# A Weighted Balanced Realization of 2-D Discrete Systems

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*Abstract*—**An innovative weighted structurally balanced realization of two-dimensional (2-D) discrete systems is proposed. Two auxiliary transfer-function matrices called the weighted-input-to-state and the state-to-weighted-output transfer-function matrices are first introduced. Based on these matrices, the controllability and observability grammians of the weighted system are defined and the existence of the grammians is justified. The resulting 2-D weighted structurally balanced realization method can be applied for the reduction of the system order of 2-D discrete systems.**

## I. Introduction

The balanced approximation is an effective and numerically economical method for the reduction of the system order in one-dimensional (1-D) and 2-D systems. It has, in addition, several desirable properties such as a bounded error and preservation of stability [1]-[4] of the original system. If the requirements imposed on the approximation error are different for different frequency ranges, improved reduced-order systems can be obtained by applying a weighted balanced approximation method. The continuous-time 1-D case is considered in [1] and the discrete-time 1-D case is considered in [2, 3].

The goal of this paper is to develop a weighted balanced realization for 2-D discrete systems. To this end, two auxiliary transfer-function matrices called the *weighted-input-to-state* and the *state-to-weighted-output* transfer-function matrices are first introduced. It is shown that these matrices are quadratically stable (Q-stable) provided that the input and output weights and the system are Q-stable. Here quadratic stability (Q-stability) is the stability associated with the constant 2-D Lyapunov inequalities [4], which is known to be stronger than the bounded-input-bounded-output (BIBO) stability [5].

On the basis of these auxiliary transfer-function matrices, the *structured* controllability and observability grammians of the weighted system are defined and the existence of the grammians is justified. The grammians are the solutions of two 2-D Lyapunov inequalities, and solving them amounts to solving two *unconstrained* optimization problems. The resulting weighted structurally balanced realization can be used in the reduction of the system orders of 2-D discrete systems.

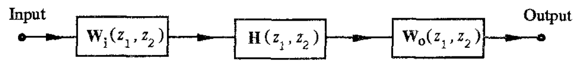


Figure 1: A weighted 2-D discrete system

## II. Auxiliary transfer-function matrices

The system configuration to be considered here is shown in Figure 1, where $\mathbf{H}(z_1, z_2) \in R^{p\times q}$ is the transfer-function matrix of the system, and $\mathbf{W}_i(z_1, z_2) \in R^{q\times t}$ and $\mathbf{W}_o(z_1, z_2) \in R^{s\times p}$ are the transfer-function matrices of the input and output weights, respectively.

If we use the Roesser state-space model [5] to describe the system, then

$$\mathbf{H}(z_1, z_2) = \mathbf{C}\,[\mathbf{I}(z_1, z_2)-\mathbf{A}\,]^{-1}\,\mathbf{B}+\mathbf{D} \quad (1a)$$

$$\mathbf{W}_i(z_1, z_2) = \mathbf{C}_i\,[\mathbf{I}(z_1, z_2)-\mathbf{A}_i\,]^{-1}\,\mathbf{B}_i+\mathbf{D}_i \quad (1b)$$

$$\mathbf{W}_o(z_1, z_2) = \mathbf{C}_o\,[\mathbf{I}(z_1, z_2)-\mathbf{A}_o\,]^{-1}\,\mathbf{B}_o+\mathbf{D}_o \quad (1c)$$

The matrices involved can be partitioned as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix},$$

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{A}_{i1} & \mathbf{A}_{i2} \\ \mathbf{A}_{i3} & \mathbf{A}_{i4} \end{bmatrix}, \qquad \mathbf{B}_i = \begin{bmatrix} \mathbf{B}_{i1} \\ \mathbf{B}_{i2} \end{bmatrix},$$

$$\mathbf{A}_o = \begin{bmatrix} \mathbf{A}_{o1} & \mathbf{A}_{o2} \\ \mathbf{A}_{o3} & \mathbf{A}_{o4} \end{bmatrix}, \qquad \mathbf{B}_o = \begin{bmatrix} \mathbf{B}_{o1} \\ \mathbf{B}_{o2} \end{bmatrix}$$

$$\mathbf{C}_i = \begin{bmatrix} \mathbf{C}_{i1} & \mathbf{C}_{i2} \end{bmatrix}, \qquad \mathbf{C}_o = \begin{bmatrix} \mathbf{C}_{o1} & \mathbf{C}_{o2} \end{bmatrix},$$

where $\mathbf{A} \in \Re^{(n+m)\times(n+m)}$, $\mathbf{B} \in \Re^{(n+m)\times q}$, $\mathbf{C} \in \Re^{p\times(n+m)}$, $\mathbf{A}_i \in \Re^{(n_i+m_i)\times(n_i+m_i)}$, $\mathbf{B}_i \in \Re^{(n_i+m_i)\times t}$,

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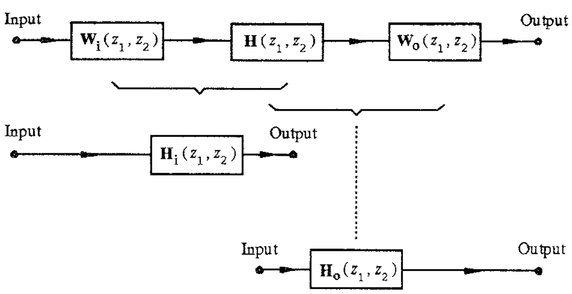


Figure 2: Auxiliary Transfer-function matrices

$\mathbf{C}_i \in \Re^{q\times(n_i+m_i)}$, $\mathbf{A}_o \in \Re^{(n_o+m_o)\times(n_o+m_o)}$, $\mathbf{B}_o \in \Re^{(n_o+m_o)\times p}$, $\mathbf{C}_o \in \Re^{s\times(n_o+m_o)}$, and

$$\mathbf{I}(z_1, z_2) = z_1\mathbf{I} \oplus z_2\mathbf{I}$$

In this equation, the symbol $\oplus$ denotes the direct sum, and $\mathbf{I}$ is the identity matrix. In the rest of the paper, we assume that $\mathbf{H}(z_1, z_2)$, $\mathbf{W}_i(z_1, z_2)$, and $\mathbf{W}_o(z_1, z_2)$ are 2-D Q-stable, namely, there exist positive definite matrices $\mathbf{S}$, $\mathbf{S}_i$, and $\mathbf{S}_o$, and positive definite, block-diagonal matrices

$$\mathbf{G} = \mathbf{G}_1 \oplus \mathbf{G}_2, \quad \mathbf{G}_i = \mathbf{G}_{i1} \oplus \mathbf{G}_{i2}, \quad \mathbf{G}_o = \mathbf{G}_{o1} \oplus \mathbf{G}_{o2}$$

such that

$$\mathbf{A}\mathbf{G}\mathbf{A}^T - \mathbf{G} = -\mathbf{S} \qquad (2a)$$

$$\mathbf{A}_i\mathbf{G}_i\mathbf{A}_i^T - \mathbf{G}_i = -\mathbf{S}_i \qquad (2b)$$

$$\mathbf{A}_o\mathbf{G}_o\mathbf{A}_o^T - \mathbf{G}_o = -\mathbf{S}_o \qquad (2c)$$

*A. Definition of* $\mathbf{H}_i(z_1, z_2)$ *and* $\mathbf{H}_o(z_1, z_2)$

**Definition 1** Based on the configuration in Figure 1, the *weighted-input-to-state* transfer-function matrix, $\mathbf{H}_i(z_1, z_2)$, and the *state-to-weighted-output* transfer-function matrix, $\mathbf{H}_o(z_1, z_2)$, are defined as

$$\mathbf{H}_i(z_1, z_2) = [\mathbf{I}(z_1, z_2) - \mathbf{A}]^{-1}\mathbf{B}\,\mathbf{W}_i(z_1, z_2) \qquad (3a)$$

$$\mathbf{H}_o(z_1, z_2) = \mathbf{W}_o(z_1, z_2)\,\mathbf{C}\,[\mathbf{I}(z_1, z_2) - \mathbf{A}]^{-1} \qquad (3b)$$

These definitions are illustrated in Figure 2.

By performing permutations for certain blocks of the matrices in equations (3a) and (3b), $\mathbf{H}_i(z_1, z_2)$ and $\mathbf{H}_o(z_1, z_2)$ can be written as

$$\mathbf{H}_i(z_1, z_2) = \hat{\mathbf{C}}_i\left[\mathbf{I}(z_1, z_2) - \hat{\mathbf{A}}_i\right]^{-1}\hat{\mathbf{B}}_i \qquad (4a)$$

$$\mathbf{H}_o(z_1, z_2) = \hat{\mathbf{C}}_o\left[\mathbf{I}(z_1, z_2) - \hat{\mathbf{A}}_o\right]^{-1}\hat{\mathbf{B}}_o \qquad (4b)$$

respectively, where

$$\hat{\mathbf{A}}_i = \begin{bmatrix} \hat{\mathbf{A}}_{i1} & \hat{\mathbf{A}}_{i2} \\ \hat{\mathbf{A}}_{i3} & \hat{\mathbf{A}}_{i4} \end{bmatrix} = \left[\begin{array}{cc|cc} \mathbf{A}_1 & \mathbf{B}_1\mathbf{C}_{i1} & \mathbf{A}_2 & \mathbf{B}_1\mathbf{C}_{i2} \\ 0 & \mathbf{A}_{i1} & 0 & \mathbf{A}_{i2} \\ \hline \mathbf{A}_3 & \mathbf{B}_2\mathbf{C}_{i1} & \mathbf{A}_4 & \mathbf{B}_2\mathbf{C}_{i2} \\ 0 & \mathbf{A}_{i3} & 0 & \mathbf{A}_{i4} \end{array}\right]$$

$$\hat{\mathbf{B}}_i = \left[\,\hat{\mathbf{B}}_{i1}^T \ \ \hat{\mathbf{B}}_{i2}^T\,\right]^T = \left[\,\mathbf{D}_i^T\mathbf{B}_1^T \ \ \mathbf{B}_{i1}^T \ \mid \ \mathbf{D}_i^T\mathbf{B}_2^T \ \ \mathbf{B}_{i2}^T\,\right]^T$$

$$\hat{\mathbf{C}}_i = \left[\,\hat{\mathbf{C}}_{i1} \ \ \hat{\mathbf{C}}_{i2}\,\right] = \left[\begin{array}{cc|cc} \mathbf{I} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \end{array}\right]$$

$$\hat{\mathbf{A}}_o = \begin{bmatrix} \hat{\mathbf{A}}_{o1} & \hat{\mathbf{A}}_{o2} \\ \hat{\mathbf{A}}_{o3} & \hat{\mathbf{A}}_{o4} \end{bmatrix} = \left[\begin{array}{cc|cc} \mathbf{A}_1 & 0 & \mathbf{A}_2 & 0 \\ \mathbf{B}_{o1}\mathbf{C}_1 & \mathbf{A}_{o1} & \mathbf{B}_{o1}\mathbf{C}_2 & \mathbf{A}_{o2} \\ \hline \mathbf{A}_3 & 0 & \mathbf{A}_4 & 0 \\ \mathbf{B}_{o2}\mathbf{C}_1 & \mathbf{A}_{o3} & \mathbf{B}_{o2}\mathbf{C}_2 & \mathbf{A}_{o4} \end{array}\right]$$

$$\hat{\mathbf{B}}_o = \left[\,\hat{\mathbf{B}}_{o1}^T \ \ \hat{\mathbf{B}}_{o2}^T\,\right]^T = \left[\begin{array}{cc|cc} \mathbf{I} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \end{array}\right]^T$$

$$\hat{\mathbf{C}}_o = \left[\,\hat{\mathbf{C}}_{o1} \ \ \hat{\mathbf{C}}_{o2}\,\right] = \left[\,\mathbf{D}_o\mathbf{C}_1 \ \ \mathbf{C}_{o1} \ \mid \ \mathbf{D}_o\mathbf{C}_2 \ \ \mathbf{C}_{o2}\,\right]$$

As indicated in Figure 2, transfer-function matrix $\mathbf{H}_i(z_1, z_2)$ in (4a) relates a weighted input signal to the state of $\mathbf{H}(z_1, z_2)$, which takes the input weight $\mathbf{W}_i(z_1, z_2)$ into account. Similarly, transfer-function matrix $\mathbf{H}_o(z_1, z_2)$ in (4b) relates a state to the weighted output, which takes the output weight $\mathbf{W}_o(z_1, z_2)$ into account.

*B. Quadratic Stability of* $\mathbf{H}_i(z_1, z_2)$ *and* $\mathbf{H}_o(z_1, z_2)$

**Lemma 1** If transfer-function matrices $\mathbf{H}(z_1, z_2)$ and $\mathbf{W}_i(z_1, z_2)$ and $\mathbf{W}_o(z_1, z_2)$ are all 2-D Q-stable, then the weighted-input-to-state-output and the state-input-to-weighted-output transfer-function matrices $\mathbf{H}_i(z_1, z_2)$ and $\mathbf{H}_o(z_1, z_2)$ defined by (3a) and (3b) are also 2-D Q-stable.

Proof: If $\mathbf{H}(z_1, z_2)$ and $\mathbf{W}_o(z_1, z_2)$ are 2-D Q-stable transfer-function matrices, then $\mathbf{H}(z_1, z_2)$ and output weight $\mathbf{W}_o(z_1, z_2)$ satisfy the 2-D Lyapunov equations (2a) and (2b), respectively. By Lemma 4 of [4], (2a) and (2b) imply that there exist block-diagonal matrices

$$\mathbf{Y} = \mathbf{Y}_1 \oplus \mathbf{Y}_2 > 0 \quad \text{and} \quad \mathbf{Y}_i = \mathbf{Y}_{i1} \oplus \mathbf{Y}_{i2} > 0$$

such that

$$\mathbf{\Psi} = \mathbf{A}\mathbf{Y}\mathbf{A}^T - \mathbf{Y} + \mathbf{B}\mathbf{B}^T < 0 \qquad (5)$$

$$\mathbf{\Psi}_i = \mathbf{A}_i\mathbf{Y}_i\mathbf{A}_i^T - \mathbf{Y}_i + \mathbf{B}_i\mathbf{B}_i^T < 0 \qquad (6)$$

Define

$$\hat{\mathbf{Y}} = \left[\begin{array}{cc|cc} \mathbf{Y}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_{i1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Y}_{i2} \end{array}\right] = \hat{\mathbf{Y}}_1 \oplus \hat{\mathbf{Y}}_2$$

and

$$\hat{\boldsymbol{\Psi}} = \hat{\mathbf{A}}_i \hat{\mathbf{Y}} \hat{\mathbf{A}}_i^T - \hat{\mathbf{Y}} + \hat{\mathbf{B}}_i \hat{\mathbf{B}}_i^T$$

As $\boldsymbol{\Psi}_i$ is negative definite, $\hat{\boldsymbol{\Psi}}$ can be block diagonalized as

$$\mathbf{X}_2(\mathbf{X}_1 \hat{\boldsymbol{\Psi}} \mathbf{X}_1^T)\mathbf{X}_2^T = \begin{bmatrix} \mathbf{AYA}^T - \mathbf{Y} + \mathbf{B}\hat{\mathbf{K}}\mathbf{B}^T & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi}_i \end{bmatrix}$$

where

$$\mathbf{X}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} \mathbf{I} & -\mathbf{BK}\boldsymbol{\Psi}_i^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{K} = \mathbf{C}_i \mathbf{Y}_i \mathbf{A}_i^T + \mathbf{D}_i \mathbf{B}_i^T$$

and

$$\hat{\mathbf{K}} = \mathbf{C}_i \mathbf{Y}_i \mathbf{C}_i^T + \mathbf{D}_i \mathbf{D}_i^T - \mathbf{K}\boldsymbol{\Psi}_i^{-1}\mathbf{K}^T \qquad (7)$$

Hence, matrix $\mathbf{X}_2(\mathbf{X}_1 \hat{\boldsymbol{\Psi}} \mathbf{X}_1^T)\mathbf{X}_2^T$ is negative definite (and hence $\hat{\boldsymbol{\Psi}} < \mathbf{0}$) if and only if

$$\mathbf{AYA}^T - \mathbf{Y} + \mathbf{B}\hat{\mathbf{K}}\mathbf{B}^T \;<\; \mathbf{0} \qquad (8)$$

With a fixed $\mathbf{Y}_i$ that satisfies (6), matrix $\hat{\mathbf{K}}$ in (7) is a known matrix, and so $||\hat{\mathbf{K}}|| \leq \beta$ for some $\beta > \mathbf{0}$. From (5) it follows that for any scalar $\alpha > 1$

$$\begin{aligned} \mathbf{A}(\alpha\mathbf{Y})\mathbf{A}^T - \alpha\mathbf{Y} + \mathbf{B}\hat{\mathbf{K}}\mathbf{B}^T &= \alpha\boldsymbol{\Psi} + \mathbf{B}(\hat{\mathbf{K}} - \alpha\mathbf{I})\mathbf{B}^T \\ &\leq \alpha\boldsymbol{\Psi} + \mathbf{B}\hat{\mathbf{K}}\mathbf{B}^T \end{aligned}$$

Since $\boldsymbol{\Psi} < \mathbf{0}$, if $\alpha$ is chosen such that

$$\alpha \;>\; \frac{\beta \, ||\mathbf{B}||^2}{|\rho_{\max}(\boldsymbol{\Psi})|} \qquad (9)$$

then we have

$$\mathbf{A}(\alpha\mathbf{Y})\mathbf{A}^T - \alpha\mathbf{Y} + \mathbf{B}\hat{\mathbf{K}}\mathbf{B}^T < \mathbf{0} \qquad (10)$$

In other words, (8) holds if $\mathbf{Y}$ is scaled to $(\alpha\mathbf{Y})$ as is seen in (10). We conclude that

$$\hat{\mathbf{A}}_i \hat{\mathbf{P}} \hat{\mathbf{A}}_i^T - \hat{\mathbf{P}} + \hat{\mathbf{B}}_i \hat{\mathbf{B}}_i^T < \mathbf{0}$$

where

$$\hat{\mathbf{P}} = \left[\begin{array}{cc|cc} \alpha\mathbf{Y}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_{i1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \alpha\mathbf{Y}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Y}_{i2} \end{array}\right]$$

with $\alpha$ satisfying (9) and, therefore, $\mathbf{H}_i(z_1, z_2)$ is 2-D Q-stable.

## III. Weighted Balanced Realization

We now propose a weighted structurally balanced realization for 2-D systems which is essentially an extension of the (unweighted) structurally balanced realization proposed in [4].

### *A. Definition of Weighted Balanced Realization*

Following Lemma 1, $\mathbf{H}_i(z_1, z_2)$ and $\mathbf{H}_o(z_1, z_2)$ are 2-D Q-stable. Consequently, there exist block-diagonal, positive definite matrices

$$\hat{\mathbf{P}} = \hat{\mathbf{P}}_1 \oplus \hat{\mathbf{P}}_2 = \left[\begin{array}{cc|cc} \hat{\mathbf{P}}_{11} & \hat{\mathbf{P}}_{21} & \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{P}}_{21}^T & \hat{\mathbf{P}}_{31} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \hat{\mathbf{P}}_{12} & \hat{\mathbf{P}}_{22} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{P}}_{22}^T & \hat{\mathbf{P}}_{32} \end{array}\right]$$

$$\hat{\mathbf{Q}} = \hat{\mathbf{Q}}_1 \oplus \hat{\mathbf{Q}}_2 = \left[\begin{array}{cc|cc} \hat{\mathbf{Q}}_{11} & \hat{\mathbf{Q}}_{21} & \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{Q}}_{21}^T & \hat{\mathbf{Q}}_{31} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \hat{\mathbf{Q}}_{12} & \hat{\mathbf{Q}}_{22} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{Q}}_{22}^T & \hat{\mathbf{Q}}_{32} \end{array}\right]$$

that satisfy the 2-D Lyapunov inequalities

$$\hat{\mathbf{A}}_i \hat{\mathbf{P}} \hat{\mathbf{A}}_i^T - \hat{\mathbf{P}} + \hat{\mathbf{B}}_i \hat{\mathbf{B}}_i^T \;<\; \mathbf{0} \qquad (11a)$$

$$\hat{\mathbf{A}}_o^T \hat{\mathbf{Q}} \hat{\mathbf{A}}_o - \hat{\mathbf{Q}} + \hat{\mathbf{C}}_o^T \hat{\mathbf{C}}_o \;<\; \mathbf{0} \qquad (11b)$$

From (4a) and (4b), it is observed that the four blocks of the system matrix of $\mathbf{H}(z_1, z_2)$, namely, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, and $\mathbf{A}_4$ of matrix $\mathbf{A}$ defined in (1a), occur in both $\hat{\mathbf{A}}_i$ and $\hat{\mathbf{A}}_o$ as the (1, 1), (1, 3), (3, 1), and (3, 3) blocks. Therefore, the *structured controllability and observability grammians of the weighted system*, denoted as $\mathbf{P}$ and $\mathbf{Q}$, respectively, can be obtained from $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ as

$$\mathbf{P} = \hat{\mathbf{I}} \, \hat{\mathbf{P}} \, \hat{\mathbf{I}}^T, \qquad \mathbf{Q} = \hat{\mathbf{I}} \, \hat{\mathbf{Q}} \, \hat{\mathbf{I}}^T$$

where

$$\hat{\mathbf{I}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}$$

that is,

$$\mathbf{P} = \begin{bmatrix} \hat{\mathbf{P}}_{11} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{P}}_{12} \end{bmatrix}_{(n+m)\times(n+m)} \equiv \mathbf{P}_1 \oplus \mathbf{P}_2$$

$$\mathbf{Q} = \begin{bmatrix} \hat{\mathbf{Q}}_{11} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Q}}_{12} \end{bmatrix}_{(n+m)\times(n+m)} \equiv \mathbf{Q}_1 \oplus \mathbf{Q}_2$$

which are positive definite, block-diagonal matrices. Hence a nonsingular, block-diagonal matrix $\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2$ can be found [6] such that

$$\begin{aligned} \mathbf{T}_1^{-1}\mathbf{P}_1\mathbf{T}_1^{-T} &= \mathbf{T}_1^T\mathbf{Q}_1\mathbf{T}_1 = \mathbf{\Sigma}_1 \\ &= \text{diag}\,(\sigma_1, \sigma_2, \cdots, \sigma_n) \\ \mathbf{T}_2^{-1}\mathbf{P}_2\mathbf{T}_2^{-T} &= \mathbf{T}_2^T\mathbf{Q}_2\mathbf{T}_2 = \mathbf{\Sigma}_2 \\ &= \text{diag}\,(\mu_1, \mu_2, \cdots, \mu_m) \end{aligned}$$

where

$$\sigma_1 \geq \cdots \geq \sigma_n \geq 0 \quad \text{and} \quad \mu_1 \geq \cdots \geq \mu_m \geq 0$$

Having found balancing transformation matrix $\mathbf{T}$, a weighted balanced realization of a system in the Roesser state-space model can be characterized by the set $\{\mathbf{A}_b, \mathbf{B}_b, \mathbf{C}_b, \mathbf{D}\}$ with

$$\mathbf{A}_b = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \quad \mathbf{B}_b = \mathbf{T}^{-1}\mathbf{B}, \quad \mathbf{C}_b = \mathbf{C}\mathbf{T}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are given in (1a).

### *B. Computation of Weighted Grammians*

The Cholesky factorizations of the positive definite matrices $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ are given by

$$\hat{\mathbf{P}} = \hat{\mathbf{L}}_p\hat{\mathbf{L}}_p^T, \qquad \hat{\mathbf{Q}} = \hat{\mathbf{L}}_q\hat{\mathbf{L}}_q^T \tag{12}$$

where $\hat{\mathbf{L}}_p$ and $\hat{\mathbf{L}}_q$ are block-diagonal lower-triangular matrices. Matrices $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ that satisfy (11a) and (11b) can be obtained by solving the *unconstrained* convex minimization problems

$$\text{minimize}\, ||\, \hat{\mathbf{L}}_p^{-1}\,[\,\hat{\mathbf{A}}_i\hat{\mathbf{L}}_p \quad \hat{\mathbf{B}}_i\,]\, || \tag{13a}$$

$$\text{minimize}\, ||\, \hat{\mathbf{L}}_q^{-1}\,[\,\hat{\mathbf{A}}_o^T\hat{\mathbf{L}}_q \quad \hat{\mathbf{C}}_o^T\,]\, || \tag{13b}$$

Obviously, local minimum points of (13a) and (13b), $\hat{\mathbf{L}}_p$ and $\hat{\mathbf{L}}_q$, are acceptable if and only if

$$||\, \hat{\mathbf{L}}_p^{-1}\,[\,\hat{\mathbf{A}}_i\hat{\mathbf{L}}_p \quad \hat{\mathbf{B}}_i\,]\, || < 1$$

$$||\, \hat{\mathbf{L}}_q^{-1}\,[\,\hat{\mathbf{A}}_o^T\hat{\mathbf{L}}_q \quad \hat{\mathbf{C}}_o^T\,]\, || < 1$$

This is because only then matrices $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$, formed by (12), will satisfy the Lyapunov inequalities (11a) and (11b).

## V. CONCLUSION

A weighted structurally balanced approximation for 2-D discrete systems has been proposed, which is based on two auxiliary transfer-function matrices. The weighted structured controllability and observability grammians are the solutions of two 2-D Lyapunov inequalities, and can be obtained by solving two unconstrained optimization problems. The weighted structurally balanced approximation can be used for the reduction of the system order of 2-D discrete systems.


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