An Iterative Algorithm for the Design of 2-D Nonseparable Diamond-Shaped Filter Banks

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Abstract — An iterative method for the design of two-dimensional (2-D) nonseparable diamond-shaped filter banks is proposed. Matrix techniques are used to re-formulate a least-squares objective function as a standard fourth-order function of the filter coefficients. A linearization procedure is then applied to obtain a modified quadratic objective function. As a result, the major step in the iterative algorithm proposed is to minimize a standard unconstrained quadratic function, which can be done with high computation efficiency. Compared with other existing design methods, our method is a direct approach that entails more flexibility in achieving design specifications and improved computational efficiency, as will be demonstrated by design examples.

I. INTRODUCTION

One of the most important classes of 2-D filter banks is the class of nonseparable diamond-shaped filter banks, in which significant horizontal and vertical frequency components are retained while less important diagonal frequency components are rejected. Most of the existing methods for the design of 2-D nonseparable diamond-shaped filter banks start by designing a one-dimensional (1-D) prototype filter and then applying a suitable transformation to get a 2-D filter [1]-[3].

Many 2-D filter design techniques, like the window method, McClellan transformation method, and the optimization method, have been developed [4]. In the design of filter banks, in addition to specifications on passband ripple and stopband attenuation, the perfect reconstruction condition must be taken into account. The perfect reconstruction condition leads to a nonlinear minimization problem where the objective function is of fourth-order. In this paper a linearization technique is applied to this fourth-order function to modify it into a quadratic function. As a result, the major step in the iterative algorithm proposed is to solve a standard unconstrained minimization of a quadratic function, which can be done with high computation efficiency. Compared with the other existing methods, the proposed method is a direct 2-D design approach which does not require any transformations, and is more flexible in adjusting the filter characteristics. In addition, the design efficiency is improved as will be demonstrated by the design examples.

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II. PROPOSED ITERATIVE METHOD AND THE DESIGN PROCEDURE

A. Formulation of the objective function

A 2-D nonseparable diamond-shaped filter bank can be designed as shown in Fig. 1. The ideal frequency response of the analysis lowpass filter H_0 is illustrated in Fig. 2(a). After analysis filtering, the signals are quincunx downsampled as shown in Fig. 2(b), where effectively half of the samples are discarded. At the reconstruction end, upsampling will replace the discarded samples with zero values. The input and output relationship of the filter bank can be expressed as

$$\begin{aligned}\hat{X}(z_1,z_2) = {} & \tfrac{1}{2}[H_0(z_1,z_2)F_0(z_1,z_2)+H_1(z_1,z_2)F_1(z_1,z_2)] \\ & \cdot X(z_1,z_2)+\tfrac{1}{2}[H_0(-z_1,-z_2)F_0(z_1,z_2) \\ & +H_1(-z_1,-z_2)F_1(z_1,z_2)]\cdot X(-z_1,-z_2) \quad (1)\end{aligned}$$

where the first term represents the input signal component and the second term represents the aliasing component. By assuming that

$$H_1(z_1,z_2) = z_1^{-1}H_0(-z_1,-z_2) \quad (2a)$$

$$F_0(z_1,z_2) = H_0(z_1,z_2) \quad (2b)$$

$$F_1(z_1,z_2) = z_1 H_0(-z_1,-z_2) \quad (2c)$$

the aliasing term in Eqn. (1) is cancelled and (1) becomes

$$\hat{X}(z_1,z_2) = \tfrac{1}{2}[H_0^2(z_1,z_2)+H_0^2(-z_1,-z_2)]\cdot X(z_1,z_2) \quad (3)$$

So if the condition

$$H_0^2(\omega_1, \omega_2) + H_0^2(\omega_1 + \pi, \omega_2 + \pi) = 1 \tag{4}$$

is satisfied for $-\pi \le \omega_1 \le \pi, -\pi \le \omega_2 \le \pi$, the output will be a replica of the input.

In our design the region of support of H_0 is assumed to be an $N \times N$ square centered at origin where N is an odd number. If the impulse response of H_0, $h(n_1, n_2)$, for $-N \le n_1 \le N, -N \le n_2 \le N$, has octagonal symmetry, then the frequency response is of zero-phase and can be written as [4]

$$\begin{aligned} H_0(\omega_1, \omega_2) = & \sum_{n_2=0}^{(N-3)/2} \sum_{n_1=n_2+1}^{(N-1)/2} a(n_1, n_2) \\ & \cdot [\cos(n_1\omega_1)\cos(n_2\omega_2) + \cos(n_2\omega_1)\cos(n_1\omega_2)] \\ & + \sum_{n=0}^{(N-1)/2} a(n,n)\cos(n\omega_1)\cos(n\omega_2) \end{aligned} \tag{5}$$

where

$$\begin{aligned} & a(0,0) = h(0,0) \\ & a(n_1, 0) = 2h(n_1, 0) && \text{for } 1 \le n_1 \le (N-1)/2 \\ & a(n_1, n_2) = 4h(n_1, n_2) && \text{for } 1 \le n_2 \le (N-1)/2, \\ & && \text{and } n_2 \le n_1 \le (N-1)/2 \end{aligned}$$

For convenience we write the frequency response in a matrix form as

$$H_0(\boldsymbol{\omega}) = \mathbf{c}(\boldsymbol{\omega})^T \mathbf{y}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2)$, $\mathbf{y}$ is a column vector formed by $a(n_1, n_2)$ in the order shown in Fig. 3, and $\mathbf{c}(\boldsymbol{\omega})$ denoted as $[T_0(\boldsymbol{\omega})\ T_1(\boldsymbol{\omega})\ \cdots\ T_{K-1}(\boldsymbol{\omega})]^T$ with $K = (N+3)(N+1)/8$ is a column vector with $T_i(\boldsymbol{\omega})$ being the corresponding multiplier of $a(n_1, n_2)$ in (5).

The design can be accomplished by minimizing the objective function given by

$$E = E_1 + \alpha E_2 \tag{6}$$

where α is a positive weight,

$$E_1 = \sum_{\boldsymbol{\Omega}} \left[H_0^2(\boldsymbol{\omega}) + H_0^2(\boldsymbol{\omega} + \pi) - 1\right]^2$$

where $\boldsymbol{\Omega} = [\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots, \boldsymbol{\omega}_p]$ are chosen from the shaded area shown in Fig. 4(a), and

$$E_2 = \sum_{\boldsymbol{\Omega}_s} H_0^2(\boldsymbol{\omega})$$

where $\boldsymbol{\Omega}_s = [\boldsymbol{\omega}_{s1}, \boldsymbol{\omega}_{s2}, \ldots, \boldsymbol{\omega}_{sm}]$ are chosen from the shaded area in Fig. 4(b). On comparing (6) with (4), we see that term E_1 in (6) deals with the reconstruction requirement whereas E_2 is used to reduce the intra-band aliasing effects.

B. The iterative approach

The objective function E in (6) is a fourth-order function with respect to the coefficients in $\mathbf{y}$. Instead of minimizing the objective function directly, which is a nonlinear optimization problem, an iterative method is adopted in which the error function in (6) is modified into

$$E' = E_1' + \alpha E_2' \tag{7}$$

where

$$\begin{aligned} E_1' &= \sum_{\boldsymbol{\Omega}} [H_0(\boldsymbol{\omega})G_0(\boldsymbol{\omega}) + H_0(\boldsymbol{\omega} + \pi)G_0(\boldsymbol{\omega} + \pi) - 1]^2 \\ E_2' &= \sum_{\boldsymbol{\Omega}_s} G_0^2(\boldsymbol{\omega}) \end{aligned}$$

Like H_0, G_0 is an $N \times N$ lowpass FIR filter with octagonal symmetrical impulse response, and $G_0(\boldsymbol{\omega})$ can be expressed as

$$G_0(\boldsymbol{\omega}) = \mathbf{c}(\boldsymbol{\omega})^T \mathbf{x}$$

where $\mathbf{x}$ is a column vector formed by the coefficients of G_0. To start the iteration, we first design a diamond-shaped lowpass filter H_0, and write term E' in (7) as

$$E' = (\mathbf{Ux} - \mathbf{I})^T(\mathbf{Ux} - \mathbf{I}) + \alpha(\mathbf{U_S x})^T(\mathbf{U_S x}) \tag{8}$$

where

$$\mathbf{U}_t(\boldsymbol{\Omega}) = \begin{bmatrix} T_0(\boldsymbol{\omega}_1) & T_1(\boldsymbol{\omega}_1) & \cdots & T_{K-1}(\boldsymbol{\omega}_1) \\ T_0(\boldsymbol{\omega}_2) & T_1(\boldsymbol{\omega}_2) & \cdots & T_{K-1}(\boldsymbol{\omega}_2) \\ \vdots & & & \vdots \\ T_0(\boldsymbol{\omega}_p) & T_1(\boldsymbol{\omega}_p) & \cdots & T_{K-1}(\boldsymbol{\omega}_p) \end{bmatrix} \tag{9a}$$

$$\mathbf{U}_s = \begin{bmatrix} T_0(\boldsymbol{\omega}_{s1}) & T_1(\boldsymbol{\omega}_{s1}) & \cdots & T_{K-1}(\boldsymbol{\omega}_{s1}) \\ T_0(\boldsymbol{\omega}_{s2}) & T_1(\boldsymbol{\omega}_{s2}) & \cdots & T_{K-1}(\boldsymbol{\omega}_{s2}) \\ \vdots & & & \vdots \\ T_0(\boldsymbol{\omega}_{sm}) & T_1(\boldsymbol{\omega}_{sm}) & \cdots & T_{K-1}(\boldsymbol{\omega}_{sm}) \end{bmatrix} \tag{9b}$$

$$\mathbf{H}(\boldsymbol{\Omega}) = \mathrm{diag}[H_0(\boldsymbol{\omega}_1), \ldots, H_0(\boldsymbol{\omega}_s), \ldots, H_0(\boldsymbol{\omega}_p)] \tag{9c}$$

$$\mathbf{U} = \mathbf{H}(\boldsymbol{\Omega})\mathbf{U}_t(\boldsymbol{\Omega}) + \mathbf{H}(\boldsymbol{\Omega} + \pi)\mathbf{U}_t(\boldsymbol{\Omega} + \pi) \tag{9d}$$

$$\mathbf{I} = [\,1\ 1\ \cdots\cdots\ 1]^T \in \mathbf{R}^{p \times 1} \tag{9e}$$

Since $\mathbf{U}^T\mathbf{U} + \alpha\mathbf{U}_s^T\mathbf{U}_s$ is positive-definite, E' has a global minimum point given by

$$\mathbf{x} = (\mathbf{U}^T\mathbf{U} + \alpha\mathbf{U_S}^T\mathbf{U_S})^{-1} \cdot (\mathbf{U}^T\mathbf{I}) \tag{10}$$

After obtaining $\mathbf{x}$, $\mathbf{y}$ is updated using

$$\mathbf{y} := (1 - \tau)\mathbf{y} + \tau\mathbf{x} \tag{11}$$

where $0 < \tau < 1$ is a smoothing parameter. The above process is repeated until $||\mathbf{x} - \mathbf{y}||$ is less than a prescribed tolerance. A step-by-step description of this design method is as follows:

Algorithm

Step 1 Use a conventional method (e.g., SVD method) to design a 2-D $N \times N$ diamond-shaped FIR filter and use its coefficients to form the initial $\mathbf{y}$.

Step 2 Calculate $\mathbf{U}_t(\mathbf{\Omega})$, $\mathbf{U}_s$ and $\mathbf{I}$ using (9a), (9b), and (9e), respectively.

Step 3 Use (9c) and (9d) to form $\mathbf{H}(\mathbf{\Omega})$ and $\mathbf{U}$, and compute $\mathbf{x}$ in (10).

Step 4 If $\| \mathbf{y} - \mathbf{x} \| < \epsilon$, where ϵ is a prescribed tolerance, output $\mathbf{x}$ as the design result and stop. Otherwise, update $\mathbf{y}$ using (11) and repeat from Step 3.

III. DESIGN EXAMPLES

By applying the proposed iterative algorithm, a set of 2-D nonseparable diamond-shaped filter banks were designed. The parameters used are $\alpha = 10^{-5}$, $\tau = 0.7$, $\epsilon = 10^{-3}$ in three design examples where $N = 7,\ 9,\ 11$, respectively. The impulse responses of the obtained H_0 are listed in Table I. They are in the order indicated in Fig. 3.

TABLE I
THE IMPULSE RESPONSES OF THE OBTAINED H_0

Coef No.	N=7	N=9	N=11
1	6.303933e-01	6.133241e-01	5.988531e-01
2	1.550647e-01	1.682721e-01	1.778429e-01
3	−1.414847e-02	−1.245620e-02	−1.010828e-02
4	−7.125124e-04	1.426121e-03	1.940527e-03
5	−3.333147e-02	−1.104196e-03	−1.265525e-03
6	−1.279112e-02	−3.472240e-02	−2.571798e-04
7	2.544806e-03	−2.329296e-02	−3.521509e-02
8	5.145418e-03	4.409431e-03	−3.190801e-02
9	3.895804e-04	2.493733e-04	6.252349e-03
10	−7.061552e-05	1.102501e-02	−3.512157e-04
11	-	1.911315e-03	3.730406e-04
12	-	−5.020628e-04	1.576852e-02
13	-	−1.062799e-03	4.451412e-03
14	-	−1.620485e-04	−1.196672e-03
15	-	2.189800e-05	−9.916944e-05
16	-	-	−3.143924e-03
17	-	-	−3.937832e-04
18	-	-	6.164737e-05
19	-	-	1.429489e-04
20	-	-	9.356728e-05
21	-	-	−1.059957e-05

The 3-D plots of the amplitude responses of H_0 and corresponding analysis highpass filter H_1 with $N = 9$ are shown in Fig. 5 and Fig. 6, respectively. To evaluate the reconstruction performance of the filter bank, a parameter, called peak reconstruction error (PRE), is defined as

$$\mathrm{PRE} = \max_{\boldsymbol{\omega}}\{|20 \log_{10} \left[H_0^2(\boldsymbol{\omega}) + H_0^2(\boldsymbol{\omega} + \pi)\right] |\}$$

where $\boldsymbol{\omega}$ varies over the entire frequency domain. The PRE value as well as the number of iterations used for each design are listed in Table II. From Table II, it is quite obvious that good designs can be achieved after a small number of iterations.

TABLE II
PRE VALUES AND NO. OF ITERATIONS

	N=7	N=9	N=11
PRE values(dB)	0.0135	0.0067	0.0090
No. of iterations	7	8	7

IV. CONCLUSIONS

An iterative method for the design of nonseparable diamond-shaped 2-D filter banks has been proposed. Several design examples have been provided to demonstrate the computational efficiency and design flexibility of the design method proposed.

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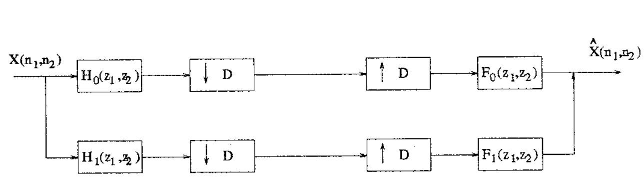

FIGURE 1
DIAGRAM OF A 2-D NONSEPARABLE DIAMOND-SHAPED FILTER BANK

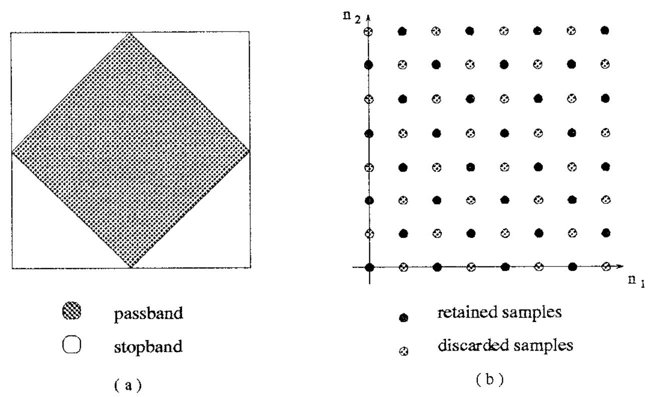

FIGURE 2
BAND CHARACTERISTICS AND QUINCUNX SAMPLING

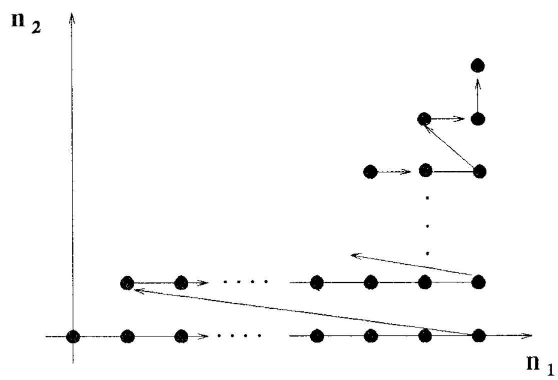

FIGURE 3
THE ORDER OF THE $a(n_1, n_2)$ IN FORMING VECTOR $\mathbf{y}$

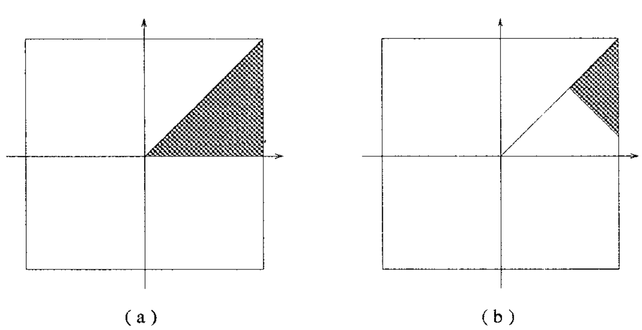

FIGURE 4
SAMPLING AREAS

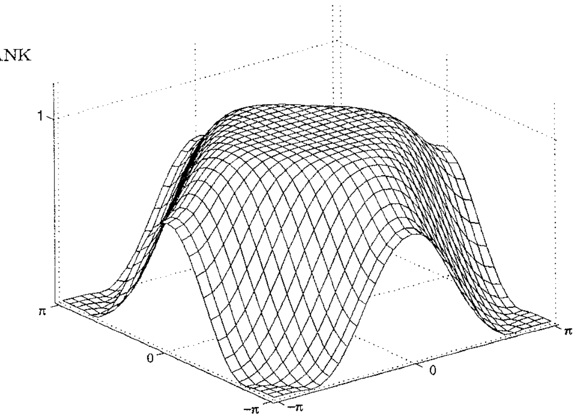

FIGURE 5
AMPLITUDE RESPONSE OF LOWPASS FILTER H_0 WITH $N = 9$

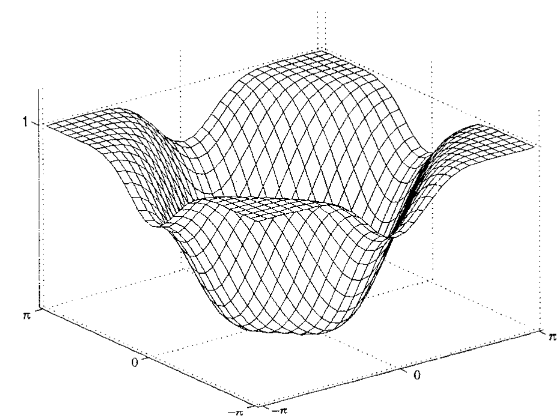

FIGURE 6
AMPLITUDE RESPONSE OF HIGHPASS FILTER H_1 WITH $N = 9$