New Algorithm for Structurally Balanced Model Reduction of 2-D Discrete Systems

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Abstract—**A new structurally balanced model reduction algorithm that leads to a stable reduced-order system with improved approximation error is proposed. The algorithm is developed by formulating the problem at hand as an unconstrained optimization problem in which the objective function includes a term that depends on the sum of discarded 2-D Hankel singular values. An example is given to illustrate the performance of the reduced-order system obtained using the new algorithm.**

I. Introduction

The balanced approximation is an effective and numerically economical method for the reduction of the system order in one-dimensional (1-D) [1] and two-dimensional (2-D) systems [2]-[5]. Three different 2-D balanced realizations have been proposed in the past, namely, *pseudo-balanced*, *quasi-balanced*, and *structurally balanced* realizations [2]-[4], respectively. The pseudo-balanced and quasi-balanced realizations may lead to unstable reduced-order systems [3]. The structurally balanced realization, on the other hand, yields reduced-order systems that are guaranteed to be stable [4]. An algorithm to obtain structurally balanced realizations was proposed in [5]. Unfortunately, the reduced-order system obtained by truncating the resulting balanced realization is sometimes unsatisfactory with respect to the approximation error.

The goal of this paper is to present a structurally balanced model reduction algorithm that yields a stable reduced-order system with improved approximation error. Our method is based on the structurally balanced realization reported in [4]. In this approach, the structured grammians, as defined in [4], are not unique and, consequently, a certain degree of freedom is available which can be used to improve the algorithm proposed in [5]. In the new algorithm, we modify the objective function of the unconstrained optimization problem in [5] by including a term which is dependent on the sum of the discarded 2-D Hankel singular values. The paper concludes with an example which illustrates the merits of the new algorithm.

II. Basic principles

Consider a 2-D discrete system represented by the Roesser state-space model, that is,

$$\begin{aligned}\begin{bmatrix}\mathbf{x}^h(l+1,\,k)\\ \mathbf{x}^v(l,\,k+1)\end{bmatrix} &= \begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_3 & \mathbf{A}_4\end{bmatrix}\begin{bmatrix}\mathbf{x}^h(l,\,k)\\ \mathbf{x}^v(l,\,k)\end{bmatrix}+\begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix}\mathbf{u}(l,\,k)\\ &= \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u} \qquad (1a)\\ \mathbf{y}(l,\,k) &= \begin{bmatrix}\mathbf{C}_1 & \mathbf{C}_2\end{bmatrix}\begin{bmatrix}\mathbf{x}^h(l,\,k)\\ \mathbf{x}^v(l,\,k)\end{bmatrix}+\mathbf{D}\mathbf{u}(l,\,k)\\ &= \mathbf{C}\mathbf{x}+\mathbf{D}\mathbf{u} \qquad (1b)\end{aligned}$$

where $\mathbf{x}^h \in \mathcal{R}^n$ and $\mathbf{x}^v \in \mathcal{R}^m$ are the horizontal and vertical state vectors, and $\mathbf{u} \in \mathcal{R}^q$ and $\mathbf{y} \in \mathcal{R}^p$ are input and output vectors, respectively. The necessary condition for the structurally balanced realization is that the original 2-D system is quadratically stable (Q-stable). The necessary and sufficient condition for the quadratic stability (Q-stability) of a 2-D discrete system is given by the following lemma.

Lemma 1 [4]
The 2-D system defined by (1a) and (1b) is *Q-stable* if and only if there exist block-diagonal positive matrices

$$\mathbf{P} = \mathbf{P}_1 \oplus \mathbf{P}_2 > \mathbf{0} \quad \text{and} \quad \mathbf{Q} = \mathbf{Q}_1 \oplus \mathbf{Q}_2 > \mathbf{0}$$

where $\oplus$ denotes the direct sum of the matrices, $\mathbf{P}_1,\ \mathbf{Q}_1 \in \mathcal{R}^{(n\times n)}$ and $\mathbf{P}_2, \mathbf{Q}_2 \in \mathcal{R}^{(m\times m)}$ such that the following inequalities hold

$$\mathbf{A}\mathbf{P}\mathbf{A}^T - \mathbf{P} + \mathbf{B}\mathbf{B}^T < \mathbf{0} \qquad (2a)$$
$$\mathbf{A}^T\mathbf{Q}\mathbf{A} - \mathbf{Q} + \mathbf{C}^T\mathbf{C} < \mathbf{0} \qquad (2b)$$

In principle, matrices $\mathbf{P}$ and $\mathbf{Q}$ that satisfy *2-D Lyapunov inequalities* (2a) and (2b) can be obtained by

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solving the following *constrained* convex minimization problems [3]

$$\text{minimize } \rho_{\max}(\mathbf{APA}^T - \mathbf{P} + \mathbf{BB}^T) \quad (3a)$$
$$\text{minimize } \rho_{\max}(\mathbf{A}^T\mathbf{QA} - \mathbf{Q} + \mathbf{C}^T\mathbf{C}) \quad (3b)$$

such that

$$\rho_{\max}(\mathbf{APA}^T - \mathbf{P} + \mathbf{BB}^T) < 0$$
$$\rho_{\max}(\mathbf{A}^T\mathbf{QA} - \mathbf{Q} + \mathbf{C}^T\mathbf{C}) < 0$$

and

$$\mathbf{P} > 0 \quad \text{and} \quad \mathbf{Q} > 0$$

where $\rho_{\max}()$ denotes the largest eigenvalue of the matrix in the bracket (). In general, if there are solutions $\mathbf{P}$ and $\mathbf{Q}$, then there are infinitely many solutions. The definition of a structurally balanced system, assuming that matrices $\mathbf{P}$ and $\mathbf{Q}$ are available, is as follows:

Definition 1 [4]
If the 2-D system satisfies inequalities (2a) and (2b) and there exists a non-singular matrix $\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2$ such that

$$\begin{aligned} \mathbf{T}_1^{-1}\mathbf{P}_1\mathbf{T}_1^{-T} &= \mathbf{T}_1^T\mathbf{Q}_1\mathbf{T}_1 = \mathbf{\Sigma}_1 \\ &= \text{diag}(\sigma_1, \ldots, \sigma_n) \geq 0 \quad (4a) \\ \mathbf{T}_2^{-1}\mathbf{P}_2\mathbf{T}_2^{-T} &= \mathbf{T}_1^T\mathbf{Q}_2\mathbf{T}_2 = \mathbf{\Sigma}_2 \\ &= \text{diag}(\mu_1, \ldots, \mu_m) \geq 0 \quad (4b) \end{aligned}$$

then the system is said to be *structurally balanced*, and the matrices $\mathbf{P}$ and $\mathbf{Q}$ are called the *structured grammians* of the 2-D system.

III. NEW ALGORITHM FOR STRUCTURALLY BALANCED MODEL REDUCTION

As $\mathbf{P}$ and $\mathbf{Q}$ are positive definite matrices, Cholesky factorization can be used to express them as

$$\mathbf{P} = \mathbf{L}_p\mathbf{L}_p^T \quad \text{and} \quad \mathbf{Q} = \mathbf{L}_q\mathbf{L}_q^T \quad (5)$$

where $\mathbf{L}_p = \mathbf{L}_{p1} \oplus \mathbf{L}_{p2}$ and $\mathbf{L}_q = \mathbf{L}_{q1} \oplus \mathbf{L}_{q2}$ are block-diagonal lower triangular matrices. Then, (5) and inequalities in (2a) and (2a) imply that

$$\left[\ \mathbf{L}_p^{-1}\mathbf{AL}_p \quad \mathbf{L}_p^{-1}\mathbf{B}\ \right]\left[\ \mathbf{L}_p^{-1}\mathbf{AL}_p \quad \mathbf{L}_p^{-1}\mathbf{B}\ \right]^T < \mathbf{I}$$
$$\left[\ \mathbf{L}_q^{-1}\mathbf{A}^T\mathbf{L}_q \quad \mathbf{L}_q^{-1}\mathbf{C}^T\ \right]\left[\ \mathbf{L}_q^{-1}\mathbf{A}^T\mathbf{L}_q \quad \mathbf{L}_q^{-1}\mathbf{C}^T\ \right]^T < \mathbf{I}$$

For a 1-D system, the Hankel singular values, defined as the square roots of the eigenvalues of the product of the controllability and observability grammians, are system invariants. That is, they are invariant under state-variable transformations. For a 2-D system, let $\mathbf{P}$ and $\mathbf{Q}$ be positive definite solutions of Lyapunov inequalities (2a) and (2b) and define the square roots of the eigenvalues of $\mathbf{PQ}$, i.e.,

$$\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_n,\ \mu_1,\ \mu_2,\ \cdots,\ \mu_m$$

as the Hankel singular values (h.s.v.'s) of the system. Then, these parameters are not, in general, invariant under state-variable transformations since the solutions of inequalities (2a) and (2b) are not unique. In other words, the h.s.v.'s so defined are $\mathbf{P}$ and $\mathbf{Q}$ dependent. However, this dependence provides an approach to obtain a suitable solution of (2a) and (2b) so as to achieve small reduction error. More importantly, it is shown in [4] that the reduction error introduced by the structurally balanced realization is bounded by twice the sum of the discarded Hankel singular values. Therefore, instead of the constrained minimization problem in (3a) and (3b), we consider the following *unconstrained* minimization problem:

$$\begin{aligned} \text{minimize } f(\mathbf{L}_p, \mathbf{L}_q) = \text{minimize}\,[\ & f_1(\mathbf{L}_p) + k_1 f_2(\mathbf{L}_q) \\ & + k_2 f_e(\mathbf{L}_p, \mathbf{L}_q)\ ] \quad (6) \end{aligned}$$

such that

$$f_1(\mathbf{L}_p) < 1 \quad \text{and} \quad f_2(\mathbf{L}_q) < 1 \quad (7)$$

and

$$\mathbf{L}_p > 0 \quad \text{and} \quad \mathbf{L}_q > 0$$

where

$$f_1(\mathbf{L}_p) = \|\ \mathbf{L}_p^{-1}\left[\ \mathbf{AL}_p \quad \mathbf{B}\ \right]\ \|$$
$$f_2(\mathbf{L}_p) = \|\ \mathbf{L}_q^{-1}\left[\ \mathbf{A}^T\mathbf{L}_q \quad \mathbf{C}^T\ \right]\ \|$$
$$f_e(\mathbf{L}_p, \mathbf{L}_q) = \sum_{i=r_1+1}^{n} \sigma_i + \sum_{j=r_2+1}^{m} \mu_j$$

and $k_1 > 0$ and $k_2 > 0$ are scalars. By minimizing the objective function $f(\mathbf{L}_p, \mathbf{L}_q)$ with appropriate selection of k_1 and k_2, block-diagonal matrices $\mathbf{P}$ and $\mathbf{Q}$ can be found such that the constraints in (7) are satisfied. In this way, the stability of the reduced-order system is guaranteed and the reduction error is expected to be small as well. Listed below is a step-by-step summary of the new algorithm.

Algorithm

Step 1: Find a local minimum of $f(\mathbf{L}_p, \mathbf{L}_q)$ in (6) such that the inequalities in (7) hold. The unconstrained optimization can be carried out using established numerical optimization techniques [6]. Scalars

k_1 and k_2 can be adjusted to satisfy the inequalities in (7).

Step 2: Compute structured grammians

$$\mathbf{P} = \mathbf{L}_p\mathbf{L}_p^T \text{ and } \mathbf{Q} = \mathbf{L}_q\mathbf{L}_q^T$$

Step 3: Find nonsingular matrix $\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2$ by using known algorithms [7] such that (4a) and (4b) are satisfied.

Step 4: Obtain the structurally balanced realization $\{\mathbf{A}_b, \mathbf{B}_b, \mathbf{C}_b, \mathbf{D}\}$ with

$$\mathbf{A}_b = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \quad \mathbf{B}_b = \mathbf{T}^{-1}\mathbf{B}, \quad \mathbf{C}_b = \mathbf{C}\mathbf{T}$$

and partition it as

$$\mathbf{A}_b = \left[\begin{array}{cc|cc} \mathbf{A}_{1r} & \mathbf{A}_{12} & \mathbf{A}_{2r} & \mathbf{A}_{22} \\ \mathbf{A}_{13} & \mathbf{A}_{14} & \mathbf{A}_{23} & \mathbf{A}_{24} \\ \hline \mathbf{A}_{3r} & \mathbf{A}_{32} & \mathbf{A}_{4r} & \mathbf{A}_{42} \\ \mathbf{A}_{33} & \mathbf{A}_{34} & \mathbf{A}_{43} & \mathbf{A}_{44} \end{array}\right], \quad \mathbf{B}_b = \left[\begin{array}{c} \mathbf{B}_{1r} \\ \mathbf{B}_{12} \\ \hline \mathbf{B}_{2r} \\ \mathbf{B}_{22} \end{array}\right]$$

$$\mathbf{C}_b = \left[\begin{array}{cc|cc} \mathbf{C}_{1r} & \mathbf{C}_{12} & \mathbf{C}_{2r} & \mathbf{C}_{22} \end{array}\right]$$

where $\mathbf{A}_{1r} \in R^{r_1\times r_1}, \mathbf{B}_{1r} \in R^{r_1\times q}, \mathbf{C}_{1r} \in R^{p\times r_1}, \mathbf{A}_{4r} \in R^{r_2\times r_2}, \mathbf{B}_{2r} \in R^{r_2\times q}, \mathbf{C}_{2r} \in R^{p\times r_2}$.

Step 5: The reduced-order 2-D system of order (r_1, r_2) can now be formed as $\{\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}\}$ with

$$\mathbf{A}_r = \begin{bmatrix} \mathbf{A}_{1r} & \mathbf{A}_{2r} \\ \mathbf{A}_{3r} & \mathbf{A}_{4r} \end{bmatrix}, \mathbf{B}_r = \begin{bmatrix} \mathbf{B}_{1r} \\ \mathbf{B}_{2r} \end{bmatrix}, \mathbf{C}_r = \begin{bmatrix} \mathbf{C}_{1r} & \mathbf{C}_{2r} \end{bmatrix}$$

IV. Example

In this section, the lowpass filter discussed in [3] is used to illustrate the proposed new model reduction algorithm. The state-space model of the filter is given by (1a) and (1b) with

$$\mathbf{A}_1 = \begin{bmatrix} 0.5370 & -0.0688 & 0.9855 & 0.5039 \\ 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0.5388 & -0.0666 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} -0.3907 & 0.2450 & -0.4836 & -0.2473 \\ 0.2512 & -0.1452 & 0.0270 & 0.0138 \\ 1.2705 & 1.1068 & 0.1981 & 0.1013 \\ 1.7964 & 0.4220 & 0.5921 & 0.3027 \\ 0 & 0 & -0.3934 & 0.2425 \\ 0 & 0 & 0.2520 & -0.1453 \\ 0 & 0 & 1.2708 & 1.1072 \\ 0 & 0 & 1.7975 & 0.4233 \end{bmatrix}$$

$$\mathbf{A}_4 = \begin{bmatrix} 0.4907 & 1 & 0 & 0 & 0.4907 & 0 & -0.4907 & 0 \\ -0.0274 & 0 & 0 & 0 & -0.0274 & 0 & 0.0274 & 0 \\ -0.2011 & 0 & 0 & 1 & -0.2011 & 0 & 0.2011 & 0 \\ -0.6008 & 0 & 0 & 0 & -0.6008 & 0 & 0.6008 & 0 \\ 0 & 0 & 0 & 0 & 0.4912 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0282 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.2011 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -0.6008 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}_1 = 10^{-3}\begin{bmatrix} 1.34044 & 0 & 1.34044 & 0 \end{bmatrix}^T$$

$$\mathbf{B}_{21} = 10^{-3}\begin{bmatrix} -0.6578 & 0.0367 & 0.2695 & 0.8054 \end{bmatrix}^T$$

$$\mathbf{B}_{22} = 10^{-3}\begin{bmatrix} -0.6585 & 0.0378 & 0.2695 & 0.8054 \end{bmatrix}^T$$

$$\mathbf{B}_2 = \begin{bmatrix} \mathbf{B}_{21}^T & \mathbf{B}_{22}^T \end{bmatrix}^T$$

$$\mathbf{C}_1 = \begin{bmatrix} 0.9837 & 0.5016 & 0.9856 & 0.5039 \end{bmatrix}$$

$$\mathbf{C}_2 = \begin{bmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{D} = 10^{-3} \times 1.34044$$

The amplitude response of the original filter of order (4, 8) is depicted in Figure 1. We first apply the algorithm in [5] to obtain a reduced-order filter of order (4, 4), represented by $\mathbf{H}_r(z_1, z_2)$, whose amplitude response is shown in Figure 2. Then, the new algorithm is used to obtain a reduced-order filter of order (4, 4), represented by $\mathbf{H}_{mr}(z_1, z_2)$, whose amplitude response is shown in Figure 3.

The performance of the reduced-order models obtained using pseudo-balanced, quasi-balanced, and structurally balanced (algorithm in [5]) model reductions and the proposed modified structurally balanced model reduction is summarized in Table 1. Column 3 of Table 1 gives the l_∞ norms of the reduction error matrices, which are defined as

$$e_\infty = \max_{0\le\omega_1,\omega_2\le 1} |\mathbf{H}(e^{j2\pi\omega_1}, e^{j2\pi\omega_2}) - \mathbf{H}_r(e^{j2\pi\omega_1}, e^{j2\pi\omega_2})|$$

From Table 1, we note that the reduction error introduced by the modified structurally balanced model reduction is much smaller than those introduced by the structurally balanced model reduction obtained with the algorithm in [5] and the quasi-balanced model reduction.

Table 1: Performance of Reduced-Order Models of Order (4, 4)

Balanced Model Reductions	Stability	e_∞
pseudo-balanced	stable	0.1107
quasi-balanced	stable	0.2783
structurally balanced	stable	1.0682
modified structurally balanced	stable	0.1595

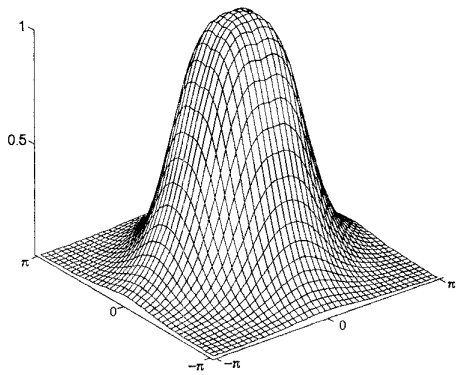

Figure 1: Amplitude response of the original filter of order (4, 8).

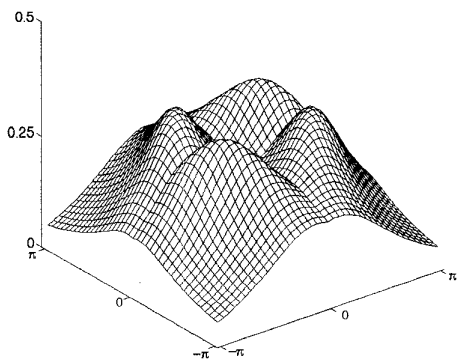

Figure 2: Amplitude response filter of order (4, 4) obtained with the algorithm in [6].

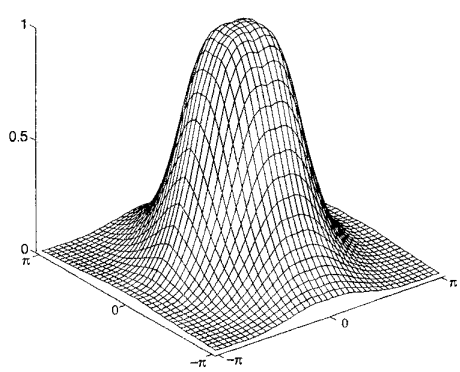

Figure 3: Amplitude response of the filter of order (4, 4) obtained with the new algorithm.

V. Conclusion

A new algorithm for structurally balanced model reduction of 2-D discrete systems has been proposed, in which the objective function of the unconstrained minimization problem includes a term which is related to the sum of discarded 2-D Hankel singular values. The reduced-order system obtained by truncating the structurally balanced realization is guaranteed to be stable and, further, reduced approximation error is achieved.

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