Linear Parameterization of Orthogonal Wavelets

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Abstract

This paper describes a new method for the parameterization of compactly supported orthogonal wavelet filters. The well-known Daubechies orthogonal wavelets can be viewed as a subset in the parameterized orthogonal wavelet class, which processes maximum number of vanishing movements for a given filter length. Unlike the existing parameterizations of orthogonal wavelets, the proposed method does the parameterization through a linear characterization of all halfband filters. The paper also includes examples of optimal designs of orthogonal wavelets obtained using this parameterization technique in conjunction with efficient linear programming or quadratic programming, and application of these wavelets to signal compression and signal denoising.

1. Introduction

Compactly supported orthogonal wavelets discovered by I. Daubechies [1] have been a subject of study since 1988. The Daubechies wavelets are not only orthogonal but also having maximum number of vanishing moments, which means that the lowpass analysis associated with the Daubechies scaling functions have maximum degree of flatness at $\omega = \pi$ (as well as $\omega = 0$). It is well known [2] that there is a direct connection of the vanishing moments to the accuracy of approximating a smooth function (signal) using a linear combination of orthogonal basis functions $\phi(t-k)$, $k \in Z$ where $\phi(t)$ is the mother scaling function. On the other hand, there are other occasions where one may wish to trade a certain number of vanishing moments with other features that turn out to be more desirable for a particular DSP task. Features of this kind include for example improved frequency responses of the wavelet filters for signal compression and noise reduction. It would therefore be desirable to parameterize *all* orthogonal wavelets so that (i) the class of Daubechies filters are obtained when the dimension of the "free" parameters shrinks to zero; (ii) more importantly, for non-maximum flat wavelet filters these parameters can be used to carry out an optimal design that is more suitable for the application at hand.

Parameterizing orthogonal wavelets is not new. Let F be a subfield of C closed under complex conjugation and $U(2,\ F(z,\ 1/z])$ be the multiplicative group of two-by-two unitary matrices over the ring $F[z,\ 1/z]$ with $|z| = 1$. Let $SU_I(2,\ F[z,\ 1/z])$ be the subgroup of such unitary matrices whose determinants are equal to 1 and which are equal to the identity matrix at $z = 1$. Reference [3] describes a factorization theorem for $SU_I(2,\ F[z,\ 1/z])$ and shows that the theorem can be used to parameterize various classes of wavelet families. Reference [4] presents a parameterization of $M \times M$ causal FIR lossless systems of degree N as

$$H_N(z) = V_N(z) \ \cdots \ V_1(z) H_0$$

where V_k are $M \times M$ degree-one FIR lossless matrices and H_0 is a constant unitary matrix. In [5], the results of [4] were extended to characterize compactly supported dyadic orthogonal wavelets. For example, orthogonal wavelet (lowpass) filters with length $= 4$ can be parameterized as

$$\begin{aligned} c_0 &= \sin\theta_1 \sin\left(\theta_1 - \frac{\pi}{4}\right) \\ c_1 &= \sin\theta_1 \sin\left(\theta_1 + \frac{\pi}{4}\right) \\ c_2 &= \cos\theta_1 \sin\left(\theta_1 + \frac{\pi}{4}\right) \\ c_3 &= \cos\theta_1 \sin\left(\frac{\pi}{4} - \theta_1\right) \end{aligned}$$

and the D_4 lowpass filter is obtained with $\theta_1 = \pi/12$. Note that the filter coefficients c_i depend on parameter θ_1 *nonlinearly.* As is evidenced in Eqns. (27) and (28) of [5], the degree of nonlinearity increases with the filter length.

In this paper, the problem of parametering *all* compactly supported orthogonal wavelets is tackled using a different approach. We describe an elementary method for the characterization of halfband filters that are associated with the

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orthogonal wavelets from a purely linear algebraic perspective. The halfband filters are parameterized *linearly* with a "free" parameter vector of dimension $\frac{N+1}{2} - L$ where $N+1$ and L denote the filter length and the number of vanishing moments that the wavelets shall possess, respectively. As the halfband filter is related to the wavelet filter in an extremely simple manner [2], this linear parameterization leads immediately to several efficient methods for the design of optimal orthogonal wavelets where the optimization may be performed using for example linear or quadratic programming [6][7].

2. Characterization of nonnegative halfband filters

Consider the zero-phase transfer function

$$\begin{aligned} P(z) &= \left(\frac{1+z^{-1}}{2}\right)^L \left(\frac{1+z}{2}\right)^L B(z) \\ &= \left(\frac{z^{-1}}{4} + \frac{1}{2} + \frac{z}{4}\right)^L \sum_{k=-K}^{K} \hat{b}_k z^k \end{aligned} \quad (1)$$

with $\hat{b}_k = \hat{b}_{-k}$ for $1 \le k \le K$. It follows that $P(e^{j\omega}) \ge 0\ \forall\omega$ if

$$B(z) = B_1(z) B_1(z^{-1}) \quad (2)$$

for some causal FIR $B_1(z)$. In this case we can write

$$P(z) = C(z)C(z^{-1}) \quad (3)$$

where

$$C(z) = \left(\frac{1+z^{-1}}{2}\right)^L B_1(z) \quad (4)$$

has at least L zeros at $\omega = \pi$. An FIR $P(z)$ in (3) is said to be a *halfband* filter if

$$P(z) + P(-z) = 2 \qquad \text{for all } z \quad (5)$$

In this case $C(z)$ in (3) is actually an orthogonal, lowpass, analysis wavelet filter with gain $C(e^{j0}) = \sqrt{2}$ and at least L vanishing moments, see Chapter 5 of [2].

A key step in our approach to the parameterization of wavelet filters $C(z)$ is a linear characterization of $P(z)$ satisfying (5). Denote

$$B(z) = \sum_{k=-K}^{K} \hat{b}_k z^k = z^{-K} \sum_{k=0}^{2K} b_k z^k, \ b_{2K-k} = b_k \quad (6)$$

and

$$\left(\frac{z^{-1}}{4} + \frac{1}{2} + \frac{z}{4}\right)^L = z^{-L} \sum_{l=0}^{2L} a_l z^l, \ a_{2L-l} = a_l \quad (7)$$

$P(z)$ in (1) can be written as

$$P(z) = z^{-(L+K)} \sum_{s=0}^{2N} \left(\sum_{\substack{l+k=s \\ l\ge 0, k\ge 0}} a_l b_k \right) z^{s-N}$$

where $N = K + L$ is assumed to be an odd integer. For $P(z)$ to meet condition (2), it is necessary and sufficient to have

$$\sum_{\substack{l+i=s \\ l\ge 0, k\ge 0}} a_l b_k = \begin{cases} 1 & s = N \\ 0 & s = 1,\ 3,\ \ldots,\ N-2 \end{cases} \quad (8)$$

With a_l's specified by (7), (8) presents $(N+1)/2$ *linear* constraints on coefficient vector $b = [b_0\ b_1\ \cdots\ b_K]^T$. These constraints can be put together into a linear system of equations of the form

$$Ab = m \quad (9)$$

where $A \in R^{\frac{N+1}{2}\times(K+1)}$ is determined by a_l's in (7), $b \in R^{(K+1)\times 1}$ characterizes $B(z)$ in (6), and $m = [0\ \ldots\ 0\ 1]^T \in R^{\frac{(N+1)}{2}\times 1}$. For example, with $L = 3$ and $K = 6$, A in (9) is given by

$$A = \begin{bmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \\ 0 & 0 & 0 & a_0 + a_6 & a_1 + a_5 & a_2 + a_4 & a_3 \end{bmatrix}$$

The rank of A is $(N+1)/2$, and all the b's satisfying (9) are parameterized by

$$b = a_l^* + V^*\phi \quad (10)$$

where a_l^* is the last column of the Moore-Penrose Pseudo inverse of A, V^* comes from the SVD of A [8], i.e., $A = U\Sigma V^T$:

$$V^* = [v_{\frac{n+3}{2}}\ \cdots\ v_{K+1}] \in R^{(K+1)\times\eta}$$

with $\eta = (K - L + 1)/2$ and v_i = the ith column of V, and $\phi \in R^{\eta\times 1}$ is a *free* parameter vector which can be utilized, for example, to shape up the frequency response of an orthogonal wavelet filter. In summary, all FIR $P(z)$ of form (1) satisfying (5) are parameterized by (10), where a_l^* and V^* can be pre-determined by a_l's in (7) as long as the number of vanishing moments of the associated wavelet, L, is given.

3. Design of orthogonal wavelet filters

We propose two methods for the design of non-maximum flat orthogonal wavelet filters with a given number of vanishing moments that minimize a frequency-response-related error measure.

3.1 A linear-programming-based method

A key requirement in a halfbased-filter-based design of orthogonal wavelet filters is the nonnegativity of $P(e^{j\omega})$. By (5), $P(e^{j\omega})$ is odd symmetrical with respect to $\omega = \pi/2$. If $B_1(z)$ is a lowpass filter, then $C(z)$ in (4) is also a lowpass filter, and so is $P(e^{j\omega})$. It is therefore quite clear that a halfband lowpass $P(e^{j\omega})$ is nonnegative if

$$P(e^{j\omega}) \geq 0 \qquad \text{for } \omega \in [\omega_a,\ \pi] \tag{11}$$

where ω_a is the stopband edge of $P(z)$. As $P(e^{j\omega}) = |C(e^{j\omega})|^2$, shaping up the frequency response of the wavelet filter $C(z)$ is directly related to designing a $P(z)$ with satisfactory $P(e^{j\omega})$ subject to (11). To this end we seek to find ϕ in (10) such that

$$J_1(\phi) = \int_0^{\omega_p} W(\omega)[2 - P(e^{j\omega})]d\omega \tag{12a}$$

is minimized subject to

$$P(e^{j\omega}) \geq 0 \qquad \text{for } \omega \in [\omega_a,\ \pi] \tag{12b}$$

where $W(\omega) \geq 0$ is a weighting function, and ω_p denotes the normalized passband edge.
Denote

$$R(z) = R_1(z)R_1(z^{-1})$$

with

$$R_1(z) = \left(\frac{1+z^{-1}}{2}\right)^L$$

we can write

$$P(e^{j\omega}) = R(e^{j\omega})B(e^{j\omega})$$

where

$$R(e^{j\omega}) = 2^{-L}(1+\cos\omega)^L$$

and

$$\begin{aligned} B(e^{j\omega}) &= c^T(\omega)b \\ c(\omega) &= [1 \;\; 2\cos\omega \;\; \cdots \;\; 2\cos K\omega]^T \end{aligned}$$

Using the parameterization of b given by (10), (12a) now becomes

$$J_1(\phi) = \text{const} - d^T\phi \tag{13a}$$

where

$$d^T = 2^{-L}\left[\int_0^{\omega_p} W(\omega)(1+\cos\omega)^L c^T(\omega)d\omega\right]V^*$$

and const denotes a scalar independent of ϕ, and the linear constraint (12b) can be approximated with a set of linear inequalities with respect to ϕ:

$$-c^T(\omega)V^*\phi \leq c^T(\omega)a_l^* \tag{13b}$$

where ω varies over an equally spaced dense grid points on $[\omega_a,\ \pi)$. Minimizing (13a) subject to (13b) is a typical linear programming (LP) problem for which many efficient solution methods are available [6].

Once the minimizer ϕ^* of problem (13) is obtained, the coefficients of $B(z) = B^*(z)$ is given by (10) with $\phi = \phi^*$: $b^* = a_1^* + V^*\phi^*$, and the optimal, nonnegative halfband filter is $P^*(z) = R(z)B^*(z)$. Since $P^*(e^{j\omega})$ is nonnegative, so is $B^*(e^{j\omega})$. Hence a minimum-phase $B_1^*(z)$ can be found such that $B^*(z) = B_1^*(z)B_1^*(z^{-1})$, and the optimal orthogonal lowpass wavelet filter is given by

$$C^*(z) = R_1(z)B_1^*(z)$$

Note that the presence of $R_1(z)$ ensures that orthogonal wavelet filter $C^*(z)$ has (at least) L vanishing moments.

3.2 A quadratic-programming-based method

An alternative to the LP method in Section 3.1 is to use quadratic programming (QP) optimization to accomplish the design. A QP algorithm minimizes a (usually convex) quadratic objective function subject to linear constraints. Efficient algorithms for solving QP problems have been available in the literature [6][7][9]. One of the reasonable quadratic objective functions for our design problem is

$$J_2(\phi) = \int_0^{\omega_p} W(\omega)[P(e^{j\omega}) - 2]^2 d\omega \tag{14}$$

which is to be minimized subject to constraints (12b). Following the notation used in Sec. 3.1, simple manipulations give

$$J_2(\phi) = \phi^T Q\phi + \phi^T q + \text{const} \tag{15}$$

where

$$\begin{aligned} Q &= 2^{-2L}V^{*^T} \cdot \int_0^{\omega_p} W(\omega)(1+\cos\omega)^{2L}c(\omega)c^T(\omega)d\omega \cdot V^* \\ q &= V^{*^T}[2^{-2L+1}\int_0^{\omega_p} W(\omega)(1+\cos\omega)^{2L}c(\omega)c^T(\omega)d\omega \cdot a_l^* \\ &\quad -2^{-L+2}\int_0^{\omega_p} W(\omega)(1+\cos\omega)^L c(\omega)d\omega] \end{aligned}$$

Minimizing $J_2(\phi)$ in (15) subject to constraints in (13b) is a typical QP program. Once the minimizer ϕ^* of problem (15), (13b) is obtained, the optimal orthogonal lowpass wavelet filter $C^*(z)$ can be found by following the steps described at the end of Sec. 3.1.

4. Examples

To illustrate the design methods, we use Daubechies orthogonal filter D_8 as a prototype in comparison with a family of orthogonal wavelets of length 8 with $L = 2$ vanishing moments. By giving up two vanishing moments,

$\eta = \frac{K-L+1}{2} = 2$ degrees of freedom are obtain to shape up the frequency response of $C(z)$. In the use of the LP method, 26 constraints are used over the frequency range $[\omega_a,\ \pi)$ where ω_a varies from 0.7π to 0.8π with an increment of 0.02π for each design. The six frequency responses of the wavelet lowpass filters designed are depicted and compared to the magnitude response of the D_8 filter (shown in dotted line). With $L = 2,\ K = 5$, and the same set of stopband edges, the QP method is applied to design six orthogonal wavelet filters. The number of constraints used on $[\omega_a,\ \pi)$ is 7. The design results obtained are shown in Fig. 2 where for comparison purposes the magnitude response of D_8 is also plotted (dotted line).

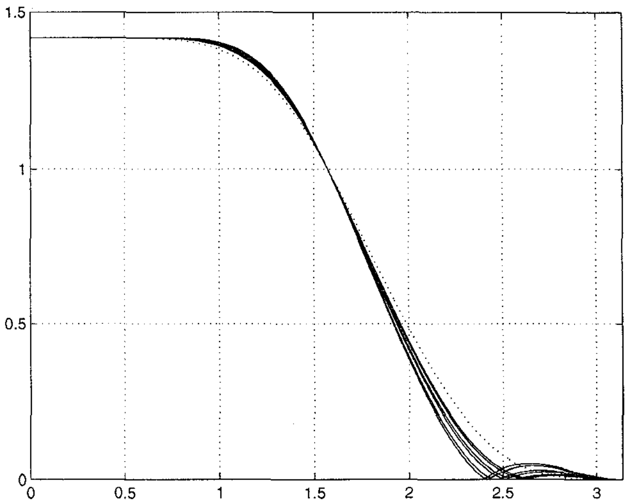

Figure 1: Magnitude responses of orthogonal lowpass wavelet filters with $\omega_a = 0.7\pi,\ 0.72\pi,\ \ldots,\ 0.8\pi$ (real lines) designed using the LP method, and the D_8 lowpass filter.

As an application of the orthogonal wavelets obtained by the proposed parameterization, we apply the $L/K = 2/5$ filter designed using the LP method with $\omega_a = 0.75\pi$ to compress the test signal [10]

$$x(t) = e^{-10t} \sin 100t$$

which is discretized on [0, 1] with 512 samples. The impulse response of the orthogonal filters in the analysis bank are listed in Table 1.

The discrete-time signal then undergoes a nine-scale multiresolution decomposition in which the $L/K = 2/5$ wavelets are used. With a given tolerance ε, the coefficients of the decomposition pass if larger than ε, or set to be zero otherwise. The coefficients so modified are then sent off to the receiver end to reconstruct the signal. The ratio of the total number of coefficients to the number of nonzero coefficients after the thresholding, K, is a good indication of compression ratio achievable by this wavelet based system. Table 2 shows the compression results for tolerance ε varying from 0.02 to 0.08. The term error in the table is defined as

$$\text{error} = \frac{||x - x_r||_2}{||x||_2}$$

where x and x_r denotes the input and reconstructed discrete-time signals, respectively. For comparison, the compression results obtained by using the D_8 filters are also listed in Table 2.

Another application of the wavelets is noise removal. We apply the same $L/K = 2/5$ wavelet filters to the test signal "Doppler" [11] which has 1024 samples. The denoising method employed here was proposed by Donoho [11][12] in which the wavelet coefficients undergo a "soft" shrinkage with a threshold ε. In our simulations, the hyperbolic shrinkage proposed in [13] was employed,

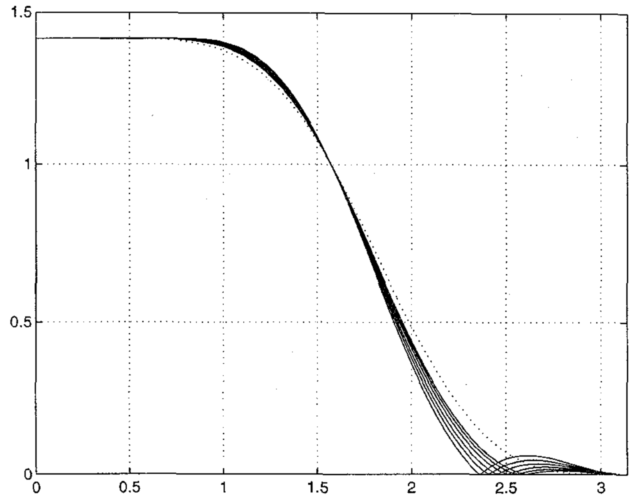

Figure 2: Magnitude responses of orthogonal lowpass wavelet filters with $\omega_a = 0.7\pi,\ 0.72\pi,\ \ldots,\ 0.8\pi$ (real lines) designed using the QP method, and the D_8 lowpass filter.

Table 1. Impulse response of the analysis wavelet filters

h_0	h_1
−0.01763201482469	−0.28526059019532
0.04588107436382	0.74228966354408
0.03764458954295	−0.56882238821696
−0.19285727158956	−0.05519545707580
−0.05519545707580	0.19285727158956
0.56882238821696	0.03764458954295
0.74228966354408	−0.04588107436382
0.28526059019532	−0.01763201482469

Table 2 Compression results of the sine-exponential signal

ε	D_8		$L/K = 2/5$	
	K	error	K	error
0.02	8.8276	0.0156	10.4490	0.0198
0.03	10.2400	0.0254	12.1905	0.0273
0.04	11.6364	0.0347	13.1282	0.0327
0.05	12.1905	0.0387	14.2222	0.0401
0.06	14.2222	0.0537	15.0588	0.0459
0.07	15.0588	0.0600	16.0000	0.0530
0.08	16.0000	0.0669	16.5161	0.0576

and both the universal thresholding proposed by Donoho [12] and the cross-validation-based thresholding proposed by Nason [14] are tried. The denoising results using the $L/K = 2/5$ filters as well as the D_8 filters are listed in Table 3, where SNR1 and SNR2 are the signal-to-noise ratio before and after the denoising, respectively.

Table 3 Denoising results for the Doppler test signal

		Universal thresholding	Cross-validation thresholding
D_8	SNR1	14.0750	14.0750
	SNR2	19.6312	19.5450
L/K	SNR1	14.0750	14.0750
$= 2/5$	SNR2	19.8708	19.9394

From the above simulations, it appears that the family of orthognal wavelet filters generated by the proposed parameterization contains promissing members for improved performance in several important application fields.

5. Conclusion

We have proposed a purely algebraic method for the parameterization of compactly supported orthogonal wavelet filters. The parameterization provides a means to search wavelet filters more suitable for specific applications. As the parameterization is done through a *linear* characterization of the associated halfband filters, such a search can be accomplished efficiently by using fast optimization techniques such as linear and quadratic programming.

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