Design Methods for Partial Filter Banks Using Interpolation

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ABSTRACT—Two iterative algorithms for the design of partial filter banks using interpolation are described. The design problems are formulated as a 4th-order nonlinear optimization problems in which the objective function is a weighted sum of a reconstruction-error term and an aliasing-error term. The optimization problem is solved by iteratively minimizing a pair of quadratic functions. Explicit formulas for evaluating the minimums of these quadratic functions are described, which lead to an efficient and fast algorithm. Two design examples are presented to illustrate the proposed algorithms. The designed partial filter banks are used to approximate bathymetric waveforms and electrocardiagrams (ECGs).

I. INTRODUCTION

Filter banks based on multirate digital signal processing (DSP) systems have recently gained great popularity in applications such as digital audio/speech coding, progressive image coding, spectrum analysis, time-varying system identification, and subband adaptive filtering [1]-[4]. Fig. 1 illustrates a typical M-channel maximally decimated filter bank where $K = M$, and $H_k(z)$ and $G_k(z)$ $(k = 0, \cdots, K)$ are the transfer functions of the analysis and synthesis filters, respectively. The analysis filters split the input signal into M subband signals which are then downsampled by a factor M. At the other end of the system, the subband signals are upsampled first, filtered by the synthesis filters, and are then combined to reconstruct the input signal. The function blocks which perform the subsampling and upsampling are referred to as downsampler and expander, respectively. If the filter bank is a perfect reconstruction system, the reconstructed signal is an exact copy of the input signal although a delay is introduced.

Partial filter banks (PFB) are multirate systems in which only some channels of a conventional filter bank are utilized in order to achieve reduced implementation cost and improved processing efficiency. A typical PFB has the structure shown in Fig. 1 where $K < M$. Such a system will be referred to as an M/K *partial filter bank* and the K channels involved are called *principal channels.*

In this paper, we consider an M-band PFB using interpolation as shown in Fig. 2. In this M/K PFB, each zero-pading expander in Fig. 1 is replaced by an interpolation block which interpolates the decimated subsignal in a sample-and-hold or linear manner. For signals which represent continuous waveforms with only a small number of peaks, interpolation of this type yields improved approximation accuracy or higher compression ratio than a conventional M/K PFB.

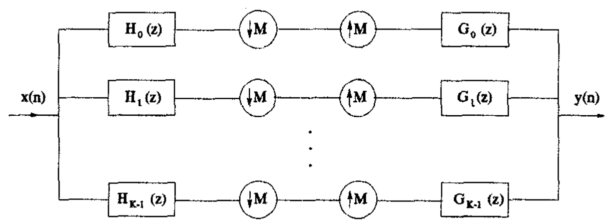

Fig. 1. An M-band filter bank $(K = M)$ or an M/K PFB $(K < M)$.

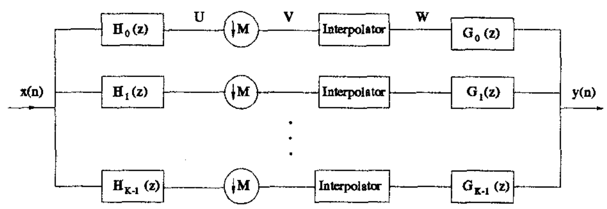

Fig. 2. Partial filter bank with interpolation blocks.

We propose two algorithms for the design of partial filter banks in which the interpolation blocks use sample-and-hold or linear interpolation strategies. The design problems are formulated as 4th-order nonlinear optimization problems in which the objective function is a weighted sum of a reconstruction-error term and an aliasing-error term. The optimization problem is solved by iteratively minimizing a pair of quadratic functions. Explicit formulas for evaluating these quadratic functions are derived, which lead to an efficient and fast algorithm.

II. DESIGN OF PFB WITH INTERPOLATION EXPANDER

A. Objective Function

The transfer function of a downsampling unit which takes one sample and discards the $M-1$ subsequent sam-

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ples is given by

$$V(z) = \frac{1}{M} \sum_{k=0}^{M-1} U(z^{1/M} e^{2\pi jk/M})$$

For the PFB depicted in Fig. 2, each zero-padding expander is replaced by an interpolation block which interpolates the decimated subsignal with either a zero-order hold or a 1st-order hold. The transfer functions of a zero-order and 1st-order interpolator are given by

$$W(z) = V(z^m)\Phi(z)$$

where

$$\Phi(z) = \begin{cases} \sum_{i=0}^{M-1} z^{-i} & \text{for zero-order hold} \\ \Psi(z) & \text{for 1st-order hold} \end{cases}$$

$$\Psi(z) = 1 + \frac{1}{M} \sum_{i=1}^{M-1} (iz^{M-i} - iz^{-i} + Mz^{-i})$$

It follows that

$$W(z) = \frac{\Phi(z)}{M} \sum_{k=0}^{M-1} U(ze^{2\pi jk/M})$$

Therefore, the input-output relation in an M/K PFB is given by

$$\begin{aligned} &Y(z) \\ &= \sum_{k=0}^{K-1} Y_k(z) \\ &= \frac{\Phi(z)}{M} \sum_{k=0}^{K-1} G_k(z) \sum_{l=0}^{M-1} H_k(ze^{2\pi jl/M}) X(ze^{2\pi jl/M}) \\ &= \left[\frac{\Phi(z)}{M} \sum_{k=0}^{K-1} G_k(z) H_k(z) \right] X(z) + \\ &\left[\frac{\Phi(z)}{M} \sum_{l=1}^{M-1} \sum_{k=0}^{K-1} G_k(z) H_k(ze^{2\pi jl/M}) \right] X(ze^{2\pi jl/M}) \quad (1) \end{aligned}$$

In (1), the first term is the reconstruction-error term while the second term is an aliasing-error term, and the design is to find $H_k(z)$ and $G_k(z)$ for $1 \le k \le K$ such that

$$\frac{\Phi(z)}{M} \sum_{k=0}^{K-1} G_k(z) H_k(z) \approx z^{-d}$$

$$\frac{1}{M} \sum_{l=1}^{M-1} |\Phi(z) \sum_{k=0}^{K-1} G_k(z) H_k(ze^{2j\pi l/M})|^2 \approx 0$$

where d is the system delay. If the analysis and synthesis filters are symmetric FIR filters with odd length N, then $d = N - 1$. A weighted objective function can be constructed as

$$E = E_1 + \alpha E_2 \quad (2a)$$

where α is a weight and

$$E_1 = \int_0^\pi \left| \frac{\Phi}{M} \sum_{k=0}^{K-1} G_k H_k - e^{-jd\omega} \right|^2 d\omega \quad (2b)$$

$$E_2 = \sum_{l=1}^{M-1} \int_0^\pi |\Phi \sum_{k=0}^{K-1} G_k H_k(e^{j\mu})|^2 d\omega \quad (2c)$$

$$\mu = \omega + 2\pi l/M \quad (2d)$$

B. Closed-Form Formulas for E_1 and E_2

By (2), minimizing E in (2a) is a typical unconstrained, 4th-order optimization problem. The problem can be solved by iteratively minimizing a pair of quadratic functions whose minimum points can be obtained analytically. To this end, we write

$$E_1 = \frac{1}{M^2} E_{11} - \frac{2}{M} \text{ real } \{E_{12}\} + \pi \quad (3a)$$

where

$$E_{11} = \int_0^\pi |\Phi \sum_{k=0}^{K-1} G_k H_k|^2 d\omega \quad (3b)$$

$$E_{12} = \int_0^\pi \Phi \sum_{k=0}^{K-1} G_k H_k e^{-jd\omega} d\omega \quad (3c)$$

By expressing the frequency response of filter G_i as

$$G_i(e^{j\omega}) = e^{-j\omega(N-1)/2} \mathbf{g}_i^T \mathbf{c}(\omega)$$

with

$$\mathbf{g}_i^T = [g_{i,\frac{N-1}{2}} \;\; 2g_{i,\frac{N-3}{2}} \cdots 2g_{i1} \;\; 2g_{i0}]$$

$$\mathbf{c}(\omega) = [1 \;\; \cos\omega \cdots \cos(N-1)\omega/2]^T$$

We have

$$E_{11} = \mathbf{g}^T \mathbf{P_1}(\mathbf{h}) \, \mathbf{g} \quad (4a)$$

$$E_{12} = \mathbf{g}^T \mathbf{b}(\mathbf{h}) \quad (4b)$$

$$E_2 = \mathbf{g}^T \mathbf{P_2}(\mathbf{h}) \mathbf{g} \quad (4c)$$

where $\mathbf{P_1}(\mathbf{h})$, $\mathbf{b}(\mathbf{h})$, and $\mathbf{P_2}(\mathbf{h})$ are defined as

$$\mathbf{P_1}(\mathbf{h}) = \int_0^\pi W(\omega) W(\omega)^H \Phi(\omega) d\omega \quad (4d)$$

$$\mathbf{b}(\mathbf{h}) = \int_0^\pi W(\omega) e^{-j(d - \frac{N-1}{2})\omega} d\omega \quad (4e)$$

$$\mathbf{P_2}(\mathbf{h}) = \sum_{l=1}^{M-1} \int_0^\pi W_l(\omega) W_l(\omega)^H \Phi(\omega) d\omega \quad (4f)$$

$$C(\omega) = \begin{bmatrix} c(\omega) & 0 & \cdots & 0 \\ 0 & c(\omega) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & c(\omega) \end{bmatrix} \tag{4g}$$

$$\mathbf{g}^T = [\mathbf{g}_0^T \ \mathbf{g}_1^T \ \cdots \ \mathbf{g}_{K-1}^T] \tag{4h}$$

$$W(\omega) = C(\omega)[H_0(e^{-j\omega}) \cdots H_{K-1}(e^{-j\omega})]\Phi(e^{j\omega})$$

$$W_l(\omega) = C(\omega)[H_0(e^{-j\mu}) \cdots H_{K-1}(e^{-j\mu})]\Phi(e^{j\omega})$$

From (4), the objective function in (2a) can be expressed as

$$E(\mathbf{g}) = \mathbf{g}^T \left[\frac{\mathbf{P_1(h)}}{M^2} + \alpha \mathbf{P_2(h)}\right] \mathbf{g} - \mathbf{g}^T \frac{2}{M}\mathbf{b(h)} + \pi \tag{5a}$$

For a fixed $\mathbf{h}$, this is a quadratic function. The $\mathbf{g}$ that minimizes $E(\mathbf{g})$ is given by

$$\mathbf{g} = \frac{1}{M}\left[\frac{\mathbf{P_1(h)}}{M^2} + \alpha \mathbf{P_2(h)}\right]^{-1} \mathbf{b(h)} \tag{5b}$$

Likewise, for a fixed $\mathbf{g}$, the objective function in (2a) can be expressed as

$$E(\mathbf{h}) = \mathbf{h}^T \left[\frac{\mathbf{P_1(g)}}{M^2} + \alpha \mathbf{P_2(g)}\right] \mathbf{h} - \mathbf{h}^T \frac{2}{M}\mathbf{b(g)} + \pi \tag{6a}$$

Hence the $\mathbf{h}$ that minimizes $E(\mathbf{h})$ is given by

$$\mathbf{h} = \frac{1}{M}\left[\frac{\mathbf{P_1(g)}}{M^2} + \alpha \mathbf{P_2(g)}\right]^{-1} \mathbf{b(g)} \tag{6b}$$

C. Evaluation of $\mathbf{P_1}, \mathbf{P_2}$, *and* $\mathbf{b}$

The purpose of this subsection is to show that matrices $\mathbf{P_1}, \mathbf{P_2}$, and $\mathbf{b}$ used in (5b) and (6b) can be evaluated efficiently.

Writing

$$H_i(e^{j\omega}) = e^{-j\omega(N-1)/2}\mathbf{h}_i^T \mathbf{c}(\omega),$$

with

$$\mathbf{h}_i^T = [h_{i\,\frac{N-1}{2}} \ \ 2h_{i\,\frac{N-3}{2}} \cdots 2h_{i1} \ \ 2h_{i0}]$$

it can be verified that

$$\mathbf{P_1(h)} = \mathbf{A}\left[\int_0^\pi |\Phi(w)c_1(\omega)|^2 d\omega\right]\mathbf{A}^T \tag{7a}$$

$$\mathbf{b(h)} = \mathbf{A}\left[\int_0^\pi \Psi(w)c_1(\omega) d\omega\right]\mathbf{A}^T \tag{7b}$$

$$\mathbf{P_2(h)} = \sum_{l=1}^{M-1} \mathbf{A}\left[\int_0^\pi |\Phi(w)c_2(\omega)|^2 d\omega\right]\mathbf{A}^T \tag{7c}$$

$$c_1(\omega) = c(\omega) \otimes c(\omega)^T \tag{7d}$$

$$c_2(\omega) = c(\omega) \otimes c(\mu)^T \tag{7e}$$

where $\otimes$ denotes the Kronecker product. $\mathbf{A}$ is a $(N+1)K/2 \times [(N+1)/2]^2$ matrix defined by

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_1} \\ \mathbf{A_2} \\ \vdots \\ \mathbf{A_{K-1}} \end{bmatrix}$$

$$\mathbf{A_i} = \begin{bmatrix} \mathbf{h}_i^T & 0 & \cdots & 0 \\ 0 & \mathbf{h}_i^T & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \mathbf{h}_i^T \end{bmatrix}$$

$$|\Phi(w)|^2 = \begin{cases} M + 2\xi & \text{for zero-order hold} \\ \left[1 + \frac{2}{M}\xi\right]^2 & \text{for 1st-order hold} \end{cases}$$

$$\Psi(w) = \begin{cases} \sum_{n=0}^{M-1} \cos(nw) & \text{for zero-order hold} \\ 1 + \frac{2}{M}\xi & \text{for 1st-order hold} \end{cases}$$

$$\xi(\omega) = \sum_{n=1}^{M-1} (M-n)\cos(nw)$$

D. Iterative Algorithm

In the ith iteration, the iterative algorithm starts with a fixed $\mathbf{h} = \mathbf{h}^{(i-1)}$. By evaluating $\mathbf{P_1(h)}, \mathbf{P_2(h)}$, and $\mathbf{b(h)}$, $\mathbf{g}^{(i)}$ can be computed using (5b). The $\mathbf{g}^{(i)}$ is then updated as

$$\mathbf{g}^{(i)} = \tau \mathbf{g}^{(i)} + (1-\tau)\mathbf{g}^{(i-1)} \tag{8}$$

where τ is a smoothing parameter between 0 and 1. Likewise, with a fixed $\mathbf{g} = \mathbf{g}^{(i-1)}$, $\mathbf{P_1(g)}, \mathbf{P_2(g)}$, and $\mathbf{b(g)}$ can be found and $\mathbf{h}^{(i)}$ can be computed using (6b). The $\mathbf{h}^{(i)}$ is then updated as

$$\mathbf{h}^{(i)} = \tau \mathbf{h}^{(i)} + (1-\tau)\mathbf{h}^{(i-1)} \tag{9}$$

The algorithm terminates when $||\mathbf{g}^{(i)} - \mathbf{g}^{(i-1)}|| + ||\mathbf{h}^{(i)} - \mathbf{h}^{(i-1)}||$ is less than a prescribed tolerance ϵ. A step-by-step describption of the algorithm is given below:

Algorithm

Step 1 Using a conventional method (e.g., the window method), design K linear-phase bandpass FIR filters of odd-length N, and construct an initial coefficient vector $\mathbf{h}^{(0)}$ and an initial $\mathbf{g}^{(0)}$.

Step 2 Use (7) and (5b) to compute $\mathbf{g}^{(i)}$.

Step 3 Use (8) to update $\mathbf{g}^{(i)}$

Step 4 Use (7) and (6b) to compute $\mathbf{h}^{(i)}$.

Step 5 Use (9) to update $\mathbf{h}^{(i)}$

Step 6 If $||\mathbf{g}^{(i)} - \mathbf{g}^{(i-1)}|| + ||\mathbf{h}^{(i)} - \mathbf{h}^{(i-1)}|| < \epsilon$, output $\mathbf{h} = \mathbf{h}^{(i)}$ and $\mathbf{g} = \mathbf{g}^{(i)}$ as the design results and stop. Otherwise, set $i = i + 1$ and repeat from Step 2.

III. DESIGN EXAMPLES AND APPLICATIONS

The proposed algorithms were implemented with MATLAB to design two 4/1 PFB's with design parameters

$N = 13$, $\alpha = 0.1$, $\tau = 0.5$, $\epsilon = 10^{-6}$. The results are summarized in Tables I and II.

TABLE I
COEFFICIENTS OF THE PFB WITH ZERO-ORDER INTERPOLATOR

i	h_i	g_i
0	-0.03427807250118	-0.03607030976537
1	-0.01832235281052	-0.01982350663395
2	0.09284169189923	0.09573630936347
3	-0.04083776503526	-0.04306650229149
4	0.17601125854477	0.18100443196475
5	0.29908018284526	0.30854700383988
6	0.02811156877670	0.02843840125223

TABLE II
COEFFICIENTS OF THE PFB WITH 1ST-ORDER INTERPOLATOR

i	h_i	g_i
0	-0.03313605020188	-0.04232378020502
1	-0.04108901109332	-0.05250207108116
2	-0.01005818042629	-0.01306375437982
3	0.05226778252075	0.06620308150825
4	0.13674122157149	0.17373367535213
5	0.20722577776126	0.26347107018263
6	0.23194334187114	0.29493678406580

The partial filter banks were applied to compress bathymetric signals and electrocardiagrams (ECGs). As a motivation of the approximation, we note that a typical electrocardiagram monitoring device can generate a large amount of data. The approximated signals achieved with compression ratio of 4 are illustrated in Fig. 3. The mean-square errors between the original and the compressed signals are given in Table III.

It is worthwhile to note that an $M/1$ PFB system involves only one channel and, hence it can be implemented efficiently. For applications such as bathymetric data processing, an $M/1$ system can be split into two parts: the function from the input to point S (see Fig. 4) can be performed on site and can be implemented as part of the bathymeter in the aircraft in order to reduce the data storage by a factor of M. The rest of the system can be implemented at the ground station. Note that for each digitized waveform, the on-site part of the system would involve only discrete convolution and subsampling which can be reaily performed in real time.

IV. CONCLUSION

We have proposed two iterative algorithms for the design of partial filter banks in which the interpolation blocks use zero-order or linear interpolation strategies. The design problem is formulated as a 4th-order nonlinear optimization problem in which the objective function is constructed as a weighted sum of a reconstruction-error term and an aliasing-error term. The objective function is converted into a pair of quadratic functions and solved by iteratively minimizing the pair of quadratic functions. Explicit formulas for evaluating the integrals involed are derived for fast and precise implementation. The algorithm is illustrated by designing two PFBs which are used to approximate bathymetric and ECG waveforms.

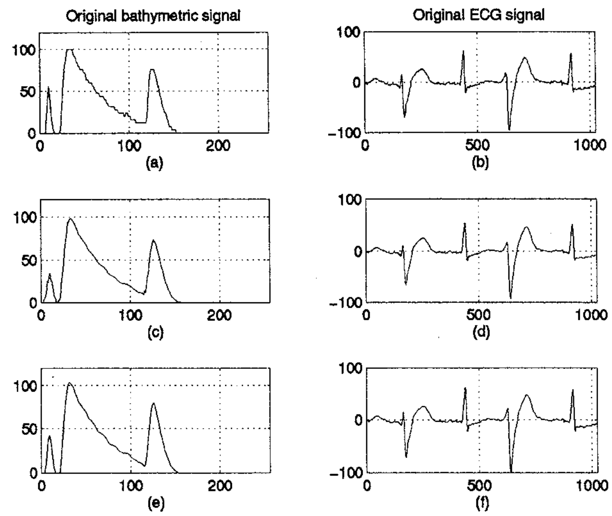

Figure 3: Test signals and their approximations. (a),(b) original test signals; (c),(d) outputs of the PFB with sample-and-hold interpolation; (e),(f) outputs of the PFB with linear interpolation.

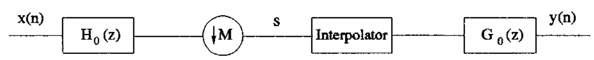

Figure 4: An $M/1$ partial filter bank.

TABLE III
MEAN-SQUARE ERROR OF COMPRESSED SIGNALS

Figure	Mean-Square Error
c	0.0820
d	0.0909
e	0.0728
f	0.0801

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