Design of Perfect Reconstruction QMF Banks: A Parameterization Method

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Abstract

This paper describes a two-step filter bank design method that (1) parameterizes, for a given lowpass analysis filter of order N, all lowpass synthesis filters of order M that will form 2-channel, linear-phase, perfect reconstruction QMF filter banks in terms of a 'free' parameter vector in the null-space of a matrix characterized by the impulse response of the given lowpass analysis filter; and (2) utilizes this parameter vector to shape up the frequency response of the lowpass synthesis filter using a quadratic programming technique.

1. Introduction

Two channel perfect reconstruction QMF banks find applications in a variety of DSP tasks, and several methods for the design of this class of filter banks are available in the literature [1]–[3]. This paper describes a parameterization method for the design of two-channel, linear-phase, FIR filter banks with the perfect reconstruction (PR) property. The design method consists of two steps: a null-space parameterization that characterizes all QMF filter banks of a given order that satisfy the PR condition; and a quadratic programming (QP) based optimization to ensure that the filters involved in the multirate system have desired frequency characteristics. Relevant to this work, a null-space projection method for QMF bank design can be found in [4]. There are two drawbacks in the method proposed in [4]: it cannot be applied to odd length cases; and the least-squares design used in [4] may lead to unsatisfactory frequency responses for the lowpass filter in the synthesis bank. On the other hand, the present method is applicable to both even length and odd length cases, and the QP optimization technique enables the user to design individual filters of the multirate system with equal ripple passband and peak-constrained least-squares stopband [5].

2. The Design Method

2.1. Filter Bank Parameterization

Consider the 2-channel filter bank shown in Fig. 1. The input and output relation of the system is given by

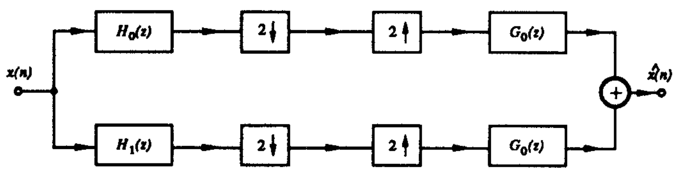

Figure 1. 2-channel filter bank.

$$\hat{X}(z) = \frac{1}{2}[H_0(z)G_0(z) + H_1(z)G_1(z)]X(z) + \frac{1}{2}[H_0(-z)G_0(z) + H_1(-z)G_1(z)]X(-z)$$

By assigning $G_1(z) = -H_0(-z)$ and $H_1(z) = G_0(-z)$, the aliasing term is eliminated and the PR property holds if

$$H_0(z)G_0(z) - H_0(-z)G_0(-z) = z^{-k_d} \qquad (1)$$

where k_d is the system delay. Let

$$H_0(z) = \sum_{i=0}^{N-1} h_i z^{-i}, \quad G_0(z) = \sum_{i=0}^{M-1} g_i z^{-i}$$

then the system delay is

$$k_d = \frac{N+M}{2} - 1$$

if linear phase response of the individual filters are required. The basic idea of designing a perfect reconstruction filter bank using the proposed parameterization technique is to first design a half-band, linear-phase lowpass FIR filter $H_0(z)$ and then use $H_0(z)$ to design $G_0(z)$ whose order is higher than that of $H_0(z)$ so that the null space of a matrix associated with (1) becomes nontrivial and can be used to reshape $G_0(z)$.

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Case 1: M and N are even integers.
This is the case studied in [4], from which we know that $N+M$ has to be a multiple of 4, and the dimension of the null-space characterized by the PR condition is $\frac{M-N}{4}$. See [4] for the details.

Case 2: M and N are odd integers.
Again, we assume $(M+N)/4$ is an integer. Note that this assumption in conjunction with assuming N is odd implies that $\eta \equiv \frac{M-N+2}{4}$ is an integer. Later we will show that η is the degree of freedom of the system that can be used to shape up filter $G_0(z)$. Another note before we proceed: if we let $M=N+L$ and let $L=2P$, then $(M+N)/4=(N+P)/2$. Hence if N and M are odd integers, then $(M+N)/4$ is an integer if and only if $M-N=2P$ with P odd. The PR condition in this case can be expressed as

$$Hg = m \tag{2}$$

where $H = [b_1 + b_M \;\; b_2 + b_{M-1} \;\; \cdots \;\; b_{(M-1)/2} + b_{(M+3)/2} \;\; b_{(M+1)/2}]$ is a matrix of size $(M+N)/4 \times (M+1)/2$, where b_i is the ith column of the matrix

$$B = \left[\begin{array}{ccccccc|c} h_1 & h_0 & 0 & 0 & \dots & 0 & 0 & \\ h_3 & h_2 & h_1 & h_0 & \dots & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\ h_{N-2} & h_{N-3} & & & & & & \mathbf{0} \\ 0 & h_{N-1} & h_{N-2} & h_{N-3} & \dots & 0 & 0 & \\ 0 & 0 & 0 & h_{N-1} & \dots & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & h_1 & h_0 & \end{array}\right]$$

$$g = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{(M-1)/2} \end{bmatrix}, \quad m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^{[(M+N)/4] \times 1}$$

where $B \in R^{[(M+N)/4] \times M}$. If H has full row rank, i.e., rank $(H) = (M+N)/4$, then the dimension of the null space of operator H is η. And the general solution of (2) is given by

$$g = H^+ m + (I - H^+ H)\phi^* \tag{3}$$

where H^+ denotes the Moore-Penrose pseudo-inverse of H, and ϕ^* is an arbitrary column vector of dimension $(M+1)/2$. Note that

$$H(I - H^+ H)\phi^* = 0$$

i.e., the second term on the right-hand side of (3) represents a vector in the null space of H. In other words, $(I - H^+ H)$ projects an arbitrary ϕ^* onto the null space of H. Let

$$H = U \left[\begin{array}{ccc|c} \sigma_1 & 0 & & \\ & \ddots & & 0 \\ 0 & & \sigma_{\frac{M+N}{4}} & \end{array}\right] V^T$$

be the singular value decomposition of H [6]. It can be shown that g in (3) can be written as

$$g = s_c + V^* \phi \tag{4}$$

where

$$\begin{aligned} s_c &= H^+ m \\ &= V \begin{bmatrix} \sigma_1^{-1} & & \\ & \ddots & \\ & & \sigma_{\frac{M+N}{4}}^{-1} \\ & 0 & \end{bmatrix} U^T m \\ V^* &= [v_{\frac{M+N}{4}+1} \;\; \cdots \;\; v_{\frac{M+1}{2}}] \\ \phi &= [\phi_1 \; \dots \; \phi_\eta]^T \end{aligned}$$

v_i for $i = (M+N)/4+1, \; \cdots, \; (M+1)/2$ is the ith column of matrix V, and ϕ contains η free parameters which can be used to shape up $G_0(e^{j\omega})$. Details on this matter are given in Sec. 2.2.

2.2. Shaping Up $G_0(e^{j\omega})$: A Quadratic Programming Approach

The parameter vector ϕ in (4) can be used to shape up the frequency response of $G_0(z)$. Specifically, let $M_g(\omega) = 2g^T c(\omega)$ with

$$c(\omega) = \begin{cases} [\cos((M-1)\omega/2) \;\; \cdots \;\; \cos(\omega/2)]^T, & M \text{ even} \\ [\cos((M-1)\omega/2) \;\; \cdots \;\; \cos\omega \;\; 1/2]^T, & M \text{ odd} \end{cases}$$

One seeks to find ϕ in (4) such that

$$E = \int_0^{\omega_p} [M_g(\omega) - 1]^2 d\omega + \alpha \int_{\omega_s}^{\pi} M_g^2(\omega) d\omega \tag{5}$$

is minimized subject to

$$|M_g(\omega) - 1| \le \delta_p \quad \text{for } \omega \in [0, \; \omega_p] \tag{6}$$

$$|M_g(\omega)| \le \delta_s \quad \text{for } \omega \in [\omega_s, \; \pi] \tag{7}$$

With $M_g(\omega) = 2g^T c(\omega)$ where g is given in (4), the objective function E in (5) can be expressed as

$$E = \phi^T Q \phi + 2\phi^T p + \text{const} \tag{8}$$

where

$$
\begin{aligned}
Q &= V^{*^T} Y V^* \\
p &= V^{*^T}(Ys_c - y) \\
Y &= 4\int_0^{\omega_p} c(\omega)c^T(\omega)d\omega + 4\alpha\int_{\omega_s}^{\pi} c(\omega)c^T(\omega)d\omega \\
y &= 2\int_0^{\omega_p} c(\omega)d\omega
\end{aligned}
$$

Note that the integrals involved in Y and y can be evaluated in closed form. The constraints in (6), (7) are usually implemented at grid points on the associated frequency ranges as

$$
\begin{aligned}
1-\delta_p \le 2c^T(\omega_i)g &\le 1+\delta_p & \omega_i &\in [0,\ \omega_p] \\
-\delta_s \le 2c^T(\omega_j)g &\le \delta_s & \omega_j &\in [\omega_s,\ \pi]
\end{aligned}
$$

By (4), the above constraints can be expressed as

$$2c^T(\omega_i)V^*\phi \le 1+\delta_p - 2c^T(\omega_i)s_c \quad (9a)$$
$$-2c^T(\omega_i)V^*\phi \le -1+\delta_p + 2c^T(\omega_i)s_c \quad (9b)$$
$$2c^T(\omega_j)V^*\phi \le \delta_s - 2c^T(\omega_j)s_c \quad (9c)$$
$$-2c^T(\omega_j)V^*\phi \le \delta_s + 2c^T(\omega_j)s_c \quad (9d)$$

where $\omega_i \in [0,\ \omega_p]$, $\omega_j \in [\omega_s, \pi]$. If n_p grid points in the passband and n_s grid points in the stopband are used, then (9) can be written as

$$A\phi \le b \quad (10)$$

where $A \in R^{2(n_p+n_s)\times\eta}$, $b \in R^{2(n_p+n_s)\times 1}$. Minimizing (8) with respect to ϕ subject to constraints (10) is a typical quadratic programming (QP) problem. It can readily be shown that the Hessian matrix of E, which is the matrix $Q = V^{*^T}YV^*$, is always positive definite. Consequently, the QP problem has a unique global minimum, and efficient algorithms for computing the minimum exist [7]. By taking a sufficiently large weight α in (8) (see the expression of Y), the solution of this QP problem is a linear-phase lowpass filter with virtually equiripple passband and peak-constrained least squares stopband.

3. Design Examples

We now present two design examples to illustrate the proposed algorithm. The first example is to design a linear-phase, perfect-reconstruction QMF bank with $N = 23$, $M = 53$, $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$ (the Nyquist frequency has been normalized to π). The half-band lowpass analysis filter H_0 was designed using the method proposed in [5]. The result is a linear-phase FIR filter with equiripple passband and peak-constrained least squares stopband. The first 12 coefficients of $H_0(z)$ are listed in Table 1, and the amplitude response of $H_0(z)$ is depicted in Fig. 2a. The degree of freedom created by the order difference is $\eta = (M-N+2)/4 = 8$. With the 8-dimensional parameter vector ϕ in (4), the lowpass synthesis filter G_0 was designed using the proposed method. Its passband ripple and stopband attenuation were well controlled by parameters δ_p, δ_s, n_p, n_s, and α. With $\delta_p = 3\times 10^{-2}$, $\delta_s = 5\times 10^{-3}$, $n_p = 20$, $n_s = 30$, and $\alpha = 10^3$, $G_0(z)$ was designed to obtain a linear-phase FIR filter with frequency response comparable to that of $H_0(z)$, see Fig. 2a where the highpass amplitude response is just a mirror image of $|G_0(e^{j\omega})|$. The first 27 coefficients of $G_0(z)$ are listed in Table 1. As is expected, the reconstruction error of the filter bank, which is defined as $e(\omega) = |H_0(e^{j\omega})G_0(e^{j\omega}) + H_1(e^{j\omega})G_1(e^{j\omega}) - e^{-jk_d\omega}|$, is virtually zero as can be observed from Fig. 2b.

Table 1: Example 1

n	$H_0(z)$	$G_0(z)$
0	−3.8996090e−03	−2.4709327e−05
1	3.5660407e−03	−2.2595718e−05
2	1.4005533e−02	1.5276840e−04
3	−2.1636543e−03	7.2257428e−05
4	−2.5110868e−02	−5.1678119e−04
5	3.8838177e−03	−1.7693195e−04
6	4.7913403e−02	1.3601796e−03
7	−4.3907051e−03	3.3216074e−04
8	−9.5943813e−02	−3.1157688e−03
9	4.8369791e−03	−5.4524916e−04
10	3.1479952e−01	7.6104303e−03
11	4.9500672e−01	−2.1952476e−03
12		−1.2424316e−02
13		6.5716989e−03
14		1.9823505e−02
15		−1.2612712e−02
16		−3.0349349e−02
17		1.7562517e−02
18		3.7010645e−02
19		−3.1983807e−02
20		−4.5797634e−02
21		5.2920174e−02
22		5.2313684e−02
23		−9.9182181e−02
24		−5.7202397e−02
25		3.1651175e−01
26		5.5791165e−01

The purpose of the second example is to demonstrate the proposed method can handle higher order design problems with satisfactory results. We use the algorithm to design an FIR QMF bank with $N = 31$, $M = 81$, $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$. Other parameters involved in the design are $\delta_p = 3\times 10^{-2}$, $\delta_s =$

5×10^{-4}, $n_p = 20$, $n_s = 36$ and $\alpha = 1000$. The first 16 coefficients of $H_0(z)$ and first 41 coefficients of $G_0(z)$ are listed in Table 2, and their amplitude responses as well as the reconstruction error are depicted in Fig. 3.

4. Concluding Remarks

We have proposed a parameterization approach to designing QMF's with the perfect reconstruction property. Since the QP problem involved can be carried out effectively even for designs of fairly large sizes, the algorithm is fast compared to many existing design methods that are iterative in nature. With sufficient degrees of freedom in the null space, it is also possible to use the proposed technique to design QMF systems that have other desirable properties such as regularity and minimized finite wordlength effects. Results on this matter will be reported elsewhere.

Acknowledgment

The author is grateful to Micronet, Networks of Centres of Excellence Program and to the Natural Sciences and Engineering Research Council of Canada, for supporting this work.

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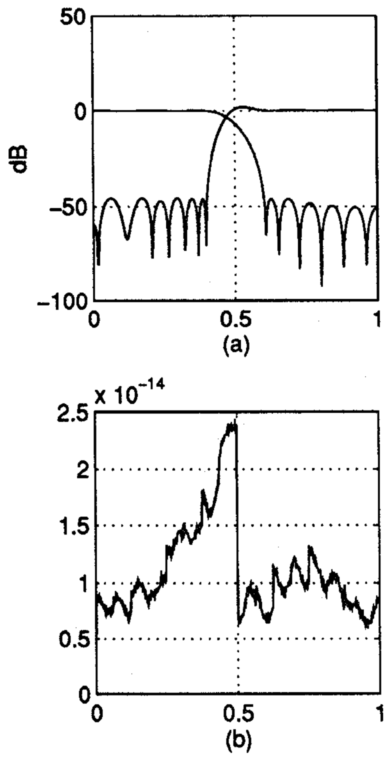

Figure 2. Example 1: (a) amplitude responses of $H_0(z)$ and $H_1(z)$; (b) reconstruction error.

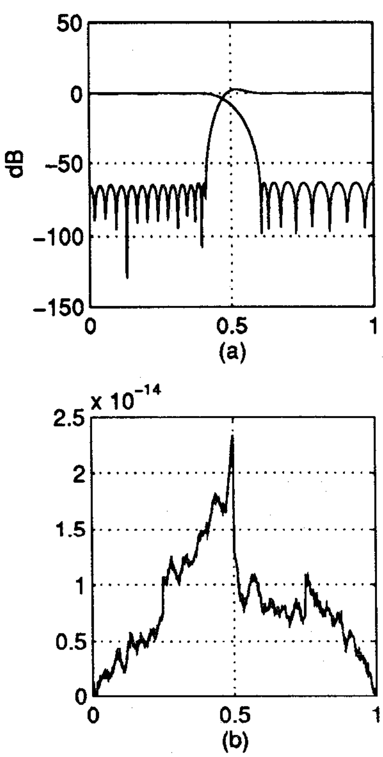

Figure 3. Example 2: (a) amplitude responses of $H_0(z)$ and $H_1(z)$; (b) reconstruction error.

Table 2: Example 2

n	$H_0(z)$	$G_0(z)$
0	−1.6336229e−05	1.3451681e−08
1	2.7471598e−03	2.2620803e−06
2	3.6886293e−03	−3.0730080e−06
3	−3.3505722e−03	−8.7620745e−06
4	−7.8492446e−03	1.4602258e−05
5	5.1707332e−03	2.4789105e−05
6	1.5465005e−02	−4.7774439e−05
7	−7.1011861e−03	−5.8679257e−05
8	−2.8146537e−02	1.2911532e−04
9	8.8688098e−03	1.2369855e−04
10	5.0106635e−02	−3.1064070e−04
11	−1.0409727e−02	−2.3929509e−04
12	−9.7487438e−02	6.9783357e−04
13	1.1427981e−02	4.9167090e−04
14	3.1536429e−01	−1.6770064e−03
15	4.8824260e−01	−3.8888611e−04
16		2.9642692e−03
17		1.6857353e−04
18		−4.9678165e−03
19		5.3225548e−04
20		7.7523937e−03
21		−1.9875071e−03
22		−1.1442007e−02
23		4.6148290e−03
24		1.6194410e−02
25		−9.3241976e−03
26		−2.4125195e−02
27		1.2659500e−02
28		3.0593599e−02
29		−1.9174958e−02
30		−3.8420945e−02
31		2.7507637e−02
32		4.6134346e−02
33		−3.9468619e−02
34		−5.3008140e−02
35		5.9537617e−02
36		5.8568417e−02
37		−1.0368954e−01
38		−6.2145032e−02
39		3.1750967e−01
40		5.6336138e−01