ON THE USE OF DISCRETE LAPLACIAN OPERATORS IN IMAGE RESTORATION

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ABSTRACT

The role of the stabilizing functional in the Tikhonov regularization is as crucial as that of the regularization parameter. However, a 3×3 discrete differential operator is usually used to generate the stabilizing functional. In this paper, it is demonstrated that the restoration quality is practically independent of the size of the generating stabilizing operator. As a result, the choice of small size of the generating stabilizing operator is justified. Moreover, this observation leads to an advantage that spatial-domain implementing filters can be designed within a reasonable small order.

I. Introduction

Due to the imperfection of image systems, recorded images are usually degraded versions of the original scenes. Image restoration is a process that attempts to reduce the degradations from a recorded image so that the resulting image will best approximate the original scene subject to some given criteria. This requires the knowledge of the image system used and the degradation mechanism involved. In this paper, image degradation is described as a spatially invariant blur plus an additive white noise process. Thus we have

$$g = Hf + n \tag{1}$$

in which g and f represent the lexicographically ordered recorded and original images, respectively, the matrix H represents the spatially invariant linear degradation, and n represents the additive white noise. It has been observed that the image restoration problem of solving (1) for f for a given image g is an ill-posed problem. The standard method for solving (1) is to used the regularization theory due to Tikhonov [1] where an approximate solution to (1) is obtained by minimizing the quadratic functional

$$Q(\hat{f}) = ||H\hat{f} - g||^2 + \lambda ||C\hat{f}||^2 \tag{2}$$

where C is a stabilizing operator and λ is a regularization parameter. The stabilizing functional $||C\hat{f}||^2$ in (2) measures the degree of regularity of the solution $\hat{f}$ and plays an important role in the regularization process. The choice of the stabilizing operator C or its generating stabilizing operator C_0 [2] is usually based on certain assumptions of the image formation process. A general theory of choosing the stabilizing functional can be found in [3]. For good restoration results, the stabilizing functional by itself is not sufficient. In the regularization process, it is the regularization parameter λ that controls the tradeoff between the solution accuracy $||H\hat{f} - g||^2$ and its degree of regularity $||C\hat{f}||^2$. The choice of λ is another important issue in the regularization process. Detailed discussions of choosing the regularization parameter and stabilizing functional are given in a separated paper [4].

If the stabilizing functional in (2) is designed to damp out the high frequencies content of the estimated image, then a discrete Laplacian is conventionally chosen. It is demonstrated in this case that the commonly used restoration performance measures such as the improvement signal-to-noise-ratio (ISNR) and the mean square error (MSE) [4] are practically independent of the order of the generating digital Laplacians chosen. This provides a strong support to the use of the simplest 3×3 discrete Laplacian

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{3}$$

as the generating stabilizing operator.

II. More about the Stabilizing Operator

In regularization, the stabilizing functional $||C\hat{f}||^2$ is chosen as a measure of smoothness. The reason behind this choice can be justified as follows.

A. *Matching the Wiener Filter*

It is easy to see that a regularized solution of (1) formulated as the solution of (2) is given by

$$f_\lambda = (H^T H + \lambda C^T C)^{-1} H^T g \tag{4}$$

Matrices H and C are block circulant. An important property of block circulant matrices is that they have the same set of eigenvectors. To be specific, there exists an unitary matrix U such that any block circulant matrix A can be expressed as

$$A = U\Lambda U^* \tag{5}$$

where Λ is a diagonal matrix determined by the eigenvalues of A and $*$ denotes the operator of complex conjugate transposition. It can be shown [2] that this U acts like a discrete Fourier transform (DFT) and consequently solution (4) can be obtained in the frequency domain as follows:

$$f_\lambda(\omega_1,\omega_2) = \frac{H^*(\omega_1,\omega_2)g(\omega_1,\omega_2)}{|H(\omega_1,\omega_2)|^2 + \lambda|C(\omega_1,\omega_2)|^2} \tag{6}$$

Note that the classical Wiener filter solution of (1) is given by [5]

$$\hat{f}_{\text{Wiener}}(\omega_1,\omega_2) = \frac{H^*(\omega_1,\omega_2)g(\omega_1,\omega_2)}{|H(\omega_1,\omega_2)|^2 + \frac{P_n(\omega_1,\omega_2)}{P_f(\omega_1,\omega_2)}} \tag{7}$$

where $P_f(\omega_1,\omega_2)$ and $P_n(\omega_1,\omega_2)$ are the power spectra of f and n, respectively. By comparing (6) and (7), we have

$$|C(\omega_1,\omega_2)|^2 \propto P_n(\omega_1,\omega_2)/P_f(\omega_1,\omega_2) \tag{8}$$

Since the noise process is assumed to be white, $P_n(\omega_1,\omega_2)$ is a constant. Thus

$$|C(\omega_1,\omega_2)|^2 \propto 1/P_f(\omega_1,\omega_2) \tag{9}$$

For most images, the spectrum $P_f(\omega_1,\omega_2)$ is always concentrated in the low frequency region. Equation (9) indicates that C should act like as a highpass filter.

B. *An Analytical Interpretation*

First observe that a solution $\hat{f}$ of (2) must satisfy the normal equations

$$(H^T H + \lambda C^T C)\hat{f} = H^T g \tag{10}$$

Let $\Gamma(A)$ be the condition number of A. If $\Gamma(A)$ is large, then A^{-1} is sensitive to small perturbations in A, and the problem of inverting A is ill-conditioned. For a given $\lambda > 0$, the idea of regularizing the ill-conditioned problem (1) is to choose a matrix C such that the linear system (10) is no longer ill-conditioned. This will be the case if

$$\Gamma(H^T H + \lambda C^T C) \ll \Gamma(H) \tag{11}$$

It can be shown that [6]

$$\Gamma(H^T H + \lambda C^T C) = \frac{\max\limits_i(h_i^2 + \lambda c_i^2)}{\min\limits_i(h_i^2 + \lambda c_i^2)} \tag{12}$$

where $\{h_i\}$ and $\{c_i\}$ are the eigenvalues of H and C, respectively. In order to satisfy (11), (12) suggests that C should be chosen such that if h_i^2 is large, $h_i^2 + \lambda c_i^2$ is approximately the same as h_i^2, and if h_i^2 small, $h_i^2 + \lambda c_i^2$ is significantly different from zero. If the sequence h_i^2 is decreasing, the c_i^2 should be chosen as an increasing sequence. Note that H represents in general a lowpass filter, as a result C should be chosen as a highpass filter.

III. Discrete Laplacian Operators As Stabilizing Operators

The previous section explains why the discrete Laplacian operator (3), has always been used in regularization image restoration. However, it does not necessarily mean that it is an optimal choice, as can be seen from an examination on the isotropic property of the continuous Laplacian operator $(\partial^2/\partial x^2 + \partial^2/\partial y^2)$. In image processing, we often want to sharpen blurred features such as edges and lines that may occur in any direction. Therefore we want our differential operators to be isotropic, i.e., rotation-invariant (in the sense that the rotation and the differentiation are interchangeable). It is easy to see that the Laplacian operator $(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ is isotropic. Moreover, it can be shown that an isotropic linear differential operator involves only even-order derivatives. The Laplacian operator is the simplest possible isotropic operator consisting of even-order derivatives. Note that the discrete Laplacian operator (3) is only symmetric for low frequencies (see Fig. 1), and hence it is not a good approximation to the Laplacian operator. There are other ways to construct a digital 'Laplacian', a discrete approximation to the Laplacian operator. For example, one can use different sizes of neighborhood (e.g., the 3×3 neighborhood of consisting point (k,l) and its 8 horizontal, vertical, and diagonal neighboring points), or use a weighted average over a specific neighborhood. Another example is the mth order binomial filter [7]. With $m = 2$, the 2nd-order binomial filter is described by its impulse response

$$L' = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 2 & -12 & 2 \\ 1 & 2 & 1 \end{bmatrix} \tag{13}$$

Its frequency response is given by

$$\begin{aligned} L'(\omega_1,\omega_2) &= -(\omega_1^2+\omega_2^2) + \frac{(\omega_1^2+\omega_2^2)^2}{24} + \frac{\omega_1^2\omega_2^2}{6} \\ &\quad + O((\omega_1^2+\omega_2^2)^3) \end{aligned} \tag{14}$$

and the amplitude response of L' is shown in Fig. 2. At high frequencies, the amplitude response of L' as well as L (3) exhibits considerable deviation from the ideal Laplacian. But by comparing the contours of L in (3) with that of L', we conclude that L' is slightly more symmetric than L. A good approximation to the ideal Laplacian can be obtained when the order of the binomial filter is sufficiently large.

Theoretically, the larger the mask used in designing the digital Laplacian operator, the better the resulting approximation. Since our image system is assumed to be linear shift-invariant and both H and C are block circulant, the restored image can be computed through DFT implementation. This implies that the size of the generating stabilizing operator C_0 can be as large as $N \times N$, the size of the recorded image g, without any additional computation cost. It follows immediately that the $N \times N$ discrete Laplacian L^N defined implicitly as

$$\mathrm{DFT}(L^N) = \left[((1 - \frac{2(k-1)}{N-1})\pi)^2 + ((-1 + \frac{2(l-1)}{N-1})\pi)^2\right]_{(k,l)} \quad (15)$$

which is the $N \times N$ uniform sampling of the continuous Laplacian amplitude response $\omega_1^2 + \omega_2^2$, should be the most suitable discrete Laplacian operator as a generating stabilizing operator. Note that since restored images are obtained through the DFT, an explicit L^N is unnecessary.

An extensive examination was performed on the restoration performance using different test images with different blurs and noise levels has been studied. It is found that restoration performance measures, such as ISNR and MSE, are practically independent of sizes of digital Laplacians as generating stabilizing operator. This provides strong support for using the simplest discrete Laplacian operator (3) in regularized image restoration. The following example illustrate the effects of the stabilizing operators on the regularization image restoration. The sample image is the noisy and exponentially blurred image (Fig. 3(a)). Each of the discrete Laplacian operators L (3), L' (13) and L^N (15) is used in the regularization restoration process, respectively, with the regularization parameter $\lambda = 0.0008$. The optimal ISNR (MSE) values are approximately the same and are equal to 7.20 (206.70). The restored image is shown on Fig. 3(b).

IV. R-Filters

As demonstrated in previous section, the quality of restored images is generally independent of the order of the generating stabilizing operator in regularization image restoration. This observation indeed has little practical significance for those restoration filters which are implemented in frequency domain, such as the CLS and modified Wiener filters. It is because in DFT implementation, the computation time of C is independent of the size of the generating stabilizing operator C_0. Note also that these filters cannot be implemented adaptively. Recently, we have been developing a new class of restoration filters, called R-filters [8], which are implemented in spatial domain. Most important, they are suitable for adaptive design. The importance of the result in previous section will be clear in the design of 1-D R-filters. Once again, one will see that only the property but not the size of the filter $C_0(z_1, z_2)$ counts in the design of 2-D R-filters.

A. *1-D R-Filters*

Consider a 1-D linear shift-invariant discrete imaging system

$$g(k) = h(k) * f(k) + n(k) \quad (16)$$

where $\{g(k)\}$ and $\{f(k)\}$ represent the lexicographically ordered recorded and original images, respectively, $\{n(k)\}$ is the additive noise, $\{h(k)\}$ characterizes a 1-D image blurring process and is assumed to possess finite length, and $*$ denotes the convolution. A regularized approximation to $\{f(k)\}$, denoted by $\{f_\lambda(k)\}$ can be obtained by minimizing

$$\xi_\lambda(f) = \sum_{k=-\infty}^{\infty} [h(k)*f(k) - g(k)]^2 + \lambda \sum_{k=-\infty}^{\infty} [c(k)*f(k)]^2 \quad (17)$$

Let $F(\omega)$, $G(\omega)$, $H_0(\omega)$, and $C_0(\omega)$ be the z-transform of f, g, h, and c evaluated on the unit circle $z = e^{j\omega}$. (In matrix notation, (16) can be written as $g = Hf + n$, where H is generated from the matrix H_0 [2]. The relation between the stabilizing operator C in the regularization process and C_0 is similarly defined.) By Parseval's formula, (17) can be written as

$$\xi_\lambda(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e_\lambda(f, \omega) d\omega \quad (18)$$

$$e_\lambda(f,\omega) = |H_0(\omega)F(\omega) - G(\omega)|^2 + \lambda |C_0(\omega)F(\omega)|^2 \quad (19)$$

It can readily be shown that, regarding ω as a real parameter, the minimum of $e_\lambda(f,\omega)$ (and therefore the minimum of $\xi_\lambda(f)$) is achieved by

$$F_\lambda(\omega) = \frac{G(\omega)H_0(-\omega)}{|H_0(\omega)|^2 + \lambda |C_0(\omega)|^2} \quad (20)$$

In z-domain, (20) defines an R-filter with transfer function

$$R_\lambda(z) = H_0(z^{-1})S_\lambda(z) \quad (21)$$

$$S_\lambda(z) = 1/[H_0(z)H_0(z^{-1}) + \lambda C_0(z)C_0(z^{-1})] \quad (22)$$

It can be shown that

$$S_\lambda(z) = U_\lambda(z)U_\lambda(z^{-1}) \quad (23)$$

where $U_\lambda(z)$ represents a linear shift-invariant, stable, recursive, causal 1-D digital filter which is of the form

$$U_\lambda(z) = b_0/(1 + a_1 x^{-1} + \cdots + a_M z^{-M}) \quad (24)$$

where $M = \max(m,n)$ in which $m+1$ and $n+1$ are the lengths of $h(k)$ and $c(k)$, respectively. As is illustrated in previous section, a reasonable design for the filter $c(k)$ should be of length 3 or 5. In practice, the length of $h(k)$ is usually between 5 and 15, as a result the order of $U_\lambda(z)$ is always the same as of $H(z)$. Hence $U_\lambda(z)$ can be conveniently and efficiently designed in advanced with enough number of λ's and it follows that the spatial-domain implementation of the R-filter $R_\lambda(z)$ allows regions of interest processing and adaptive processing. For example if a rule for detecting edges or uniform zones is designed, then a $R_\lambda(z)$ with small λ should be used when edge is detected, and a $R_\lambda(z)$ with large λ should be used when uniform zone is detected. Although this adaptive rule is so simple, it reduces the ringing effects to a significant amount.

B. *2-D R-Filters*

The 2-D R-filter theory is just an extension of 1-D R-filter theory. The derivation of the R-filter $R_\lambda(z_1, z_2)$ starts with replacement of the discretized imaging system (1) by

$$g(k_1, k_2) = h(k_1, k_2) * f(k_1, k_2) + n(k_1, k_2) \qquad (25)$$

and ends up with

$$R_\lambda(z) = H_0(z^{-1})W_\lambda(z) \qquad (26)$$

$$W_\lambda(z) = 1/[H_0(z)H_0(z^{-1}) + \lambda C_0(z)C_0(z^{-1})] \qquad (27)$$

where $z = (z_1, z_2)$ and $z^{-1} = (z_1^{-1}, z_2^{-1})$. Due to the fundamental difference between one-variable and two-variable polynomials, there is no similar analytic decomposition (23) of the 2-D transfer function of $W_\lambda(z_1, z_2)$. However, its amplitude response

$$1/[|H_0(e^{j\omega_1}, e^{j\omega_2})|^2 + \lambda|C_0(e^{j\omega_1}, e^{j\omega_2})|^2] \qquad (28)$$

is quadrantally symmetric with respect to the origin in the (ω_1, ω_2) plane, a stable causal 2-D filter $T(z_1, z_2)$ can be designed using the combined singular-value decomposition and balanced approximation method [9,10]. Due to the designed method, the filter $T(z_1, z_2)$ so defined is usually of low order independent of the orders of $H_0(z_1, z_2)$ and $C_0(z_1, z_2)$. Therefore, an approximate regularized solution of (1) can be obtained by filtering $g(k_1, k_2)$ sequentially through $H_0(z_1^{-1}, z_2^{-1})$, $T_0(z_1, z_2)$ and $T_0(z_1^{-1}, z_2^{-1})$.

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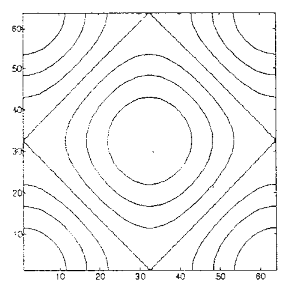


Fig. 1 Contour plot of L.

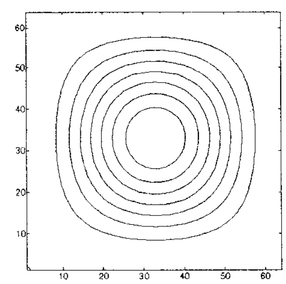
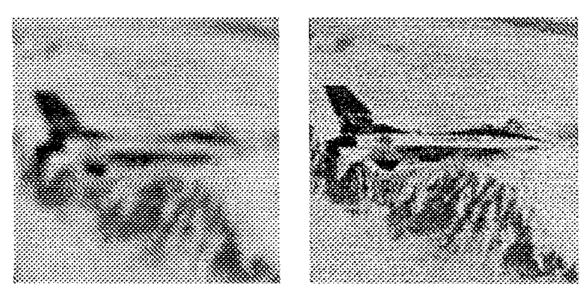


Fig. 2 Contour plot of L'.



(a) (b)

Fig. 3

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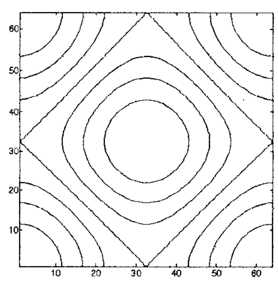

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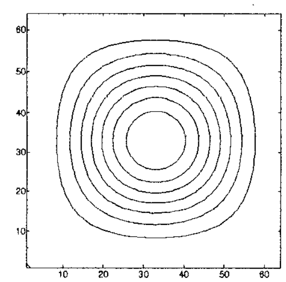

Fig. 2 Contour plot of L'.

(a)

(b)

Fig. 3