A New Algebraic Approach to the Design of Generalized QMF Banks

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Abstract — A new algebraic method for the design of conventional QMF banks is proposed. The method uses a self-convolution technique to reformulate a 4th-order objective function whose minimization leads to the design of QMF banks. It is shown that the reformulated optimization problem can be solved iteratively with improved computation efficiency compared to several existing design methods. It is also shown that the method can be extended to design QMF banks with low-reconstruction delay. Two examples are included to illustrate the design method proposed.

I. INTRODUCTION

Several methods for the design of quadrature-mirror-filter (QMF) banks have been proposed in [1]-[7]. Some of the design methods lead to near-perfect reconstruction QMF banks [1]-[4] while others lead to perfect-reconstruction QMF banks [5]-[7].

In this paper a new algebraic method for the design of conventional QMF banks with near-perfect reconstruction is proposed. The method uses a self-convolution technique to reformulate a 4th-order objective function whose minimization leads to the design of QMF banks. It is shown that the reformulated optimization problem can be solved iteratively with improved computation efficiency compared to several existing design methods. It is also shown that the method can be extended to design QMF banks with low-reconstruction delay. Two examples are included to illustrate the design method proposed.

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II. DESIGN METHOD

A. Generalized QMF Banks

Consider the two-channel filter bank shown in Fig. 1. The output and input of the system are related as

\[ \hat{X}(z) = \frac{1}{2} \{ H_0(z)G_0(z) + H_1(z)G_1(z) \} X(z) \]
\[ + \frac{1}{2} \{ H_0(-z)G_0(z) + H_1(-z)G_1(z) \} X(-z) \]

where the second term in the right-hand side represents aliasing. By assuming \( H_1(z) = H_0(-z) \), \( G_1(z) = -H_0(-z) \), and \( G_0(z) = H_0(z) \), the aliasing term is cancelled and \( I \) becomes

\[ \hat{X}(z) = \frac{1}{2} \{ H_0^2(z) - H_0^2(-z) \} X(z) \]

To reconstruct the input signal at the output, it is required that

\[ H_0^2(z) - H_0^2(-z) = z^{-k_d} \]

where \( k_d \) is the system delay.

Let

\[ H_0(z) = h_0 + h_1 z^{-1} + \cdots + h_{N-1} z^{-(N-1)} \]

with \( N \) even, and

\[ h = [ h_0 \ h_1 \ \cdots \ h_{N-1} ]^T \]

\[ Z_{2N} = [ 1 \ z^{-1} \ \cdots \ z^{-(N-1)} ]^T \]

\[ \hat{Z}_{2N} = [ 1 \ (-z)^{-1} \ \cdots \ (-z)^{-(N-1)} ]^T \]

Then we can write

\[ H_0^2(z) = g^T Z_{2N} \]

\[ H_0^2(-z) = g^T \hat{Z}_{2N} \]

where

\[ g = h \ast h \]

with \( \ast \) denoting the convolution operation. Therefore

\[ H_0^2(z) - H_0^2(-z) = g^T ( Z_{2N} - \hat{Z}_{2N} ) \equiv g \hat{Z}_{2N} \]
where
\[ \tilde{Z}_{2N} = 2 [0 \ z^{-1} \ 0 \ z^{-3} \ 0 \ \cdots \ z^{-(2N-3)} \ 0 ]^T \]
So with \( z = e^{j\omega} \), we have
\[ H_0^2(e^{j\omega}) - H_0^2(-e^{j\omega}) = 2 g^T c_0(\omega) e^{-jkd\omega} \]
where
\[ c_0(\omega) = [0 \ e^{j(k_2-1)\omega} \ 0 \ 1 \ 0 \ \cdots \ e^{j(k_2-2N-3)\omega} \ 0 ]^T \]
and (3) becomes
\[ 2 g^T c_0(\omega) = 1 \] (5)
Equation (5) suggests that a near-perfect QMF bank can be designed by minimizing the objective function
\[ E = \int_0^\pi |g^T c_0(\omega) - 1|^2 \ d\omega + \alpha \int_0^\pi |H_0(e^{j\omega})|^2 \ d\omega \] (6)
where \( \alpha > 0 \) is a weight and \( \omega_s \) is the stopband edge of the lowpass transfer function \( H_0(z) \). It can be readily verified that
\[ E = 4 g^T Q_r \ g - 4 g^T b + \alpha h^T Q_s h + \pi \]
where
\[ Q_r = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \]
\[ Q_s = \begin{bmatrix} \pi - \omega_s & -\sin \omega_s & -\frac{1}{2} \sin 2\omega_s \\ -\sin \omega_s & \pi - \omega_s & -\sin \omega_s \\ \vdots & \vdots & \vdots \\ -\frac{1}{N-1} \sin (N-1)\omega_s & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ -\frac{1}{N-2} \sin (N-2)\omega_s & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \vdots & \vdots \\ \cdots & \vdots & \vdots \end{bmatrix} \]
and \( b = [0 \ \cdots \ 0 \ \pi \ 0 \ \cdots \ 0 ]^T \) where only the \( (k_d+1) \)th entry is nonzero.

The gradient of \( E \) with respect to \( h \) is given by
\[ \nabla_h E = 8 J Q_r g - 4 J b + 2 \alpha Q_s h \]
where \( J \) is the \( N \times (2N-1) \) Jacobian of \( g \) with respect to \( h \), i.e.,
\[ J = \frac{\partial g}{\partial h} = \frac{\partial (h \ast h)}{\partial h} = \frac{\partial (H h)}{\partial h} \]
Hence
\[ \nabla_h E = 2 [(4 J Q_r H + \alpha Q_s) h - 2 J b] \] (7)

**B. Design of QMF Banks with Linear Phase**

If \( k_d = N - 1 \) and \( h \) is symmetrical with respect to its midpoint, then the filter represented by \( H_0(z) \) has linear phase response. In this case the number of design variables is reduced to \( N/2 \), and equation (7) is replaced by
\[ \nabla_h E = 2 [(4 J_s Q_r H_s + \alpha Q_s) h - 2 J_s b] \] (8)
where \( h_s = [h_0 \ h_1 \ \cdots \ h_{N/2-1} ]^T \),
\[ J_s = \begin{bmatrix} h_0 & h_1 & \cdots & h_{N/2-1} \\ h_0 & h_{N/2-1} & \cdots & h_1 \\ 0 & \cdots & \cdots & 0 \\ h_0 & \cdots & 0 & 0 \end{bmatrix} \]

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\[
H_e = \begin{bmatrix}
  h_0 & \cdots & h_0 & 0 \\
  \vdots & & \vdots \\
  h_{N/2-1} & h_{N/2-2} & \cdots & h_0 \\
  h_0 & h_1 & \cdots & 1 \\
  \vdots & & \vdots \\
  1 & 0 & 0 & 1 \\
  0 & 1 & 0 & 1 \\
\end{bmatrix}
\]

and
\[
Q_{\alpha} = [I_{N/2} 0] Q_x [I_{N/2} 0]^T
\]

By letting \( \nabla_h E = 0 \), we obtain
\[
h_e = 2(4JQ_xH_x + \alpha Q_{\alpha})^{-1}Jy
\]

To design a QMF bank, we design a half-band, linear-phase, FIR filter of length \( N \), and use the first \( N/2 \) of its coefficients to form a vector denoted by \( h_0 \). Then we calculate \( J_x \) and \( H_e \), and evaluate \( h_e \) in (9). Having obtained \( h_e \), vector \( h_0 \) is updated as
\[
h_0 = \beta h_e + (1 - \beta)h_0
\]

where the weight \( \beta \in (0,1) \) usually assumes a value of about 0.5. The updated \( h_0 \) is then used to evaluate \( h_e \) in (9) again. The iteration continues until the difference \( ||h_0 - h_e||_2 < \epsilon \) where \( \epsilon \) is a prescribed tolerance.

**C. Design of QMF Banks with Low Delay**

In a linear-phase QMF bank the reconstruction delay is \( N - 1 \), i.e., the order of the analysis and synthesis filters. In some applications, QMF banks with reconstruction delay \( k_d \) \( < N - 1 \) are desired. In such an application, we start the algorithm by designing an FIR filter of length \( N \) with group delay \( k_d/2 \). The coefficients of the filter designed are then used as an initial \( h_0 \). By computing \( J \), and \( H \) and equating \( \nabla_h E \) in (7) to 0, we obtain
\[
h = 2(4JQ_xH + \alpha Q_x)^{-1}Jy
\]

The \( h_0 \) is thereafter updated as \( h_0 = \beta h_e + (1 - \beta)h_0 \) and the above process is repeated until \( ||h_0 - h_e||_2 \) becomes less than a prescribed tolerance.

It has been observed from experiments that undesirable artifacts may occur in the transition region of filter \( H_0 \) when a low reconstruction delay \( k_d \) is required. A possible approach to reduce the artifacts is to modify the objective function in (6) by including an additional term
\[
\alpha_1 \int_{\omega_{11}}^{\omega_{12}} \left| H_0(e^{j\omega}) - e^{-j\omega k_d/2} \right|^2 d\omega
\]

where \([\omega_{11}, \omega_{12}]\) is an interval in the transition region where the artifacts occur. It can be shown that
\[
E = 4g^TQ_xg - 4g^T \alpha Q_x + \alpha_1 Q_x h + 2\alpha_1 d^T h + \alpha_1 \omega_{12} - \omega_{11}
\]

where \( Q_x = \{q_{ij}\} \) is a symmetric matrix with
\[
q_{ij} = \frac{1}{N-1}[\sin(i-j|\omega_{12}) - \sin(|i-j|\omega_{11})]
\]

for \( i, j = 1, \ldots, N \), and
\[
d = \begin{bmatrix} d_0 \\ \vdots \\ d_{N-1} \end{bmatrix}
\]

with
\[
d_i = \frac{1}{\rho_i} \left[ \sin(\rho_i \omega_{12}) - \sin(\rho_i \omega_{11}) \right]
\]

\[
\rho_i = \frac{k_d}{2} - i
\]

By setting \( \nabla_h E = 0 \), we obtain
\[
h = (4JQ_xH + \alpha Q_x + \alpha_1 Q_x)^{-1}(2Jy + \alpha_1 d)
\]

**III. DESIGN EXAMPLES**

Two design examples are given in this section to illustrate the proposed approaches. The performance of the designs are evaluated in terms of:

- the floating-point operations in millions (MFLOPS) used,
- the minimum stopband attenuation
\[
A_d = \min_{\omega_1, \omega_2, \omega_3} \left[ -20 \log_{10} |H_0(e^{j\omega})| \right]
\]
- the peak to peak passband ripple
\[
A_p = \max_{0 \leq \omega \leq \omega_p} \left[ -20 \log_{10} |H_0(e^{j\omega})| \right]
= \max_{0 \leq \omega \leq \omega_p} \left[ 20 \log_{10} |H_0(e^{j\omega})| \right]
\]

where \( \omega_p \) is the passband edge.
IV. CONCLUSION

A new algebraic method for the design of two-channel QMF banks has been proposed. The new method is efficient and can be used to design both linear-phase QMF banks and low delay QMF banks. From the design examples, it is observed that the filter banks obtained have satisfactory performance.

REFERENCES


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**Table 1**

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**Figure 1**

A two-band filter bank.
Figure 2
Example 1: Amplitude responses of the analysis filters.

Figure 3
Example 1: Magnitude of the reconstruction error.

Figure 5
Example 2: Magnitude of the reconstruction error.

Figure 4
Example 2: Amplitude responses of the analysis filters.

Figure 6
Example 2: Group delay of filter $H_0$ in the passband.