A New Algebraic Approach to the Design of Generalized QMF Banks

W.-S. Lu, H. Xu, and A. Antoniou

Department of Electrical and Computer Engineering
University of Victoria
Victoria, B.C., Canada, V8W 3P6
Email: wslu@ece.uvic.ca

Abstract— A new algebraic method for the design of conventional QMF banks is proposed. The method uses a self-convolution technique to reformulate a 4th-order objective function whose minimization leads to the design of QMF banks. It is shown that the reformulated optimization problem can be solved iteratively with improved computation efficiency compared to several existing design methods. It is also shown that the method can be extended to design QMF banks with low-reconstruction delay. Two examples are included to illustrate the design method proposed.

I. INTRODUCTION

Several methods for the design of quadrature-mirror-filter (QMF) banks have been proposed in [1]-[7]. Some of the design methods lead to near-perfect reconstruction QMF banks [1]-[4] while others lead to perfect-reconstruction QMF banks [5]-[7].

In this paper a new algebraic method for the design of conventional QMF banks with near-perfect reconstruction is proposed. The method uses a self-convolution technique to reformulate a 4th-order objective function whose minimization leads to the design of QMF banks. It is shown that the reformulated optimization problem can be solved iteratively with improved computation efficiency compared to several existing design methods. It is also shown that the method can be extended to design QMF banks with low-reconstruction delay. Two examples are included to illustrate the design method proposed.

The authors are grateful to Micronet, Networks of Centres of Excellence Program, for supporting this work.

II. DESIGN METHOD

A. Generalized QMF Banks

Consider the two-channel filter bank shown in Fig. 1. The output and input of the system are related as

$$\begin{aligned}\hat{X}(z) &= \tfrac{1}{2}[H_0(z)G_0(z) + H_1(z)G_1(z)]X(z) \\ &\quad + \tfrac{1}{2}[H_0(-z)G_0(z) + H_1(-z)G_1(z)]X(-z) \quad (1)\end{aligned}$$

where the second term in the right-hand side represents aliasing. By assuming $H_1(z) = H_0(-z)$, $G_1(z) = -H_0(-z)$, and $G_0(z) = H_0(z)$, the aliasing term is cancelled and (1) becomes

$$\hat{X}(z) = \tfrac{1}{2}\left[H_0^2(z) - H_0^2(-z)\right] X(z) \quad (2)$$

To reconstruct the input signal at the output, it is required that

$$H_0^2(z) - H_0^2(-z) = z^{-k_d} \quad (3)$$

where k_d is the system delay.

Let

$$H_0(z) = h_0 + h_1 z^{-1} + \cdots + h_{N-1} z^{-(N-1)}$$

with N even, and

$$\begin{aligned}\mathbf{h} &= [\ h_0\ h_1\ \cdots\ h_{N-1}\ ]^T \\ \mathbf{Z}_{2N} &= [\ 1\ z^{-1}\ \cdots\ z^{-2(N-1)}\ ]^T \\ \hat{\mathbf{Z}}_{2N} &= [\ 1\ (-z)^{-1}\ \cdots\ (-z)^{-2(N-1)}\ ]^T\end{aligned}$$

Then we can write

$$\begin{aligned}H_0^2(z) &= \mathbf{g}^T\,\mathbf{Z}_{2N} \\ H_0^2(-z) &= \mathbf{g}^T\,\hat{\mathbf{Z}}_{2N}\end{aligned}$$

where

$$\mathbf{g} = \mathbf{h} * \mathbf{h} \quad (4)$$

with $*$ denoting the convolution operation. Therefore

$$H_0^2(z) - H_0^2(-z) = \mathbf{g}^T(\mathbf{Z}_{2N} - \hat{\mathbf{Z}}_{2N}) \equiv \mathbf{g}\,\tilde{\mathbf{Z}}_{2N}$$

Proceedings of ASILOMAR-29

where

$$\tilde{\mathbf{Z}}_{2N} = 2\,[\; 0 \;\; z^{-1} \;\; 0 \;\; z^{-3} \;\; 0 \;\; \cdots \;\; z^{-2N+3} \;\; 0 \;]^T$$

So with $z = e^{j\omega}$, we have

$$H_0^2(e^{j\omega}) - H_0^2(-e^{j\omega}) = 2\,\mathbf{g}^T\mathbf{c}_0(\omega)\,e^{-jk_d\omega}$$

where

$$\mathbf{c}_0(\omega) = [\; 0 \;\; e^{j(k_d-1)\omega} \;\; \cdots \;\; 0 \;\; 1 \;\; 0 \;\; \cdots \;\; e^{j(k_d-2N+3)\omega} \;\; 0 \;]^T$$

and (3) becomes

$$2\,\mathbf{g}^T\mathbf{c}_0(\omega) = 1 \tag{5}$$

Equation (5) suggests that a near-perfect QMF bank can be designed by minimizing the objective function

$$E = \int_0^\pi \left|\,\mathbf{g}^T\mathbf{c}_0(\omega) - 1\,\right|^2 \,d\omega + \alpha \int_{\omega_s}^\pi |H_0(e^{j\omega})|^2 \,d\omega \tag{6}$$

where $\alpha > 0$ is a weight and ω_s is the stopband edge of the lowpass transfer function $H_0(z)$. It can be readily verified that

$$E = 4\,\mathbf{g}^T\mathbf{Q}_r\,\mathbf{g} - 4\,\mathbf{g}^T\,\mathbf{b} + \alpha\,\mathbf{h}^T\mathbf{Q}_a\mathbf{h} + \pi$$

where

$$\mathbf{Q}_r = \pi \begin{bmatrix} 0 & & & & & & \\ & 1 & & & & \mathbf{0} & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & \mathbf{0} & & & & 1 & \\ & & & & & & 0 \end{bmatrix}$$

$$\mathbf{Q}_a = \begin{bmatrix} \pi-\omega_s & -\sin\omega_s & -\frac{1}{2}\sin 2\omega_s & \cdots & -\frac{1}{N-1}\sin(N-1)\omega_s \\ -\sin\omega_s & \pi-\omega_s & -\sin\omega_s & & -\frac{1}{N-2}\sin(N-2)\omega_s \\ \vdots & & \ddots & \ddots & \vdots \\ -\frac{1}{N-1}\sin(N-1)\omega_s & \cdots & \cdots & \cdots & \pi-\omega_s \end{bmatrix}$$

and $\mathbf{b} = [\; 0 \;\cdots\; 0 \;\; \pi \;\; 0 \;\cdots\; 0 \;]^T$ where only the (k_d+1)th entry is nonzero.

The gradient of E with respect to $\mathbf{h}$ is given by

$$\nabla_{\mathbf{h}}E = 8\,\mathbf{J}\mathbf{Q}_r\mathbf{g} - 4\,\mathbf{J}\,\mathbf{b} + 2\,\alpha\,\mathbf{Q}_a\mathbf{h}$$

where $\mathbf{J}$ is the $N \times (2N-1)$ Jacobian of $\mathbf{g}$ with respect to $\mathbf{h}$, i.e.,

$$\mathbf{J} = \frac{\partial\mathbf{g}}{\partial\mathbf{h}} = \frac{\partial(\mathbf{h}*\mathbf{h})}{\partial\mathbf{h}} = \frac{\partial(\mathbf{H}\mathbf{h})}{\partial\mathbf{h}}$$

$$= 2\begin{bmatrix} h_0 & h_1 & \cdots & h_{N/2-1} & h_{N/2} & \cdots & h_{N-1} & 0 & \cdots & \cdots & 0 \\ & h_0 & \cdots & h_{N/2-2} & h_{N/2-1} & \cdots & h_{N-2} & h_{N-1} & \cdots & \cdots & 0 \\ & & \ddots & & & & & & \ddots & & \\ & & & \ddots & & & & & & \ddots & \\ \mathbf{0} & & & & & & h_0 & h_1 & \cdots & \cdots & h_{N-1} \end{bmatrix}_{N\times(2N-1)}$$

with

$$\mathbf{H} = \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & & \cdots & h_0 \\ 0 & h_{N-1} & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{N-1} \end{bmatrix}_{(2N-1)\times N}$$

Hence

$$\nabla_{\mathbf{h}}E = 2\,[\,(4\,\mathbf{J}\mathbf{Q}_r\mathbf{H} + \alpha\,\mathbf{Q}_a)\,\mathbf{h} - 2\,\mathbf{J}\mathbf{b}\,] \tag{7}$$

B. Design of QMF Banks with Linear Phase

If $k_d = N-1$ and $\mathbf{h}$ is symmetrical with respect to its midpoint, then the filter represented by $H_0(z)$ has linear phase response. In this case the number of design variables is reduced to $N/2$, and equation (7) is replaced by

$$\nabla_{\mathbf{h}_s}E = 2\,[\,(4\,\mathbf{J}_s\mathbf{Q}_r\mathbf{H}_s + \alpha\,\mathbf{Q}_{as})\,\mathbf{h}_s - 2\,\mathbf{J}_s\mathbf{b}\,] \tag{8}$$

where $\mathbf{h}_s = [\; h_0 \;\; h_1 \;\cdots\; h_{N/2-1} \;]^T$,

$$\mathbf{J}_s = 2\begin{bmatrix} h_0 & h_1 & \cdots & h_{N/2-1} & h_{N/2-1} & \cdots & h_0 & 0 & \cdots & 0 & \\ & h_0 & \cdots & h_{N/2-2} & h_{N/2-1} & \cdots & h_1 & h_0 & \cdots & 0 & \mathbf{0} \\ & & \ddots & & & & & & \ddots & & \\ \mathbf{0} & & & h_0 & h_1 & \cdots & h_{N/2-1} & h_{N/2-2} & \cdots & h_0 & \end{bmatrix}_{N/2\times(2N-1)}$$

$$
\mathbf{H}_s = \begin{bmatrix} h_0 & & & 0 \\ \vdots & h_0 & & \\ \vdots & & \ddots & \\ h_{N/2-1} & h_{N/2-2} & & h_0 \\ h_{N/2-1} & h_{N/2-1} & & \\ \vdots & & & \\ h_0 & h_1 & & \\ & h_0 & & \\ & & \ddots & \\ 0 & & & h_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ \vdots & \vdots & \ddots & \\ \vdots & \vdots & & 1 \\ \vdots & \vdots & & 1 \\ \vdots & \vdots & ⋰ & \\ 0 & 1 & & \\ 1 & & & 0 \end{bmatrix}
$$

and

$$
\mathbf{Q}_{as} = [\ \mathbf{I}_{N/2} \quad 0\ ]\, \mathbf{Q}_a \begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ \vdots & \vdots & \ddots & \\ \vdots & \vdots & & 1 \\ \vdots & \vdots & & 1 \\ \vdots & \vdots & ⋰ & \\ 0 & 1 & & \\ 1 & & & 0 \end{bmatrix}_{N \times N/2}
$$

By letting $\nabla_{\mathbf{h}_s} E = 0$, we obtain

$$
\mathbf{h}_s = 2\,(4\,\mathbf{J}_s \mathbf{Q}_r \mathbf{H}_s + \alpha\, \mathbf{Q}_{as})^{-1} \mathbf{J}_s \mathbf{b} \tag{9}
$$

To design a QMF bank, we design a half-band, linear-phase, FIR filter of length N, and use the first $N/2$ of its coefficients to form a vector denoted by $\mathbf{h}_0$. Then we calculate $\mathbf{J}_s$ and $\mathbf{H}_s$, and evaluate $\mathbf{h}_s$ in (9). Having obtained $\mathbf{h}_s$, vector $\mathbf{h}_0$ is updated as

$$
\mathbf{h}_0 = \beta\, \mathbf{h}_s + (1-\beta)\, \mathbf{h}_0 \tag{10}
$$

where the weight $\beta \in (0,1)$ usually assumes a value of about 0.5. The updated $\mathbf{h}_0$ is then used to evaluated $\mathbf{h}_s$ in (9) again. The iteration continues until the difference $\|\, \mathbf{h}_0 - \mathbf{h}_s \,\|_2 < \epsilon$ where ϵ is a prescribed tolerance.

C. Design of QMF Banks with Low Delay

In a linear-phase QMF bank the reconstruction delay is $N-1$, i.e., the order of the analysis and synthesis filters. In some applications, QMF banks with reconstruction delay $k_d < N-1$ are desired. In such an application, we start the algorithm by designing an FIR filter of length N with group delay $k_d/2$. The coefficients of the filter designed are then used as an initial $\mathbf{h}_0$. By computing $\mathbf{J}$ and $\mathbf{H}$ and equating $\nabla_{\mathbf{h}} E$ in (7) to 0, we obtain

$$
\mathbf{h} = 2\,(4\,\mathbf{J}\mathbf{Q}_r\mathbf{H} + \alpha\, \mathbf{Q}_a)^{-1} \mathbf{J}\mathbf{b} \tag{11}
$$

The $\mathbf{h}_0$ is thereafter updated as $\mathbf{h}_0 = \beta\,\mathbf{h} + (1-\beta)\,\mathbf{h}_0$ and the above process is repeated until $\|\, \mathbf{h}_0 - \mathbf{h} \,\|_2$ becomes less than a prescribed tolerance.

It has been observed from experiments that undesirable artifacts may occur in the transition region of filter H_0 when a low reconstruction delay k_d is required. A possible approach to reduce the artifacts is to modify the objective function in (6) by including an additional term

$$
\alpha_1 \int_{\omega_{t1}}^{\omega_{t2}} |H_0(e^{j\omega}) - e^{-j\omega k_d/2}|^2 \, d\omega
$$

where $[\omega_{t1},\ \omega_{t2}]$ is an interval in the transition region where the artifacts occur. It can be shown that

$$
\begin{aligned} E = {} & 4\,\mathbf{g}^T \mathbf{Q}_r\, \mathbf{g} - 4\,\mathbf{g}^T \mathbf{b} + \mathbf{h}^T (\,\alpha\, \mathbf{Q}_a + \alpha_1\, \mathbf{Q}_t\,)\mathbf{h} \\ & -2\alpha_1 \mathbf{d}^T \mathbf{h} + \pi + \alpha_1(\omega_{t2} - \omega_{t1}) \end{aligned}
$$

where $\mathbf{Q}_t = \{\mathbf{q}_{ij}\}$ is a symmetric matrix with

$$
q_{ij} = \begin{cases} \omega_{t2} - \omega_{t1} & i = j \\ \frac{1}{|i-j|}[\,\sin(|i-j|\omega_{t2}) - \sin(|i-j|\omega_{t1})\,] & i \neq j \end{cases}
$$

for $i,\ j = 1, \ldots, N$, and

$$
\mathbf{d} = \begin{bmatrix} d_0 \\ \vdots \\ d_{N-1} \end{bmatrix}
$$

with

$$
\begin{aligned} d_i &= \frac{1}{\rho_i}[\sin(\rho_i \omega_{t2}) - \sin(\rho_i \omega_{t1})] \\ \rho_i &= \frac{k_d}{2} - i \end{aligned}
$$

By setting $\nabla_{\mathbf{h}} E = 0$, we obtain

$$
\mathbf{h} = (4\,\mathbf{J}\mathbf{Q}_r\mathbf{H} + \alpha\, \mathbf{Q}_a + \alpha_1\, \mathbf{Q}_t)^{-1}(\,2\,\mathbf{J}\mathbf{b} + \alpha_1\, \mathbf{d}) \tag{12}
$$

III. DESIGN EXAMPLES

Two design examples are given in this section to illustrate the proposed approaches. The performance of the designs are evaluated in terms of:

- the floating-point operations in millions (MFLOPS) used,
- the minimum stopband attenuation
$$
A_a = \min_{\omega_s \le \omega \le \pi} [-20 \log_{10} |H_0(e^{j\omega})|]
$$
- the peak to peak passband ripple
$$
\begin{aligned} A_p = {} & \max_{0 \le \omega \le \omega_p} [20 \log_{10} |H_0(e^{j\omega})|] \\ & - \min_{0 \le \omega \le \omega_p} [20 \log_{10} |H_0(e^{j\omega})|] \end{aligned}
$$

  where ω_p is the passband edge.

- the peak reconstruction error

$$\text{PRE} = \max_{\omega} \left| H^2(e^{j\omega}) - H^2(e^{j(\omega+\pi)}) - e^{-jk_d\omega} \right|$$

- the signal-to-noise ratio

$$\begin{aligned}\text{SNR} &= 10\log_{10}\left(\frac{\text{energy of the signal}}{\text{energy of the reconstruction noise}}\right)\\ &= 10\log_{10}\left\{\frac{\Sigma\, x^2(n)}{\Sigma\, [x(n) - \hat{x}(n+k_d)]^2}\right\}\end{aligned}$$

with a random input where amplitude is uniformly distributed between 0 and 1.

As Example 1, a linear-phase QMF bank was designed with specifications $N = 64$, $\alpha = 9.5 \times 10^{-7}$, $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $\beta = 0.5$, $\epsilon = 10^{-4}$. The results are summarized in Table I. The amplitude responses of the analysis filters and the magnitude of the reconstruction error are depicted in Figs. 2 and 3, respectively. It is observed that the minimum stopband attenuation of the filter obtained is larger than 54 dB and the peak reconstruction error is of the order of 10^{-7}, which should be considered small enough for many applications.

As Example 2, an FIR QMF bank with low reconstruction delay was designed. The design parameters were $N = 80$, $k_d = 39$, $\alpha = 1$, $\alpha_1 = 1 \times 10^{-5}$, $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $\omega_{t1} = 0.45\pi$, $\omega_{t2} = 0.5\pi, \beta = 0.5$, $\epsilon = 10^{-3}$. The results are summarized in Table I. The amplitude responses of the analysis filters and magnitude of the complex reconstruction error are depicted in Figs. 4 and 5, respectively. It is observed that the filter bank obtained has satisfactory performance and the reconstruction delay is about half of that in a linear-phase QMF bank. Note that a low delay QMF bank in general does not have symmetrical impulse responses, hence linear-phase response is not guaranteed. However, from Fig. 6, it is observed that nearly constant group delay can be achieved in the passband. The phase distortion in its stopband can be negligible due to the high stopband attenuation.

TABLE I
RESULTS FOR EXAMPLES 1 AND 2.

	Example 1	Example 2
N	64	80
k_d	63	39
MFLOPS	1.52	21.5
A_a(dB)	54.09	72.01
A_p(dB)	9.00×10^{-5}	6.37×10^{-4}
PRE	4.46×10^{-7}	1.01×10^{-4}
SNR (dB)	133.0	91.4

IV. CONCLUSION

A new algebraic method for the design of two-channel QMF banks has been proposed. The new method is efficient and can be used to design both linear-phase QMF banks and low delay QMF banks. From the design examples, it is observed that the filter banks obtained have satisfactory performance.

REFERENCES

[1] J. D. Johnston, "A filter family designed for use in quadrature mirror filter banks," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, pp. 291-294, Mar. 1980.

[2] V. K. Jain and R. E. Crochiere, "Quadrature mirror filter design in the time domain," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-32, pp. 353-361, Apr. 1984.

[3] K. Nayebi, T. P. Barnwell, III, and M. J. T. Smith, "Time-domain filter bank analysis: A new design theory," *IEEE Trans. on Signal Processing*, vol. 40, pp. 1412-1429, June 1992.

[4] C.-K. Chen and J.-H. Lee, "Design of quadrature mirror filters with linear phase in the frequency domain," *IEEE Trans. Circuits Syst.*, vol. 39, pp. 593-605, Sept. 1992.

[5] T. Q. Nguyen and P. P. Vaidyanathan, "Two-channel perfect-reconstruction FIR QMF structures which yield linear-phase analysis and synthesis filters," *IEEE Trans. on Acoust., Speech, Signal Processing*, vol. 37, pp. 466-690, May 1989.

[6] B.-R. Horng and A. N. Willson, Jr., "Lagrange multiplier approaches to the design of two-channel perfect-reconstruction linear-phase FIR filter banks," *IEEE Trans. Signal Processing*, vol. 40, pp. 364-374, Feb. 1992.

[7] H. Xu, W.-S. Lu, and A. Antoniou, "Design of perfect reconstruction QMF banks by a null-space-projection method," *Proc. of IEEE Int. Symp. Circuits and Syst.*, pp. 965-968, May, 1995

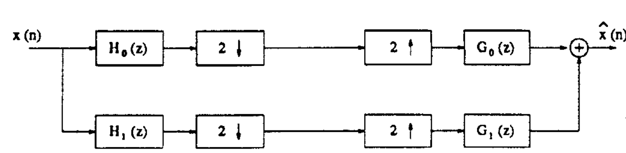

FIGURE 1
A TWO-BAND FILTER BANK.

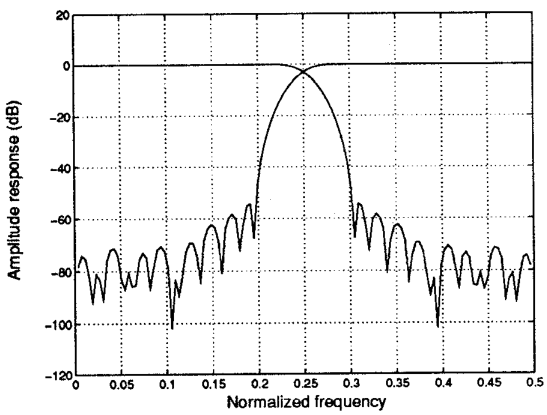

FIGURE 2
EXAMPLE 1: AMPLITUDE RESPONSES OF THE ANALYSIS FILTERS.

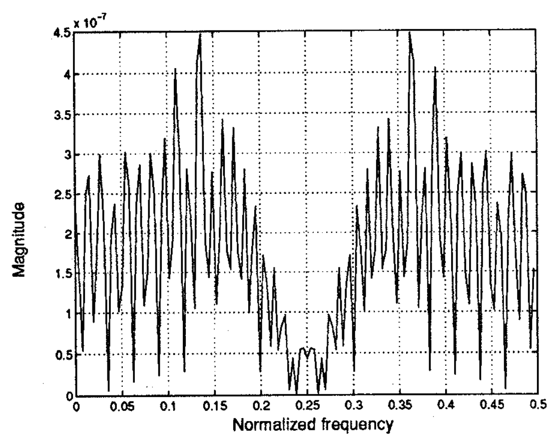

FIGURE 3
EXAMPLE 1: MAGNITUDE OF THE RECONSTRUCTION ERROR.

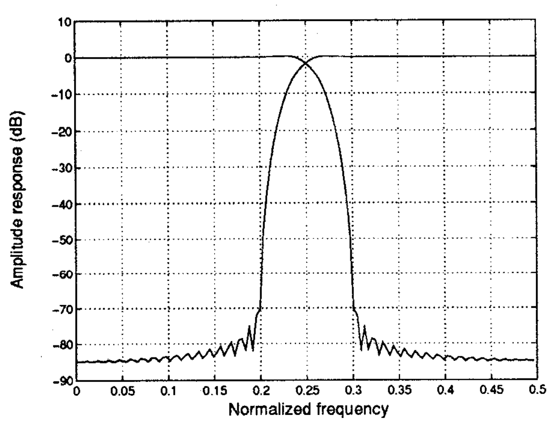

FIGURE 4
EXAMPLE 2: AMPLITUDE RESPONSES OF THE ANALYSIS FILTERS.

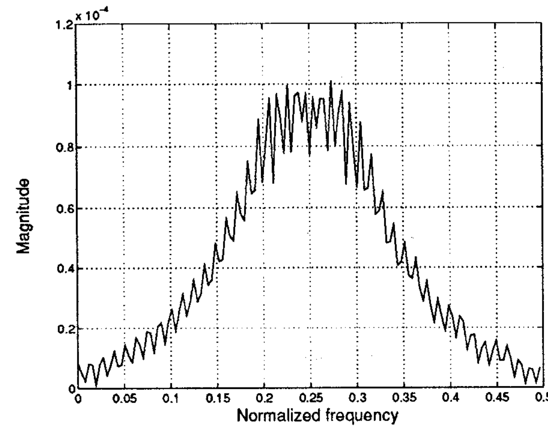

FIGURE 5
EXAMPLE 2: MAGNITUDE OF THE RECONSTRUCTION ERROR.

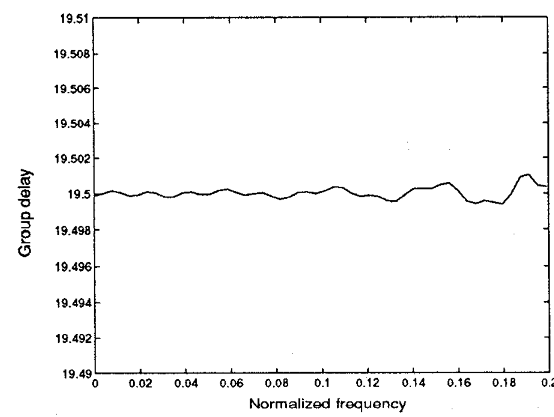

FIGURE 6
EXAMPLE 2: GROUP DELAY OF FILTER H_0 IN THE PASSBAND.