Determination of the Transfer-Function Matrix from the Two-Dimensional Fornasini-Marchesini State-Space Model

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Abstract—Two efficient algorithms for the determination of the transfer-function matrix of a two-dimensional (2-D) discrete system represented by the Fornasini-Marchesini state-space model are proposed. The development of the algorithms involves two distinct steps. First, the 2-D transfer-function matrix is reformulated in terms of the characteristic polynomials of the coefficient matrices involved. Second, the coefficient matrices are determined by using an efficient algorithm for the determination of 1-D polynomial coefficients. The simplicity and efficiency of the proposed algorithms are illustrated by examples.

I. Introduction

The representation of the transfer-function matrix of a two-dimensional (2-D) discrete system by a state-space model and vice versa are two basic problems of great importance in system analysis and design. One of the commonly used state-space models for 2-D discrete systems is the Roesser model [1]. Several algorithms for the derivation of the 2-D transfer-function matrix from the Roesser state-space model have been proposed [2]-[6]. Another popular state-space representation for 2-D discrete systems is the Fornasini-Marchesini model [7]. To date, no efficient algorithms for the determination of the 2-D transfer-function matrix from the Fornasini-Marchesini state-space representation have been reported.

This paper proposes new algorithms for the determination of the 2-D transfer-function matrix from the Fornasini-Marchesini state-space model. The algorithms are extensions of efficient algorithms for the determination of the 2-D transfer-function matrix from the Roesser state-space model [6]. First, the transfer-function matrix is reformulated in terms of the characteristic polynomials of several matrices that depend on one complex variable. Second, algorithms are proposed that identify the coefficients of a 1-D polynomial of order \( n \) when its values at \( (n + 1) \) points on the unit circle are known. Our algorithms entail solving a system of linear equations whose coefficient matrix is an unitary Vandermonde matrix. Examples are given to illustrate the efficiency of the proposed algorithms.

II. Algorithms for SISO Systems

Consider a single-input, single-output (SISO), linear, shift-invariant, discrete 2-D system represented by the Fornasini-Marchesini state-space model given by

\[
\begin{align*}
\mathbf{x}(k+1, l+1) &= \mathbf{A}_1 \mathbf{x}(k, l+1) + \mathbf{A}_2 \mathbf{x}(k+1, l) + \mathbf{b}_1 \mathbf{u}(k, l+1) + \mathbf{b}_2 \mathbf{u}(k+1, l), \\
y(k, l) &= c \mathbf{x}(k, l) + d \mathbf{u}(k, l)
\end{align*}
\]

(1a)

(1b)

where \( \mathbf{x} \in \mathbb{R}^n \) is the state vector, and \( \mathbf{u} \) and \( \mathbf{y} \) are the input and output, respectively. The transfer function of the system is given by

\[
H(z_1, z_2) = \frac{c(z_1 z_2 I - z_2 \mathbf{A}_1 - z_1 \mathbf{A}_2)^{-1}(z_2 \mathbf{b}_1 + z_1 \mathbf{b}_2) + d}{z_1 z_2 I - \mathbf{A}_2}
\]

A. Reformulation of the Transfer Function

We first reformulate the transfer function \( H(z_1, z_2) \) in terms of the characteristic polynomials of the matrices involved. By using a well-known formula for the transfer function of a 1-D SISO state-space model (see Appendix A.13 of [8]), \( H(z_1, z_2) \) can be rewritten as

\[
H(z_1, z_2) = \frac{\det(z_2 I - \mathbf{A}_2 + b_2 c) \det(z_1 I - F(z_2))}{\det(z_2 I - \mathbf{A}_2) \det(z_1 I - E(z_2))} + d - 1
\]

(2)

\[
= \frac{P(z_2, A_2 - b_2 c) P(z_1, F(z_2))}{P(z_2, A_2) P(z_1, E(z_2))} + d - 1
\]

\[
= \frac{\sum_{k=0}^{n} q_k(z_2) z_2^k}{\sum_{k=0}^{n} p_k(z_2) z_2^k}
\]

(3)

where \( p_k(z_2) \) and \( q_k(z_2) \) are polynomials in \( z_2 \) of order not greater than \( n \), and

\[
E(z_2) = z_2 A_1 (z_2 I - A_2)^{-1}
\]

(4a)

\[
F(z_2) = z_2 (A_1 - b_1 c) (z_2 I - A_2 + b_2 c)^{-1}
\]

(4b)
In (2), \( P(z_2, A_2) \), \( P(z_2, A_2 - b_2 c) \), \( P(z_1, E(z_2)) \), \( P[z_1, F(z_2)] \) are the characteristic polynomials of \( A_2 \), \( A_2 - b_2 c \), \( E(z_2) \), and \( F(z_2) \), respectively. From (2) and (3), it follows that

\[
\begin{align*}
\sum_{k=0}^{n} q_k(z_2) z_1^k &= P(z_2, A_2 - b_2 c) P[z_1, F(z_2)] \\
&\quad + (d-1)P(z_2, A_2) P[z_1, E(z_2)] \quad \text{(5a)} \\
\sum_{k=0}^{n} p_k(z_2) z_1^k &= P(z_2, A_2) P[z_1, E(z_2)] \quad \text{(5b)}
\end{align*}
\]

B. Determination of the Coefficients of a 1-D Polynomial

An efficient method for the determination of the coefficients of a 1-D polynomial will now be examined. Let

\[
p(z_2) = \alpha_0 z_2^n + \cdots + \alpha_1 z_2 + \alpha_0
\]

be a polynomial of order \( n \) with coefficients \( \alpha_0, \ldots, \alpha_1, \alpha_0 \). Also let \( \{z_2(l), 0 \leq l \leq n\} \) be \( (n+1) \) points that are uniformly distributed on the unit circle of the complex \( z_2 \) plane, i.e.,

\[
z_2(l) = e^{2\pi i l/(n+1)}, \quad 0 \leq l \leq n \quad \text{(6)}
\]

If the values \( \{p_l = p[z_2(l)], 0 \leq l \leq n\} \) are known, then the coefficients \( \{\alpha_l, 0 \leq l \leq n\} \) can be determined as

\[
\alpha = V^{-1}(z_2) \mathbf{q} \quad \text{(7)}
\]

where

\[
\alpha = [\alpha_0 \cdots \alpha_1 \alpha_0]^T, \quad \mathbf{q} = [p_0 \, p_1 \cdots p_n]^T,
\]

and \( V(z_2) \) is the \( (n+1) \times (n+1) \) Vandermonde matrix whose second to last column is

\[
z_2 = [z_2(0) \quad z_2(1) \cdots z_2(n)]^T
\]

that is,

\[
V(z_2) = \begin{bmatrix} z_2(0)^n & \cdots & z_2(0) & 1 \\
\vdots & \ddots & \vdots & \vdots \\
z_2(n)^n & \cdots & z_2(n) & 1 \end{bmatrix}
\]

Since \( z_2(l), 0 \leq l \leq n, \) are distinct, \( V(z_2) \) is always nonsingular. More important, it follows from (6) that

\[
V^H(z_2) V(z_2) = (n+1) I
\]

where \( V^H(z_2) \) denotes the complex-conjugate transpose of \( V(z_2) \). Therefore, (7) can be written as

\[
\alpha = \frac{1}{n+1} V^H(z_2) \mathbf{q} \quad \text{(8)}
\]

Equation (8) provides an efficient formula for the determination of 1-D polynomial \( p(z_2) \).

C. New Algorithms

The algorithms are based on (5a), (5b), and the efficient method for the determination of the coefficients of a 1-D polynomial. It is assumed that matrices \( A_2 \) and \( A_2 - b_2 c \) have no eigenvalues on the unit circle. The case where the matrices \( A_2 \) and/or \( A_2 - b_2 c \) have eigenvalues on the unit circle will be considered in subsection D.

Algorithm 1

Step 1: Use (4a) and (4b) to evaluate \( E(z_2) \) and \( F(z_2) \) over the set of points defined in (6).

Step 2: Compute the determinants of \( z_2 I - A_2 \) and \( z_2 I - A_2 - b_2 c \), and the characteristic equations of \( E(z_2) \) and \( F(z_2) \) for \( z_2 = z_2(l), 0 \leq l \leq n \).

Step 3: Use (5a) and (5b) to obtain the values of \( p_k[z_2(l)] \) and \( q_k[z_2(l)] \) for \( 0 \leq l \leq n, 0 \leq k \leq n \).

Step 4: For each \( k, 0 \leq k \leq n \), form vectors \( \mathbf{q} = [p_0 \cdots p_n]^T \) and \( \mathbf{q} = [q_0 \cdots q_n]^T \), and determine polynomials \( p_k(z_2) \) and \( q_k(z_2) \) by using (8).

D. Unstable and Special Cases

If \( A_2 \) has eigenvalues on the unit circle (the system is unstable) or the special case where \( A_2 - b_2 c \) has eigenvalues on the unit circle, the \( n+1 \) points defined by (6) need to be modified to

\[
z_2(l) = r e^{2\pi i l/(n+1)}, \quad 0 \leq l \leq n \quad \text{(9)}
\]

where \( r > 0 \) denotes the radius of a circle in the \( z_2 \) plane where \( A_2 \) and \( A_2 - b_2 c \) have no eigenvalues on the circle. Consequently, (8) is modified to

\[
\alpha = \frac{1}{n+1} \text{diag}\{r^{-n}, \ldots, r^{-1}, 1\} V^H(z_2) \mathbf{q} \quad \text{(10)}
\]

Note that (8) is a special case of (10) with \( r = 1 \), as may be expected.

III. Dual Algorithms

A dual algorithm to Algorithm 1 can be obtained if the roles of variables \( z_1 \) and \( z_2 \) are interchanged. By representing \( H(z_1, z_2) \) in (3) as

\[
H(z_1, z_2) = \frac{\sum_{l=0}^{n} \tilde{q}_l(z_1) z_2^l}{\sum_{l=0}^{n} \tilde{p}_l(z_1) z_2^l}
\]

where \( \tilde{p}_l(z_1) \) and \( \tilde{q}_l(z_1) \) are polynomials in \( z_1 \), it can be readily shown that

\[
\sum_{l=0}^{n} \tilde{q}_l(z_1) z_2^l = P(z_2, A_1 - b_1 c) P[z_2, F(z_1)]
\]
where
\[ \tilde{E}(z_1) = z_1A_2(z_1I-A_1)^{-1} \]  
\[ \tilde{F}(z_1) = z_1(A_2-b_2c)(z_1I-A_1+b_1c)^{-1} \]

Further, (8) needs to be modified as
\[ \alpha = \frac{1}{n+1} \bar{V}^H(z_1)q \]  
where \( z_1 = \begin{bmatrix} z_1(0) & z_1(1) & \cdots & z_1(n) \end{bmatrix}^T \) with
\[ z_1(k) = e^{j2\pi k/(n+1)}, \quad 0 \leq k \leq n \]

The dual algorithm is as follows:

**Algorithm 2**

**Step 1:** Use (12a) and (12b) to evaluate \( \tilde{E}(z_1) \) and \( \tilde{F}(z_1) \) over the set of points defined by (14).

**Step 2:** Compute the characteristic equations of \( A_1, A_1-b_1c, \tilde{E}(z_1), \) and \( \tilde{F}(z_1) \) for \( z_1 = z_1(k), 0 \leq k \leq n \).

**Step 3:** Use (11a) and (11b) to obtain the values of \( \hat{q}_l[z_1(k)] \) and \( \hat{p}_l[z_1(k)] \) for \( 0 \leq l \leq n, 0 \leq k \leq n \).

**Step 4:** For each \( l, 0 \leq l \leq n \), form vectors \( q = [\hat{q}_0 \cdots \hat{q}_n]^T \) and \( q = [\hat{q}_0 \cdots \hat{q}_n]^T \), and determine polynomials \( \hat{p}_l(z_1) \) and \( \hat{q}_l(z_1) \) by using (13).

Obviously, Algorithm 2 can be used to evaluate \( H(z_1, z_2) \) only if matrices \( A_1 \) and \( A_1-b_1c \) have no eigenvalues on the unit circle. If matrix \( A_1 \) or \( A_1-b_1c \) has eigenvalues on the unit circle, then modifications similar to (9) and (10) should be made.

**III. Algorithms for MIMO Systems**

Consider a multi-input, multi-output (MIMO), linear, shift-invariant, 2-D discrete system represented by the Fornasini-Marchesini state-space model given by
\[ x(k+1, l+1) = A_1x(k, l+1) + A_2x(k+1, l) + Bu(k, l+1) + B_2u(k, l+1) \]  
\[ y(k, l) = Cx(k, l) + Du(k, l) \]

where \( u \in \mathbb{R}^m \), \( y \in \mathbb{R}^t \), and \( D \in \mathbb{R}^{m \times t} \). The \( s \times t \) transfer-function matrix of the system can be expressed as
\[ H(z_1, z_2) = C(z_1z_2I-z_2A_1-z_1A_2)^{-1}(z_1B_1+z_1B_2)+D \]

whose entry \((k, l)\) is a scalar rational function given by
\[ H_{kl}(z_1, z_2) = C_k(z_1z_2I-z_2A_1-z_1A_2)^{-1}(z_2B_1l+z_1B_2)+D_{kl} \]

where \( C_k, B_1l, \) and \( B_2l \) are the \( k \)th row of \( C \) and the \( l \)th column of \( B_1 \) and \( B_2 \), respectively. Therefore, the transfer-function matrix \( H(z_1, z_2) \) given by (16) can be evaluated entry by entry and each entry can be treated as an SISO transfer function. Hence, (5a) associated with \( H_{kl}(z_1, z_2) \) in (17) becomes
\[ \sum_{k=0}^n q_k(z_2) z_1^k = P(z_2, A_2-B_2lC_k)P[z_1, \tilde{F}(z_2)] \]  
\[ + (D_{kl}-1)P(z_2, A_2)P[z_1, E(z_2)] \]

where
\[ \tilde{F}(z_2) = z_2(A_1-B_1lC_k)(z_2I-A_2+B_2lC_k)^{-1} \]

Therefore, Algorithm 1 can be extended to deal with the MIMO case by substituting (18) and (19) into (5a) and (4b), respectively.

Similarly, Algorithm 2 can be extended to deal with the MIMO case by modifying (11a) and (12b).

**IV. Examples**

Example 1 is a two-input two-output system of order \((2, 2)\), which was used to illustrate the algorithm in [3]. The system is represented by the Roesser model with
\[ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ - & - & + & - \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} \]
\[ B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}^T \]
\[ C = [C_1 \ C_2] = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \]

It can be represented by the Fornasini-Marchesini model [7] with
\[ A_1 = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \]
\[ A_2 = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad C = C \]

The transfer-function matrix obtained by using Algorithms 1 and 2 is
\[ H(z_1, z_2) = \begin{bmatrix} H_1(z_1, z_2) & H_2(z_1, z_2) \\ H_3(z_1, z_2) & H_4(z_1, z_2) \end{bmatrix} \]

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Table 1: Performance of the Algorithms

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Computational Complexity, Flops</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Example 1</td>
</tr>
<tr>
<td>1</td>
<td>6.195 × 10^4</td>
</tr>
<tr>
<td>2</td>
<td>6.044 × 10^4</td>
</tr>
</tbody>
</table>

where the denominator is given by the matrix

\[ D_t = \begin{bmatrix}
1 & -2 & -1 \\
4 & -10 & -2 \\
4 & -12 & 1
\end{bmatrix} \]

and the numerators are specified by \( N_{11}, \ N_{12}, \ N_{13}, \) and \( N_{14} \) as follows:

\[ N_t = \begin{bmatrix}
N_{11} & N_{12} \\
N_{13} & N_{14}
\end{bmatrix} = \begin{bmatrix}
0 & 2 & 0 & 1 & 1 \\
-1 & 6 & 7 & 0 & 5 & 3 \\
-2 & 13 & 0 & 0 & 6 & 0 \\
0 & -3 & 0 & -3 & -3 & -3 & -3 \\
3 & -3 & -14 & 1 & -1 & -5 \\
6 & -13 & -8 & 2 & -3 & -4
\end{bmatrix} \]

Example 2 is an SISO 2-D discrete system of order (16, 8), represented by the Fornasini-Marchesini state-space model given in (1a) and (1b), where each entry of \( \{A_1, A_2, b_1, b_2, c, d\} \) is a random number chosen from a normal distribution with zero mean and unit variance.

Example 3 is a four-input two-output 2-D discrete system of order (8, 16) represented by the Fornasini-Marchesini state-space model given in (15a) and (15b), where each entry of \( \{A_1, A_2, B_1, B_2, C, D\} \) is a random number chosen from a normal distribution with zero mean and unit variance.

The amounts of computation required by the algorithms for the three examples are listed in Table 1. It is evident that Algorithms 1 and 2 require different amounts of computation if the order of the system \( n_1 \neq n_2 \) \((n_1, n_2 \leq n)\). Extensive results with \( 1 \leq n_1 \leq 30 \) and \( 1 \leq n_2 \leq 30 \) have shown that Algorithm 1 requires less computation than Algorithm 2 when \( n_1 < n_2 \) (see Example 3), and Algorithm 2 requires less computation when \( n_1 > n_2 \) (see Example 2).

V. Conclusion

Two algorithms for the determination of the transfer-function matrices of 2-D discrete systems using the Fornasini-Marchesini state-space model have been proposed. These are based on a 1-D polynomial determination technique. The algorithms are efficient and reliable, and can be used for multi-input, multi-output two-dimensional discrete systems.

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Reference


