A Weighted Quasi-Balanced Model Reduction Method for 2-D Discrete Systems

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Abstract–**A new weighted quasi-balanced model reduction method for two-dimensional discrete systems with input and output weights is developed. The development of the new method can be divided into three distinct stages. First is the introduction of four auxiliary Lyapunov equations by which weighted controllability and observability quasi-gramians are defined. The second is the computation of the weighted quasi-gramians. The third is the definition of the weighted quasi-balanced realization. A 2-D filter of order (4, 5) is obtained from an original lowpass 2-D filter of order (4, 8) using the proposed method along with another lowpass filter as input weight. Comparisons of the new weighted model reduction method with the existing unweighted quasi-balanced model reduction method are included.**

I. Introduction

The balanced approximation is an effective method for the reduction of the system order in one-dimensional (1-D) and two-dimensional (2-D) systems. Among the existing balanced realization methods for 2-D discrete systems, the quasi-balanced realization method is computationally the most efficient [1]. If the requirements imposed on the approximation error are different for different frequency ranges, improved reduced-order systems can be obtained by applying a weighted balanced approximation method. The weighted structurally balanced realization for discrete 2-D systems can be carried out as described in [2]; however, the amount of computation required is very intensive.

The goal of this paper is to extend the quasi-balanced realization method [1] to a new weighted quasi-balanced model reduction method. The development of the new method can be divided into three distinct stages. First is the introduction of the four *auxiliary Lyapunov equations* by which *weighted controllability and observability quasi-gramians* are defined. The second is the computation of the weighted quasi-gramians. The third is the definition of the *weighted quasi-balanced realization* and reduced-order system. To illustrate the proposed method, a 2-D filter of order (4, 5) is obtained from an original lowpass 2-D filter of order (4, 8) using the proposed method along with another lowpass filter as input weight. Comparisons of the new weighted model reduction method with the existing unweighted quasi-balanced model reduction method are included.

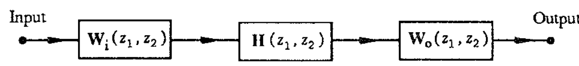

Figure 1: A weighted 2-D discrete system

II. Preliminaries

The system configuration to be considered here is shown in Figure 1, where $\mathbf{H}(z_1, z_2) \in \mathcal{C}^{p\times q}$ is the transfer-function matrix of the system of order (m, n), and $\mathbf{W}_i(z_1, z_2) \in \mathcal{C}^{q\times t}$ and $\mathbf{W}_o(z_1, z_2) \in \mathcal{C}^{s\times p}$ are the transfer-function matrices of the input and output weights of orders (m_i, n_i) and (m_o, n_o), respectively.

If we use the Roesser state-space model [3] to describe the system, then

$$\mathbf{H}(z_1, z_2) = \mathbf{C}\,[\mathbf{I}(z_1, z_2)-\mathbf{A}]^{-1}\,\mathbf{B}+\mathbf{D} \tag{1a}$$

$$\mathbf{W}_i(z_1, z_2) = \mathbf{C}_i\,[\mathbf{I}(z_1, z_2)-\mathbf{A}_i]^{-1}\,\mathbf{B}_i+\mathbf{D}_i \tag{1b}$$

$$\mathbf{W}_o(z_1, z_2) = \mathbf{C}_o\,[\mathbf{I}(z_1, z_2)-\mathbf{A}_o]^{-1}\,\mathbf{B}_o+\mathbf{D}_o \tag{1c}$$

where $\mathbf{A} \in \mathcal{C}^{(m+n)\times(m+n)}$, $\mathbf{B} \in \mathcal{C}^{(m+n)\times q}$, $\mathbf{C} \in \mathcal{C}^{p\times(m+n)}$, $\mathbf{A}_i \in \mathcal{C}^{(m_i+n_i)\times(m_i+n_i)}$, $\mathbf{B}_i \in \mathcal{C}^{(m_i+n_i)\times t}$, $\mathbf{C}_i \in \mathcal{C}^{q\times(m_i+n_i)}$, $\mathbf{A}_o \in \mathcal{C}^{(m_o+n_o)\times(m_o+n_o)}$, $\mathbf{B}_o \in \mathcal{C}^{(m_o+n_o)\times p}$, $\mathbf{C}_o \in \mathcal{C}^{s\times(m_o+n_o)}$, and

$$\mathbf{I}(z_1, z_2) = z_1\mathbf{I} \oplus z_2\mathbf{I}$$

In this equation, the symbol $\oplus$ denotes the direct sum, and $\mathbf{I}$ is the identity matrix. The matrices in-

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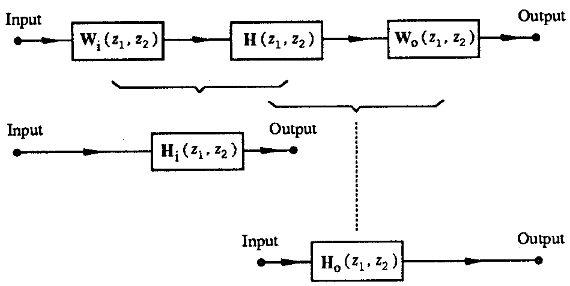

Figure 2: Auxiliary transfer-function matrices

volved in equations (1a), (1b), and (1c) can be partitioned as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_1^T \\ \mathbf{C}_2^T \end{bmatrix}^T$$

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{A}_{i1} & \mathbf{A}_{i2} \\ \mathbf{A}_{i3} & \mathbf{A}_{i4} \end{bmatrix}, \quad \mathbf{B}_i = \begin{bmatrix} \mathbf{B}_{i1} \\ \mathbf{B}_{i2} \end{bmatrix}, \quad \mathbf{C}_i = \begin{bmatrix} \mathbf{C}_{i1}^T \\ \mathbf{C}_{i2}^T \end{bmatrix}^T$$

$$\mathbf{A}_o = \begin{bmatrix} \mathbf{A}_{o1} & \mathbf{A}_{o2} \\ \mathbf{A}_{o3} & \mathbf{A}_{o4} \end{bmatrix}, \quad \mathbf{B}_o = \begin{bmatrix} \mathbf{B}_{o1} \\ \mathbf{B}_{o2} \end{bmatrix}, \quad \mathbf{C}_o = \begin{bmatrix} \mathbf{C}_{o1}^T \\ \mathbf{C}_{o2}^T \end{bmatrix}^T$$

As illustrated in Figure 2, the weighted-input-to-state and the state-to-weighted-output auxiliary transfer-function matrices, $\mathbf{H}_i(z_1, z_2)$ and $\mathbf{H}_o(z_1, z_2)$, are defined as

$$\begin{aligned} \mathbf{H}_i(z_1, z_2) &= [\mathbf{I}(z_1, z_2) - \mathbf{A}]^{-1} \mathbf{B}\, \mathbf{W}_i(z_1, z_2) \\ &= \hat{\mathbf{C}}_i \left[\mathbf{I}(z_1, z_2) - \hat{\mathbf{A}}_i \right]^{-1} \hat{\mathbf{B}}_i \qquad (2a) \\ \mathbf{H}_o(z_1, z_2) &= \mathbf{W}_o(z_1, z_2)\, \mathbf{C}\, [\mathbf{I}(z_1, z_2) - \mathbf{A}]^{-1} \\ &= \hat{\mathbf{C}}_o \left[\mathbf{I}(z_1, z_2) - \hat{\mathbf{A}}_o \right]^{-1} \hat{\mathbf{B}}_o \qquad (2b) \end{aligned}$$

where

$$\begin{aligned} \hat{\mathbf{A}}_i &= \begin{bmatrix} \hat{\mathbf{A}}_{i1} & \hat{\mathbf{A}}_{i2} \\ \hat{\mathbf{A}}_{i3} & \hat{\mathbf{A}}_{i4} \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} \mathbf{A}_1 & \mathbf{B}_1\mathbf{C}_{i1} & \mathbf{A}_2 & \mathbf{B}_1\mathbf{C}_{i2} \\ 0 & \mathbf{A}_{i1} & 0 & \mathbf{A}_{i2} \\ \hline \mathbf{A}_3 & \mathbf{B}_2\mathbf{C}_{i1} & \mathbf{A}_4 & \mathbf{B}_2\mathbf{C}_{i2} \\ 0 & \mathbf{A}_{i3} & 0 & \mathbf{A}_{i4} \end{array} \right] \\ \hat{\mathbf{B}}_i &= \left[\hat{\mathbf{B}}_{i1}^T \quad \hat{\mathbf{B}}_{i2}^T \right]^T \\ &= \left[\mathbf{D}_i^T\mathbf{B}_1^T \quad \mathbf{B}_{i1}^T \;\middle|\; \mathbf{D}_i^T\mathbf{B}_2^T \quad \mathbf{B}_{i2}^T \right]^T \\ \hat{\mathbf{C}}_i &= \left[\hat{\mathbf{C}}_{i1} \quad \hat{\mathbf{C}}_{i2} \right] = \left[\begin{array}{cc|cc} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{array} \right] \\ \hat{\mathbf{A}}_o &= \begin{bmatrix} \hat{\mathbf{A}}_{o1} & \hat{\mathbf{A}}_{o2} \\ \hat{\mathbf{A}}_{o3} & \hat{\mathbf{A}}_{o4} \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} \mathbf{A}_1 & 0 & \mathbf{A}_2 & 0 \\ \mathbf{B}_{o1}\mathbf{C}_1 & \mathbf{A}_{o1} & \mathbf{B}_{o1}\mathbf{C}_2 & \mathbf{A}_{o2} \\ \hline \mathbf{A}_3 & 0 & \mathbf{A}_4 & 0 \\ \mathbf{B}_{o2}\mathbf{C}_1 & \mathbf{A}_{o3} & \mathbf{B}_{o2}\mathbf{C}_2 & \mathbf{A}_{o4} \end{array} \right] \\ \hat{\mathbf{B}}_o &= \left[\hat{\mathbf{B}}_{o1}^T \quad \hat{\mathbf{B}}_{o2}^T \right]^T = \left[\begin{array}{cc|cc} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{array} \right]^T \\ \hat{\mathbf{C}}_o &= \left[\hat{\mathbf{C}}_{o1} \quad \hat{\mathbf{C}}_{o2} \right] \\ &= \left[\mathbf{D}_o\mathbf{C}_1 \quad \mathbf{C}_{o1} \;\middle|\; \mathbf{D}_o\mathbf{C}_2 \quad \mathbf{C}_{o2} \right] \end{aligned}$$

As indicated in Figure 2, transfer-function matrix $\mathbf{H}_i(z_1, z_2)$ in (2a) relates a weighted input signal to the state of $\mathbf{H}(z_1, z_2)$, which takes the input weight $\mathbf{W}_i(z_1, z_2)$ into account. Similarly, transfer-function matrix $\mathbf{H}_o(z_1, z_2)$ in (2b) relates a state to the weighted output, which takes the output weight $\mathbf{W}_o(z_1, z_2)$ into account.

III. Weighted Quasi-Gramians

A. Auxiliary Lyapunov Equations

We first construct the following *auxiliary Lyapunov equations*

$$\hat{\mathbf{A}}_{i1}\hat{\mathbf{P}}_1\hat{\mathbf{A}}_{i1}^T - \hat{\mathbf{P}}_1 + \hat{\mathbf{B}}_{i1}\hat{\mathbf{B}}_{i1}^T + \hat{\mathbf{A}}_{i2}\hat{\mathbf{P}}_2\hat{\mathbf{A}}_{i2}^T = 0 \quad (3a)$$

$$\hat{\mathbf{A}}_{i4}\hat{\mathbf{P}}_2\hat{\mathbf{A}}_{i4}^T - \hat{\mathbf{P}}_2 + \hat{\mathbf{B}}_{i2}\hat{\mathbf{B}}_{i2}^T + \hat{\mathbf{A}}_{i3}\hat{\mathbf{P}}_1\hat{\mathbf{A}}_{i3}^T = 0 \quad (3b)$$

$$\hat{\mathbf{A}}_{o1}^T\hat{\mathbf{Q}}_1\hat{\mathbf{A}}_{o1} - \hat{\mathbf{Q}}_1 + \hat{\mathbf{C}}_{o1}^T\hat{\mathbf{C}}_{o1} + \hat{\mathbf{A}}_{o3}^T\hat{\mathbf{Q}}_2\hat{\mathbf{A}}_{o3} = 0 \quad (3c)$$

$$\hat{\mathbf{A}}_{o4}^T\hat{\mathbf{Q}}_2\hat{\mathbf{A}}_{o4} - \hat{\mathbf{Q}}_2 + \hat{\mathbf{C}}_{o2}^T\hat{\mathbf{C}}_{o2} + \hat{\mathbf{A}}_{o2}^T\hat{\mathbf{Q}}_1\hat{\mathbf{A}}_{o2} = 0 \quad (3d)$$

and matrices

$$\hat{\mathbf{P}} = \begin{bmatrix} \hat{\mathbf{P}}_1 & 0 \\ 0 & \hat{\mathbf{P}}_2 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{P}}_{11} & \hat{\mathbf{P}}_{12} & 0 & 0 \\ \hat{\mathbf{P}}_{13} & \hat{\mathbf{P}}_{14} & 0 & 0 \\ 0 & 0 & \hat{\mathbf{P}}_{21} & \hat{\mathbf{P}}_{22} \\ 0 & 0 & \hat{\mathbf{P}}_{23} & \hat{\mathbf{P}}_{24} \end{bmatrix} \quad (4a)$$

$$\hat{\mathbf{Q}} = \begin{bmatrix} \hat{\mathbf{Q}}_1 & 0 \\ 0 & \hat{\mathbf{Q}}_2 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{Q}}_{11} & \hat{\mathbf{Q}}_{12} & 0 & 0 \\ \hat{\mathbf{Q}}_{13} & \hat{\mathbf{Q}}_{14} & 0 & 0 \\ 0 & 0 & \hat{\mathbf{Q}}_{21} & \hat{\mathbf{Q}}_{22} \\ 0 & 0 & \hat{\mathbf{Q}}_{23} & \hat{\mathbf{Q}}_{24} \end{bmatrix} \quad (4b)$$

where $(\hat{\mathbf{P}}_1, \hat{\mathbf{P}}_2)$ are the solutions of (3a) and (3b), and $(\hat{\mathbf{Q}}_1, \hat{\mathbf{Q}}_2)$ are the solutions of (3c) and (3d).

B. Definition of Weighted Quasi-Gramians

Definition 1
The *weighted controllability and observability quasi-gramians*, denoted by $\mathbf{P}_q$ and $\mathbf{Q}_q$, respectively, are defined as

$$\mathbf{P}_q = \hat{\mathbf{I}}\,\hat{\mathbf{P}}\,\hat{\mathbf{I}}^T, \qquad \mathbf{Q}_q = \hat{\mathbf{I}}\,\hat{\mathbf{Q}}\,\hat{\mathbf{I}}^T \qquad (5)$$

where $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ are given in (4a) and (4b), respectively, and

$$\hat{\mathbf{I}} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \end{bmatrix}$$

that is,

$$\mathbf{P}_q = \begin{bmatrix} \hat{\mathbf{P}}_{11} & 0 \\ 0 & \hat{\mathbf{P}}_{21} \end{bmatrix}_{(m+n)\times(m+n)} \equiv \mathbf{P}_{q1} \oplus \mathbf{P}_{q2}$$

$$\mathbf{Q}_q = \begin{bmatrix} \hat{\mathbf{Q}}_{11} & 0 \\ 0 & \hat{\mathbf{Q}}_{21} \end{bmatrix}_{(m+n)\times(m+n)} \equiv \mathbf{Q}_{q1} \oplus \mathbf{Q}_{q2}$$

are positive definite, block-diagonal matrices.

C. Computation of Weighted Quasi-Grammians

An algorithm for the computation of 2-D weighted quasi-gramians is described below. This is an extension of the algorithm developed in [4].

Algorithm 1

Step 1: Set $\hat{\mathbf{P}}_2^{(0)} = \hat{\mathbf{Q}}_2^{(0)} = 0$ and $k = 1$.

Step 2: Solve the following 1-D Lyapunov equations

$$\hat{\mathbf{A}}_{i1}\hat{\mathbf{P}}_1^{(k)}\hat{\mathbf{A}}_{i1}^T - \hat{\mathbf{P}}_1^{(k)} + \mathbf{F}_1 = 0 \quad (6a)$$

$$\hat{\mathbf{A}}_{o1}^T\hat{\mathbf{Q}}_1^{(k)}\hat{\mathbf{A}}_{o1} - \hat{\mathbf{Q}}_1^{(k)} + \mathbf{G}_1 = 0 \quad (6b)$$

for $\hat{\mathbf{P}}_1^{(k)}$ and $\hat{\mathbf{Q}}_1^{(k)}$, where

$$\mathbf{F}_1 = \hat{\mathbf{B}}_{i1}\hat{\mathbf{B}}_{i1}^T + \hat{\mathbf{A}}_{i2}\hat{\mathbf{P}}_2^{(k-1)}\hat{\mathbf{A}}_{i2}^T$$

$$\mathbf{G}_1 = \hat{\mathbf{C}}_{o1}^T\hat{\mathbf{C}}_{o1} + \hat{\mathbf{A}}_{o3}^T\hat{\mathbf{Q}}_2^{(k-1)}\hat{\mathbf{A}}_{o3}$$

Step 3: Solve the 1-D Lyapunov equations

$$\hat{\mathbf{A}}_{i4}\hat{\mathbf{P}}_2^{(k)}\hat{\mathbf{A}}_{i4}^T - \hat{\mathbf{P}}_2^{(k)} + \mathbf{F}_2 = 0 \quad (6c)$$

$$\hat{\mathbf{A}}_{o4}^T\hat{\mathbf{Q}}_2^{(k)}\hat{\mathbf{A}}_{o4} - \hat{\mathbf{Q}}_2^{(k)} + \mathbf{G}_2 = 0 \quad (6d)$$

for $\hat{\mathbf{P}}_2^{(k)}$ and $\hat{\mathbf{Q}}_2^{(k)}$, where

$$\mathbf{F}_2 = \hat{\mathbf{B}}_{i2}\hat{\mathbf{B}}_{i2}^T + \hat{\mathbf{A}}_{i3}\hat{\mathbf{P}}_1^{(k)}\hat{\mathbf{A}}_{i3}^T$$

$$\mathbf{G}_2 = \hat{\mathbf{C}}_{o2}^T\hat{\mathbf{C}}_{o2} + \hat{\mathbf{A}}_{o2}^T\hat{\mathbf{Q}}_1^{(k)}\hat{\mathbf{A}}_{o2}$$

Step 4: Set $k = k + 1$ and repeat from Step 2 until

$$||\hat{\mathbf{P}}_i^{(k)} - \hat{\mathbf{P}}_i^{(k-1)}|| < \epsilon, \quad (i = 1, 2)$$

$$||\hat{\mathbf{Q}}_i^{(k)} - \hat{\mathbf{Q}}_i^{(k-1)}|| < \epsilon, \quad (i = 1, 2)$$

where ϵ is a prescribed tolerance. Obviously, if $\hat{\mathbf{P}}_i^{(k)} \to \hat{\mathbf{P}}_i$ and $\hat{\mathbf{Q}}_i^{(k)} \to \hat{\mathbf{Q}}_i$ for $i = 1, 2$ as $k \to \infty$, then $\mathbf{P}_i$ and $\mathbf{Q}_i$ satisfy (3a), (3b), (3c) and (3d).

Step 5: Compute the *weighted quasi-gramians* of the 2-D systems from the $\hat{\mathbf{P}}_1$, $\hat{\mathbf{P}}_2$, $\hat{\mathbf{Q}}_1$, and $\hat{\mathbf{Q}}_2$ using (4a), (4b), and (5).

IV. Weighted Quasi-Balanced Realization and Model Reduction Method

We now propose a *weighted quasi-balanced realization and model reduction* method for 2-D systems. The new method is essentially an extension of the (unweighted) quasi-balanced realization and model reduction method proposed in [1].

Definition 2
The 2-D system represented by Figure 1 is said to be *weighted quasi-balanced* if the weighted quasi-gramians satisfy

$$\mathbf{P}_{q1} = \mathbf{Q}_{q1} = \mathbf{\Sigma}_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_m) \quad (7a)$$

$$\mathbf{P}_{q2} = \mathbf{Q}_{q2} = \mathbf{\Sigma}_2 = \mathrm{diag}(\mu_1, \cdots, \mu_n) \quad (7b)$$

where

$$\sigma_1 \geq \cdots \geq \sigma_m \geq 0 \quad \text{and} \quad \mu_1 \geq \cdots \geq \mu_n \geq 0$$

are called the *weighted quasi-Hankel singular values* of the system.

If the 2-D system is not weighted quasi-balanced and if the Lyapunov equations (3a), (3b), (3c) and (3d) have positive definite solutions $\mathbf{P}_q$ and $\mathbf{Q}_q$, then the algorithm in [5] can be applied to $\{\mathbf{P}_1, \mathbf{Q}_1\}$ and $\{\mathbf{P}_2, \mathbf{Q}_2\}$ to find nonsingular matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, respectively, such that

$$\mathbf{T}_1^{-1}\mathbf{P}_{q1}\mathbf{T}_1^{-T} = \mathbf{T}_1^T\mathbf{Q}_{q1}\mathbf{T}_1 = \mathbf{\Sigma}_1$$

$$\mathbf{T}_2^{-1}\mathbf{P}_{q2}\mathbf{T}_2^{-T} = \mathbf{T}_2^T\mathbf{Q}_{q2}\mathbf{T}_2 = \mathbf{\Sigma}_2$$

Having found balancing transformation matrix

$$\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2$$

a *weighted quasi-balanced realization* of the system can be characterized by the set $\{\mathbf{A}_b, \mathbf{B}_b, \mathbf{C}_b, \mathbf{D}\}$ with

$$\mathbf{A}_b = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \quad \mathbf{B}_b = \mathbf{T}^{-1}\mathbf{B}, \quad \mathbf{C}_b = \mathbf{C}\mathbf{T}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are given in (1a). One can then partition the matrices $\mathbf{A}_b$, $\mathbf{B}_b$, and $\mathbf{C}_b$ as

$$\mathbf{A}_b = \left[\begin{array}{cc|cc} \mathbf{A}_{r1} & \mathbf{A}_{12} & \mathbf{A}_{r2} & \mathbf{A}_{22} \\ \mathbf{A}_{13} & \mathbf{A}_{14} & \mathbf{A}_{23} & \mathbf{A}_{24} \\ \hline \mathbf{A}_{r3} & \mathbf{A}_{32} & \mathbf{A}_{r4} & \mathbf{A}_{42} \\ \mathbf{A}_{33} & \mathbf{A}_{34} & \mathbf{A}_{43} & \mathbf{A}_{44} \end{array}\right]$$

$$\mathbf{B}_b = \left[\begin{array}{cc|cc} \mathbf{B}_{r1}^T & \mathbf{B}_{12}^T & \mathbf{B}_{r2}^T & \mathbf{B}_{22}^T \end{array}\right]^T$$

$$\mathbf{C}_b = \left[\begin{array}{cc|cc} \mathbf{C}_{r1} & \mathbf{C}_{12} & \mathbf{C}_{r2} & \mathbf{C}_{22} \end{array}\right]$$

where $\mathbf{A}_{r1} \in \mathcal{C}^{r_1\times r_1}$, $\mathbf{A}_{r2} \in \mathcal{C}^{r_1\times r_2}$, $\mathbf{A}_{r3} \in \mathcal{C}^{r_2\times r_1}$, $\mathbf{A}_{r4} \in \mathcal{C}^{r_2\times r_2}$, $\mathbf{B}_{r1} \in \mathcal{C}^{r_1\times t}$, $\mathbf{B}_{r2} \in \mathcal{C}^{r_2\times t}$, $\mathbf{C}_{r1} \in$

$\mathcal{C}^{s\times r_1}$, $\mathbf{C}_{r2} \in \mathcal{C}^{s\times r_2}$, and the *reduced-order weighted 2-D system* of order (r_1, r_2) can be obtained as $\{\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}\}$ with

$$\mathbf{A}_r = \begin{bmatrix} \mathbf{A}_{r1} & \mathbf{A}_{r2} \\ \mathbf{A}_{r3} & \mathbf{A}_{r4} \end{bmatrix}, \ \mathbf{B}_r = \begin{bmatrix} \mathbf{B}_{r1} \\ \mathbf{B}_{r2} \end{bmatrix}, \ \mathbf{C}_r = \begin{bmatrix} \mathbf{C}_{r1}^T \\ \mathbf{C}_{r2}^T \end{bmatrix}^T$$

The transfer-function matrix of the reduced-order system is given by

$$\mathbf{H}_r(z_1, z_2) = \mathbf{C}_r\,[\mathbf{I}(z_1, z_2) - \mathbf{A}_r]\,\mathbf{B}_r + \mathbf{D}$$

V. Example

A lowpass FIR filter of order (31, 31) is used to demonstrate the proposed method. The amplitude response of the original filter is depicted in Figure 3. We first applied the (unweighted) quasi-balanced model reduction algorithm in [1] to obtain a reduced-order filter of order (12, 12), represented by $\mathbf{H}_{uw}(z_1, z_2)$, whose amplitude response is shown in Figure 4. Next, a (4, 4) lowpass filter was used as the input weight $\mathbf{W}_i(z_1, z_2)$ to emphasize the reduced-order system over the low frequency region. This input weight is modeled by (1a) with

$$\mathbf{A}_{i1} = \begin{bmatrix} 0.8653 & -0.3202 & -0.0389 & -0.0293 \\ 0.3202 & 0.6696 & -0.3457 & -0.0472 \\ -0.0389 & 0.3457 & 0.4885 & -0.4057 \\ 0.0293 & -0.0472 & 0.4057 & 0.3592 \end{bmatrix}$$

$$\mathbf{A}_{i2} = \begin{bmatrix} 0.3761 & 0.3275 & 0.1685 & 0.0574 \\ -0.3291 & -0.2866 & -0.1475 & -0.0502 \\ 0.1688 & 0.1470 & 0.0756 & 0.0258 \\ -0.0570 & -0.0496 & -0.0255 & -0.0087 \end{bmatrix}$$

$$\mathbf{A}_{i3} = \begin{bmatrix} 0.0020 & 0.0028 & 0.0031 & 0.0021 \\ -0.0020 & -0.0034 & -0.0041 & -0.0025 \\ 0.0010 & 0.0023 & 0.0033 & 0.0018 \\ -0.0012 & -0.0025 & -0.0028 & -0.0008 \end{bmatrix}$$

$$\mathbf{A}_{i4} = \begin{bmatrix} 0.8657 & -0.3205 & -0.0396 & -0.0294 \\ 0.3202 & 0.6694 & -0.3465 & -0.0465 \\ -0.0385 & 0.3456 & 0.4883 & -0.4037 \\ 0.0294 & -0.0473 & 0.4072 & 0.3591 \end{bmatrix}$$

$$\mathbf{B}_{i1} = \begin{bmatrix} -0.0149 & 0.0130 & -0.0067 & 0.0023 \end{bmatrix}^T$$
$$\mathbf{B}_{i2} = \begin{bmatrix} -0.2631 & 0.2289 & -0.1183 & 0.0398 \end{bmatrix}^T$$
$$\mathbf{C}_{i1} = \begin{bmatrix} -0.2636 & -0.2284 & -0.1181 & -0.0396 \end{bmatrix}$$
$$\mathbf{C}_{i2} = \begin{bmatrix} -0.0149 & -0.0130 & -0.0067 & -0.0023 \end{bmatrix}$$
$$\mathbf{D}_i = 10^{-4} \times 5.8826$$

The amplitude response of the input weight is depicted in Figure 5.

The proposed weighted quasi-balanced model reduction method with input weight $\mathbf{W}_i(z_1, z_2)$ and unit output weight, $\mathbf{W}_o(z_1, z_2) = \mathbf{I}$, leads to a weighted reduced-order filter of order (12, 12), represented by $\mathbf{H}_w(z_1, z_2)$, whose amplitude response is shown in Figure 6. The reduction error matrices of the unweighted and weighted quasi-balanced model-reduction can be defined as

$$\mathbf{E}_{uw} = (\,E_{uwkl}\,) \quad \text{and} \quad \mathbf{E}_w = (\,E_{wkl}\,)$$

where

$$E_{uw_{kl}} = |\mathbf{H}(e^{j2\pi k/K}, e^{j2\pi l/L}) - \mathbf{H}_{uw}(e^{j2\pi k/K}, e^{j2\pi l/L})|$$
$$E_{w_{kl}} = |\mathbf{H}(e^{j2\pi k/K}, e^{j2\pi l/L}) - \mathbf{H}_{w}(e^{j2\pi k/K}, e^{j2\pi l/L})|$$

with $k = 0, 1, \ldots, K$ and $l = 0, 1, \ldots, L$, where K and L are the numbers of frequency response samples in the normalized frequency ranges $\omega_1, \omega_2 \in [0, 1.0]$ ($\omega_1 = \frac{k}{K}$, $\omega_2 = \frac{l}{L}$). The Frobenius and l_∞ norms of $\mathbf{E}_{uw}$ and $\mathbf{E}_w$ are given by

$$e_f = ||\mathbf{E}_{uw}||_F = \left(\sum_{k=1}^{K}\sum_{l=1}^{L} E_{uw_{kl}}^2\right)^{1/2}$$
$$e_{wf} = ||\mathbf{E}_{w}||_F = \left(\sum_{k=1}^{K}\sum_{k=1}^{L} E_{w_{kl}}^2\right)^{1/2}$$
$$e_\infty = ||\mathbf{E}_{uw}||_\infty = \max_{1\le k\le K, 1\le l\le L} E_{uw_{kl}}$$
$$e_{w\infty} = ||\mathbf{E}_{w}||_\infty = \max_{1\le k\le K, 1\le l\le L} E_{w_{kl}}$$

The errors of the reduced-order models obtained using the unweighted and weighted quasi-balanced model reduction methods have been computed in three frequency ranges and are summarized in Table 1, where $\omega = \omega_1 = \omega_2$ denotes the normalized frequency.

Table 1: Errors of the Unweighted and Weighted Model Reduction

$(r_1, r_2)=$	Frequency ranges, ($\omega \in [\ ]$)		
(12, 12)	[0, 0.30]	[0.30, 0.60]	[0.60, 1.0]
e_f	0.3236	0.7262	0.5505
e_{wf}	0.3223	0.7264	0.5506
e_∞	0.0367	0.0503	0.0485
$e_{w\infty}$	0.0365	0.0504	0.0486

As is expected, Table 1 shows that the error e_{wf} ($e_{w\infty}$) from the proposed weighted quasi-balanced model reduction is smaller than the error e_f (e_∞) for the (unweighted) quasi-balanced model reduction in the low frequency range but it is larger in the intermediate and high frequency ranges due to the use of the lowpass input weight.

V. Conclusion

A new weighted quasi-balanced realization and model-reduction method for 2-D discrete systems with input and output weights has been developed. Four auxiliary Lyapunov equations are introduced and solved iteratively. The weighted controllability and observability quasi-gramians are part of the solutions of auxiliary Lyapunov equations. The proposed method leads to a better weighted reduced-order system in low frequency range than that obtained from the existing quasi-balanced model reduction method.

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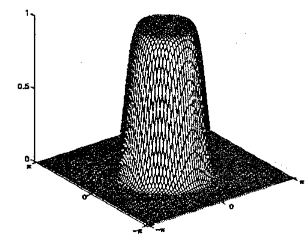

Figure 3: Amplitude response of the original FIR filter of order (31, 31).

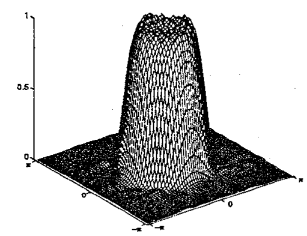

Figure 4: Amplitude response of the reduced-order filter of order (12, 12).

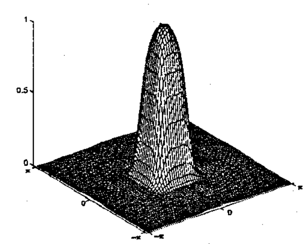

Figure 5: Amplitude response of the input weight.

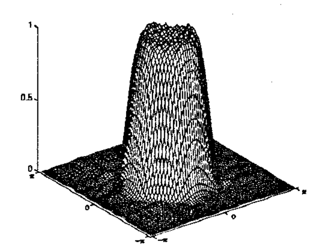

Figure 6: Amplitude response of the weighted reduced-order filter of order (12, 12).