_k \equiv 1$, then the bound $\rho_h$ becomes

$$\rho_h = \min_{\{ |z| = 1 \}} |f(z)| / (n + 1)$$

(7)

Another problem to be considered in this paper is the robust stability of $f(z)$ with its coefficients varying in a ball centered at $\{ a_k, k = 0, \ldots, n \}$ in the $(n + 1)$-dimensional parameter space. This problem will be referred to as
Ball Type Robust Stability (BRS) Problem:

Given an nth-order (nominal) Schur polynomial \( f(z) \) and assume that the perturbed coefficients \( \{\tilde{a}_k\} \) are restricted to be in the ball

\[
\left( \sum_{k=0}^{n} (\tilde{a}_k - a_k)^2 \right)^{1/2} \leq r \tag{8}
\]

find a bound \( \rho \) for \( r \) in (8) such that \( \tilde{f}(z) \) in (3) remains Schur stable if \( r < \rho \).

### 2.2. New Bounds

#### 2.2.1. Rouche’s Theorem as Related to Robust Stability

Let \( D \) be the open unit disk and \( d(z) = \tilde{f}(z) - f(z) \). Rouche’s theorem implies that all the zeros of \( f(z) \) are in \( D \) (hence the stability of \( \tilde{f}(z) \)) if

\[
|d(z)| < |f(z)| \quad \text{on} \quad T = \{z : |z| = 1\} \tag{9}
\]

With the notation in Sec. 2.1, (9) becomes

\[
\left| \sum_{k=0}^{n} (\tilde{a}_k - a_k)e^{j\omega k} \right| < |f(e^{j\omega})| \quad \text{for} \quad \omega \in [0, \pi] \tag{10}
\]

In (10), we have taken the advantage of the fact that coefficients of \( f(z) \) and \( \tilde{f}(z) \) are real, therefore parameter \( \omega \) needs only to vary from 0 to \( \pi \).

#### 2.2.2. A New Bound for the HRS Problem

Considering the HRS problem, one can write

\[
\tilde{a}_k = a_k + \lambda_k r \tag{11}
\]

with \( |\lambda_k| \leq \lambda_k \). This leads (10) to

\[
r \left| \sum_{k=0}^{n} \lambda_k e^{j\omega k} \right| < |f(e^{j\omega})| \quad \omega \in [0, \pi] \tag{12}
\]

A key step in developing improved bounds is to treat \( \sum_{k=0}^{n} \lambda_k e^{j\omega k} \) as the Euclidean norm of the vector

\[
\begin{bmatrix}
    c^T(\omega) \\
    s^T(\omega)
\end{bmatrix} \tilde{\lambda}
\]

with

\[
\begin{align*}
    c(\omega) &= \begin{bmatrix} 1 \cos \omega & \cdots & \cos n\omega \end{bmatrix}^T \\
    s(\omega) &= \begin{bmatrix} 0 \sin \omega & \cdots & \sin n\omega \end{bmatrix}^T
\end{align*}
\]

and

\[
\tilde{\lambda} = [\lambda_0 \ \lambda_1 \ \cdots \ \lambda_n]^T \in \Lambda = \{\tilde{\lambda} : |\tilde{\lambda}_k| < \lambda_k, \ \text{for} \ 0 \leq k \leq N\}
\]

where \( \Lambda \) is referred to the uncertainty polyhedron of polynomial \( \tilde{f}(z) \). Define the frequency-dependent \((n+1) \times (n+1)\) matrix

\[
H(\omega) = \begin{bmatrix}
    c^T(\omega) \\
    s^T(\omega)
\end{bmatrix} \tilde{\lambda}
\]

then we have

\[
\sum_{k=0}^{n} \lambda_k e^{j\omega k} = ||H(\omega)\tilde{\lambda}|| \tag{14}
\]

where \( \lambda = [\lambda_0 \ \lambda_1 \ \cdots \ \lambda_n]^T \) is a known vector.

By (12) and (14), one concludes that \( \tilde{f}(z) \) remains Schur stable if \( r < \rho_h \) where

\[
\rho_h = \min_{\Omega \in [0,\pi]} \frac{|f(e^{j\omega})|}{||H(\omega)\tilde{\lambda}||} \tag{15}
\]

The bound \( \rho_h \) is the global minimizer of \((n+2)-\)variable function \( |f(e^{j\omega})|/||H(\omega)\tilde{\lambda}|| \) in the region \( \omega, \tilde{\lambda} \in [0, \pi] \times \Lambda \). Two easy-to-use bounds can be derived from (15) as follows.

Note that

\[
\min_{\omega \in [0,\pi]} \frac{|f(e^{j\omega})|}{||H(\omega)\tilde{\lambda}||} = \min_{\lambda \in \Lambda} \frac{|f(e^{j\omega})|}{\max_{\lambda \in \Lambda} ||H(\omega)\tilde{\lambda}||} \tag{16}
\]

For a fixed \( \omega \), \( ||H(\omega)\tilde{\lambda}|| \) is convex with respect to \( \tilde{\lambda} \), so the maximum of \( ||H(\omega)\tilde{\lambda}|| \) is achieved at one of the vertices whose ith entry is either \( \lambda_i \) or \( \lambda_i \). A total of \( 2^{n+1} \) function evaluations are required to compute

\[
h(\omega) = \max_{\lambda \in \Lambda} ||H(\omega)\tilde{\lambda}|| = \max_{\lambda \in \Lambda} ||H(\omega)|| \tag{17}
\]

Once \( h(\omega) \) is obtained, the bound in (15) becomes

\[
\rho_h = \min_{\omega \in [0,\pi]} \frac{|f(e^{j\omega})|}{h(\omega)} \tag{18}
\]

To avoid the combinatorial optimization (17), we estimate

\[
\min_{\lambda \in \Lambda} ||H(\omega)\tilde{\lambda}|| \leq ||H(\omega)||\max_{\lambda \in \Lambda} ||\tilde{\lambda}|| = ||H(\omega)|| \cdot ||\tilde{\lambda}||
\]

where \( ||H(\omega)|| \) denotes the spectral norm of \( H(\omega) \), and \( ||\tilde{\lambda}|| \) is the Euclidean norm of \( \tilde{\lambda} \). Since

\[
\min_{\lambda \in \Lambda} |f(e^{j\omega})| \geq \frac{1}{||\tilde{\lambda}||} \min_{\lambda \in \Lambda} ||H(\omega)||
\]

one concludes that \( \tilde{f}(z) \) remains Schur stable if \( r < \rho_{h1} \) where

\[
\rho_{h1} = \frac{1}{||\tilde{\lambda}||} \min_{\omega \in [0,\pi]} f_h(\omega)
\]

with

\[
f_h(\omega) = \frac{|f(e^{j\omega})|}{||H(\omega)||} \tag{19a}
\]

If \( \lambda_k \equiv 1 \), then bound \( \rho_{h1} \) becomes

\[
\hat{\rho}_h = \frac{1}{\sqrt{n+1}} \min_{\omega \in [0,\pi]} f_h(\omega) \tag{20}
\]

From (13), it follows that [11]

\[
1 \leq ||H(\omega)|| \leq \sqrt{n+1} \tag{21}
\]

with its upper bound reached at \( \omega = 0 \) and \( \omega = \pi \). By (19) and (21), we see that

\[
\hat{\rho}_h \geq \min_{\omega \in [0,\pi]} |f(e^{j\omega})|/(n + 1) = \hat{\rho}_{h_m} \tag{22}
\]

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where $\rho_{\text{ma}}$ is the Mastorakis bound in (7). Therefore, (20) offers an improved bound over the Mastorakis bound. The examples given in Sec. 4 will demonstrate that the improvement can be substantial.

The chief reason why the improvement can be significant is that the frequencies, at which $||H(\omega)||$ is far less than $1/\sqrt{n + 1}$, consist of a large part of interval $[0, \pi]$. In fact, the curve $||H(\omega)||$ versus $\omega$ on $[0, \pi]$ has a U-shape, and as order $n$ increases, the frequency subset corresponding to the bottom part of the U-shape curve quickly dominates the $[0, \pi]$ interval. See Fig. 1 for two plots of $||H(\omega)||$ with $n = 10$ and $n = 50$.

![Figure 1. $||H(\omega)||$ on $[0, \pi]$: (a) $n = 10$; (b) $n = 50$.](image)

### 2.2.3. A Bound for the BRS Problem

Consider the perturbed coefficients

$$\tilde{a}_k = a_k + \tilde{\lambda}_k r$$

where $\tilde{\lambda} = [\tilde{\lambda}_0 \cdots \tilde{\lambda}_n]^T$ varies in the unit ball:

$$\tilde{\lambda} \in B = \{ \tilde{\lambda} : ||\tilde{\lambda}|| \leq 1 \}$$

By an argument similar to that in Sec. 2.2.2, one concludes that $f(z)$ remains Schur stable if $r < \rho_b$, where

$$\rho_b = \min_{\lambda \in \mathbb{B}} \frac{||f(e^{j\omega})||}{||H(\omega)\lambda||}$$

Since

$$\min_{\omega \in [0, \pi]} \frac{||f(e^{j\omega})||}{||H(\omega)\lambda||} = \min_{\omega \in [0, \pi]} \frac{||f(e^{j\omega})||}{\max_{\omega \in [0, \pi]} ||H(\omega)\lambda||} = \min_{\omega \in [0, \pi]} \frac{||f(e^{j\omega})||}{||H(\omega)||}$$

The bound $\rho_b$ in (24) can be expressed as

$$\rho_b = \min_{\omega \in [0, \pi]} f_b(\omega)$$

with $f_b(\omega)$ given by (19b).

### 2.2.4. Computational Complexity

First we remark that as long as the order of $f(z)$ is known $||H(\omega)||$ can be evaluated prior to the stability test. Likewise, for a given uncertainty polyhedron $\Lambda$, function $h(\omega)$ defined in (17) can be computed and used for testing robust stability of any $n$th-order polynomial as long as its uncertainty polyhedron is contained in $\Lambda$.

It is well known that function $||H(\omega)||$ is continuous but not differentiable, see for example Fig. 1. In addition, for a fixed $n$ $||H(\omega)||$ exhibits on $[0, \pi]$ exactly $n$ local minimums and $n + 1$ local maximums. Consequently, $f_b(\omega)$ is continuous but not differentiable, and in general has several local minimums. In [12], a global optimization algorithm for one-variable continuous functions is proposed. For minimizing a continuous function, the algorithm in [12] finds the global minimizer by generating a sequence of lower-bounding functions obtained from sample values of the objective function. Therefore, the minimum value obtained from the algorithm serves as a very tight lower bound of the true minimum value of the objective function. Obviously, this lower-bound (rather than upper-bound) nature makes the algorithm especially suitable for estimating bounds $\rho_b$ and $\rho_n$.

### 3. BOUNDS FOR 2-D DISCRETE SYSTEMS

Let

$$f(z_1, z_2) = \sum_{i=0}^{n} \sum_{k=0}^{m} a_{ik} z_1^i z_2^k$$

be the denominator polynomial of the transfer function of a 2-D discrete system, and let a perturbed polynomial of $f(z_1, z_2)$ be given by

$$\tilde{f}(z_1, z_2) = \sum_{i=0}^{n} \sum_{k=0}^{m} \tilde{a}_{ik} z_1^i z_2^k$$

It is well-known that the system associated with $f(z_1, z_2)$ in (26) is bounded-input-bounded-output (BIBO) stable if and only if [13, Theorem 5.6]

(a) $f_1(z) = \sum_{i=0}^{m} a_{i} z^i$ is stable and

(b) $f_2(\omega_1, \omega_2) = \sum_{i=0}^{m} a_{i} e^{j\omega_i} z_2^i$ is stable for each

$$\omega_i \in [-\pi, \pi].$$

Let us consider the HRS problem for 2-D discrete systems: Given an $(n, m)$th-order (nominal) BIBO stable polynomial $f(z_1, z_2)$ and assume the coefficients of the perturbed polynomial $\tilde{f}(z_1, z_2)$ in (27) vary in a hyper-rectangular region centered at $(a_{ik}, i = 1, \ldots, n; k = 1, \ldots, m)$

$$|\tilde{a}_{ik} - a_{ik}| \leq \lambda_{ik} r$$

find a bound $\rho_3$ such that $\tilde{f}(z_1, z_2)$ remains BIBO stable if $r$ in (29) is strictly less than $\rho_3$. Denote

$$\tilde{f}_1(z) = \sum_{i=0}^{n} \tilde{a}_{i} z^i$$

$$\tilde{f}_2(\omega_1, \omega_2) = \sum_{i=0}^{m} \left( \sum_{k=0}^{i} \tilde{a}_{ik} e^{j\omega_i} \right) z_2^i$$

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\[
\tilde{a}_{ik} = a_{ik} + \tilde{\lambda}_k r \quad \text{with} \quad |\tilde{\lambda}_k| \leq \lambda_k \quad (30c)
\]
\[
\lambda_2^{(0)} = [\lambda_{00} \cdots \lambda_{nm}]^T
\]
\[
\lambda_2 = [\lambda_{00} \cdots \lambda_{0m} \lambda_{10} \cdots \lambda_{nm}]^T
\]

The results obtained in Sec. 2.2 are directly applicable to \( f_2(z) \) in (30a) to conclude that \( f_2(z) \) remains stable if \( r < \rho_2^{(0)} \) where

\[
\rho_2^{(0)} = \frac{1}{|\lambda_2^{(0)}|} \min_{\omega \in [0, \pi]} h_2^{(0)}(\omega) \quad (31a)
\]

with

\[
h_2^{(0)}(\omega) = \frac{|f(z_1^{(0)}, e^{j\omega})|}{|H(\omega)|} \quad (31b)
\]

Based on Rouché’s theorem, an argument similar to that used in Sec. 2.2 can be adopted to claim that \( f_2(z_1, z_2) \) is stable for each \( \omega_1 \in [-\pi, \pi] \) if

\[
\sum_{k=0}^{m} \sum_{l=0}^{n} \tilde{a}_{ik} e^{j(l\omega_1 + k\omega_2)} < \sum_{k=0}^{m} \sum_{l=0}^{n} a_{ik} e^{j(l\omega_1 + k\omega_2)} \quad (32)
\]

and Eq. (32) holds if \( r < \rho_2 \) where

\[
\rho_2 = \frac{1}{|\lambda_2|} \min_{(\omega_1, \omega_2) \in S} h_2(\omega_1, \omega_2) \quad (33a)
\]

with \( S = [-\pi, \pi] \times \pi \),

\[
h_2(\omega_1, \omega_2) = \frac{|f(e^{j\omega_1}, e^{j\omega_2})|}{|H(\omega_1, \omega_2)|} \quad (33b)
\]

and

\[
H_2(\omega_1, \omega_2) = \begin{bmatrix}
1 & \cos \omega_1 & \cdots & \cos n \omega_1 & \cos (n \omega_1 + m \omega_2) \\
0 & \sin \omega_1 & \cdots & \sin n \omega_1 & \sin (n \omega_1 + m \omega_2)
\end{bmatrix}
\]

Using the stability test in (28), one now concludes that \( f(z_1, z_2) \) remains BIBO stable if \( r < \rho_2^{(0)} \) where

\[
\rho_2^{(0)} = \min(\rho_1^{(0)}, \rho_2) \quad (34)
\]

with \( \rho_2^{(0)} \) and \( \rho_2 \) determined by (31) and (33), respectively. If all \( \lambda_k \) is equal to 1, then the bound becomes

\[
\rho_2^{(0)} = \min(\rho_1^{(0)}, \rho_2) \quad (35a)
\]

with

\[
\rho_1^{(0)} = \frac{1}{\sqrt{n+1}} \min_{\omega \in [0, \pi]} h_2^{(0)}(\omega) \quad (35b)
\]

and

\[
\rho_2 = \frac{1}{\sqrt{(n+1)(m+1)}} \min_{(\omega_1, \omega_2) \in S} h_2(\omega_1, \omega_2) \quad (35c)
\]

4. EXAMPLES

Consider the Schur polynomial of order 4 that was used in [9] to illustrate the method of Mastorakis:

\[
P_4(z) = 12z^4 + 22z^3 - 2z^2 - 3z + 1
\]

and the Schur polynomial of order 8:

\[
P_8(z) = 148z^8 + 48z^7 - 44z^6 - 80z^5 + 12z^4 + 16z^3 + 5z^2 - 6z - 2
\]

Assume \( \lambda_k = 1 \), the bounds \( \rho_{\lambda n}, \rho_3, \) and \( \rho_4 \) for \( P_4(z) \) and \( P_8(z) \) are listed in Table 1.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>( \rho_{\lambda n} )</th>
<th>( \rho_3 )</th>
<th>( \rho_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_4(z) )</td>
<td>1.5131</td>
<td>2.0000</td>
<td>1.9625</td>
</tr>
<tr>
<td>( P_8(z) )</td>
<td>6.8913</td>
<td>10.0719</td>
<td>9.2538</td>
</tr>
</tbody>
</table>

Now consider the BIBO stable 2-D polynomial

\[
f(z_1, z_2) = Z_1^T C Z_2 \quad \text{with} \quad Z_1 = [z_1^2 z_1 1]^T \quad \text{and} \quad Z_2 = [z_2^2 z_2 1]^T,
\]

and

\[
C = \begin{bmatrix}
40 & -28 & 7 \\
-30 & 25 & -5 \\
6 & -4 & 1
\end{bmatrix}
\]

and assume \( \lambda_k \equiv 1 \). The \( \rho_2^{(0)} \) and \( \rho_2 \) are found to be 5.3333 and 1.2005, respectively, which lead to the bound \( \rho_2 = 1.2005 \).

ACKNOWLEDGEMENT

The author is grateful to the Natural Science and Engineering Research Council of Canada for supporting this work.

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