A New Design of 2-D Nonseparable Hexagonal Quadrature-Mirror Filter Banks

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Abstract

A new design method for 2-D nonseparable hexagonal quadrature mirror filter banks is proposed. The new method involves minimizing an objective function containing two error terms, one dealing with the perfect reconstruction condition and the other with intra-band aliasing. The method makes full use of the symmetry of hexagonal FIR filters, which leads to a considerable reduction in the amount of computation required by the optimization. This method is compared with one proposed by Simoncelli and Adelson and is shown to yield superior results.

I. Introduction

The basic idea of subband coding is to split up the frequency band of the signal and then to decimate and code each subband separately with a coder and bit rate which match the statistics of that band. A particularly useful 1-D filter bank is the quadrature-mirror filter (QMF) bank, which has found numerous applications in one-dimensional (1-D) as well as two-dimensional (2-D) signal processing.

Most applications of QMF banks to 2-D signal processing, such as image subband coding, involve separable filters, which can be designed by using 1-D techniques [1]. In this scheme the frequency spectrum of the signal is split into central lowpass, horizontal highpass, vertical highpass, and diagonal highpass bands in which the diagonal highpass band contains a mixture of the two orientations. To avoid this problem, a nonseparable hexagonal QMF bank based on hexagonal sampling was proposed by Adelson and Simoncelli [2]. The analysis and synthesis filters in [2] have a similar structure to 1-D QMF banks and it was shown that aliasing in the system output is cancelled. The filters in [2] are designed by minimizing an objective function which consists of two terms, one dealing with the perfect reconstruction condition and the other dealing with the intra-band aliasing.

In this paper, a new algorithm for the design of hexagonal QMF banks is proposed. The algorithm makes full use of the symmetry properties exhibited by the hexagonal FIR filter, both in the time and frequency domains, which results in considerable computation reduction in the optimization while achieving better designs. The paper is organized as follows : In section II, we review the structure of the nonseparable hexagonal quadrature mirror filter bank proposed. Section III describes the proposed design method. In section IV several design examples are presented and are then compared with those in [2].

II. Nonseparable Hexagonal Quadrature Mirror Filter Bank

In [2], Adelson and Simoncelli proposed a nonseparable hexagonal quadrature mirror filter bank which is based on hexagonal sampling, and showed that the mixed orientation problem in separable QMFs can be eliminated by using a nonseparable hexagonal QMF system.

The scheme of a four-band, nonseparable, hexagonal analysis-synthesis filter bank in two dimensions is illustrated in Fig. 1. In this realization, $F_i(\boldsymbol{\omega})$, $i = 1, \ldots, 4$ and $G_i(\boldsymbol{\omega})$, $i = 1, \ldots, 4$ are the transfer functions of the analysis and synthesis filters, respectively. The input $x(\mathbf{n})$ with index $\mathbf{n} = [n_1\ n_2]^T$ is a 2-D digital signal sampled on a hexagonal sampling lattice, and the filtering and subsampling are accomplished in two dimensions, i.e., $\boldsymbol{\omega} = [\omega_1\ \omega_2]^T$ is a vector, and the subsampling is parameterized by a nonsingular two-by-two *subsampling matrix*, $\mathbf{K}$, with integer entries, which is chosen here as

$$\mathbf{K} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \tag{1}$$

Based on the scheme in Fig. 1, the relation between the input $x(\mathbf{n})$ and the overall filter bank output $\hat{x}(\mathbf{n})$ can be derived in the frequency domain as

$$\hat{X}(\boldsymbol{\omega}) = \frac{1}{4}\sum_{l=0}^{3} X(\boldsymbol{\omega}+\tilde{\mathbf{k}}_l)\left[\sum_{i=0}^{3} G_i(\boldsymbol{\omega})F_i(\boldsymbol{\omega}+\tilde{\mathbf{k}}_l)\right] \quad (2)$$

where $\hat{X}(\boldsymbol{\omega})$ and $X(\boldsymbol{\omega})$ are the frequency spectra of $x(\mathbf{n})$ and $\hat{x}(\mathbf{n})$, respectively. The $\tilde{\mathbf{k}}_l$, $l = 0, \ldots, 3$ are the four modulation vectors defined by

$$\tilde{\mathbf{k}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{k}}_1 = \begin{bmatrix} 2\pi/\sqrt{3} \\ 0 \end{bmatrix}$$

$$\tilde{\mathbf{k}}_2 = \begin{bmatrix} \pi/\sqrt{3} \\ \pi \end{bmatrix}, \quad \tilde{\mathbf{k}}_3 = \begin{bmatrix} -\pi/\sqrt{3} \\ \pi \end{bmatrix}$$

It can be observed from Eqn. (2) that the output of the overall filter bank system contains the input of the system, which corresponds to $\tilde{\mathbf{k}}_0$, and the aliasing terms, which correspond to $\tilde{\mathbf{k}}_l$, $l = 1, 2, 3$. If the analysis and synthesis filters are chosen as

$$\begin{aligned} F_0(\boldsymbol{\omega}) &= G_0(-\boldsymbol{\omega}) = H(\boldsymbol{\omega}) = H(-\boldsymbol{\omega}) \\ F_1(\boldsymbol{\omega}) &= G_1(-\boldsymbol{\omega}) = e^{j\boldsymbol{\omega}\cdot\mathbf{s}_1}H(\boldsymbol{\omega}+\tilde{\mathbf{k}}_1) \\ F_2(\boldsymbol{\omega}) &= G_2(-\boldsymbol{\omega}) = e^{j\boldsymbol{\omega}\cdot\mathbf{s}_2}H(\boldsymbol{\omega}+\tilde{\mathbf{k}}_2) \\ F_3(\boldsymbol{\omega}) &= G_3(-\boldsymbol{\omega}) = e^{j\boldsymbol{\omega}\cdot\mathbf{s}_3}H(\boldsymbol{\omega}+\tilde{\mathbf{k}}_3) \end{aligned} \quad (3)$$

where H is a function that is invariant under negation of its argument and the expressions $\boldsymbol{\omega}\cdot\mathbf{s}_i$, $i = 1, 2, 3$, denote inner products of the two vectors, then the analysis-synthesis system, is called a 2-D nonseparable *hexagonal* QMF bank. If $\mathbf{s}_i$, the shifting vectors, are assigned as

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

then it can be shown that the aliasing terms in Eqn. (2) will be cancelled regardless of the choice of $H(\boldsymbol{\omega})$, and the remaining response in Eqn. (2) becomes

$$\hat{X}(\boldsymbol{\omega}) = \frac{1}{4}X(\boldsymbol{\omega})\sum_{i=0}^{3}|H(\boldsymbol{\omega}+\tilde{\mathbf{k}}_i)|^2 \quad (4)$$

Now in order to achieve perfect reconstruction of the input from the output, the design problem is reduced to finding a filter with frequency response $H(\boldsymbol{\omega})$ that satisfies the constraint

$$\sum_{i=0}^{3}|H(\boldsymbol{\omega}+\tilde{\mathbf{k}}_i)|^2 = 4 \quad (5)$$

Usually, a hexagonal lowpass filter is adopted to produce a band-splitting system.

III. Proposed Hexagonal Filter Bank Design Algorithm

The design of 2-D nonseparable hexagonal QMFs is similar to the design of 1-D QMFs. It requires the design of only one filter represented by $H(\boldsymbol{\omega})$. The whole system then can be constructed from the relations in Eqn. (3). A "good" filter is one that satisfies constraint (5), which guarantees the perfect reconstruction. In addition, many applications require that the subband signals have minimal aliasing.

As the impulse response of an FIR hexagonal filter is defined on a hexagonal grid, the size of an FIR hexagonal filter kernel can be measured in terms of the number of hexagonal "rings" that it contains. For example, a zero-ring filter contains only a single impulse located at the origin while a one-ring filter contains impulses at the origin and six more impulses around vertices of a hexagon centered at the origin. In general, the impulse response and frequency response of a hexagonal FIR system can be of 12-fold symmetry, i.e., they are symmetric about the six vertices of a hexagon and across the bisectors of the six sides, as illustrated in Fig. 2. Therefore, the independent parameters that need to be determined are defined on a wedge-shaped region covering approximately one-twelfth of the whole kernel.

In order to satisfy Eqn. (5), an error function is defined as the p-norm of the approximation errors, i.e.,

$$E_1 = \left\{\sum_{\boldsymbol{\omega}}\left[\sum_{i=0}^{3}|H(\boldsymbol{\omega}+\tilde{\mathbf{k}}_i)|^2 - 4\right]^p\right\}^{1/p} \quad (6)$$

An FIR filter with hexagonal symmetry in its impulse response has hexagonal symmetry in its frequency response. Consequently, the sampling points $\boldsymbol{\omega}$ can be chosen in a wedge-shaped region in the frequency domain, as shown by the shaded area in Fig. 3(a).

In order to reduce the intra-band aliasing, an error function, called intra-band aliasing error function, is constructed as

$$E_2 = \left[\sum_{\boldsymbol{\omega}}(|H(\boldsymbol{\omega})| - H_I(\boldsymbol{\omega}))^p\right]^{1/p} \quad (7)$$

where $H_I(\boldsymbol{\omega})$ is the amplitude response of an ideal hexagonal lowpass filter. Again, the sampling points $\boldsymbol{\omega}$ here are on a sampling grid in a wedge-shaped area, which is illustrate by shaded area in Fig. 3(b). Note that there is no sampling point in the transition band. A method similar to the one used in [3] is utilized to construct the sampling grid, which gives more samples over the regions nearby the passband and stopband edges.

The design objective is to satisfy (5) while reducing the intra-band aliasing . This can be achieved by defining an objective function that is the weighted sum of the above two error functions, i.e.,

$$E = \alpha E_1 + (1-\alpha)E_2 \qquad (8)$$

where $0 < \alpha < 1$. The weight α can be adjusted during the design.

A quasi-Newton optimization algorithm can be utilized to minimize the above objective function E with respect to the free parameters of the lowpass hexagonal filter represented by $H(\boldsymbol{\omega})$. The power p in Eqns. (6) and (7) should be chosen to be a large even number, e.g., $p = 128$, which would result in a nearly minimax design. A small weighting α should be used at the beginning of the optimization to let error function E_2 be minimized first and then α should be increased to emphasize the error function E_1.

IV. Design Examples and Comparison

By using the proposed algorithm, several non-separable hexagonal filter banks have been designed. The coefficients of the hexagonallowpass filters with 3, 4, and 5 rings are listed in Table I. The spatial positions of these coefficients in the 2-D space are illustrated in Fig. 2. The corresponding coefficients of the three highpass filters can be obtained by modulating and shifting the coefficients of the lowpass filters using Eqn. (3). For the 4-ring case, the 3-D plot of the amplitude response of the hexagonal lowpass filter is shown in Fig. 4. For comparison purposes, a 3-D plot of the amplitude response of the 4-ring filter in [2] is shown in Fig. 5. As can be seen the passbands of the two filters are almost the same but the stopband of the filter designed by the proposed method is better than that in [2]. In addition, it is found that the filter bank designed by our method satisfies the perfect reconstruction more accurately.

Table I
The impulse responses of the designed hexagonal filters

Coef	3-ring filter	4-ring filter	5-ring filter
a	0.668372	0.639332	0.593442
b	0.297021	0.305619	0.321198
c	−0.062023	−0.043298	−0.039581
d	−0.001132	0.009974	−0.004391
e	−0.006730	−0.015317	−0.006768
f	−0.006650	−0.033678	−0.036109
g		0.000359	−0.004358
h		0.008229	0.010173
i		0.020284	0.018715
j			0.001359
k			−0.003973
l			0.003823

The design method in [2] also involves optimizing an objevtive function containing two terms to deal with the perfect reconstruction condition and the intra-band aliasing. However, the objective function is evaluated on a grid of sample points over the whole frequency baseband. Our method makes use of the symmetry property of the filter and takes into consideration only the samples in a wedge-shaped area, which is one-twelfth of the total area. This leads to a significant improvement in the efficiency of the optimization involved. In [2], the error function dealing with the intra-band aliasing is in the form of a product of the frequency response of the hexagonal lowpass filter with its nonzero modulation version, which is not very sensitive to the characteristics of the filter in the passband and the stopband. Our method, on the other hand, uses a p-norm approximation error as the intra-band aliasing error function associated with a nonuniform sampling grid which is quite sensitive to the filter characteristics in the passband and stopband, including band edges. This is believed to be the reason why the designed filters have better frequency responses, especially in the stopband.

V. Conclusion

A new method for designing 2-D non-separable hexagonal quadrature-mirror filter banks is proposed. Several filter banks have been designed and compared with those in [2]. The comparisons show that the proposed method is superior relative to that in [2] in terms of the quality of the filter bank it designed as well as the computation amount it needed in the design.

Acknowledgement

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References

[1] A. Antoniou, *Digital Filters: Analysis, Design, and Applications*, Second Edition, New York : McGraw-Hill, 1993.

[2] E. P. Simoncelli and E. Adelson, "Non-Separable Extension of Quadrature Mirror Filters of Multiple Dimension", *Proc. IEEE*, vol.78, pp. 652-663, April 1990.

[3] C. Charalambous, "The performance of an algorithm for minimax design of two-dimensional linear phase FIR digital filters", *IEEE Trans. Circuits and Syst.*, vol. CAS-32, pp. 1016-1028, Oct. 1985.

[4] W.-S. Lu and A. Antoniou, *Two-Dimensional Digital Filters*, New York : Marcel Dekker,Inc., 1992.

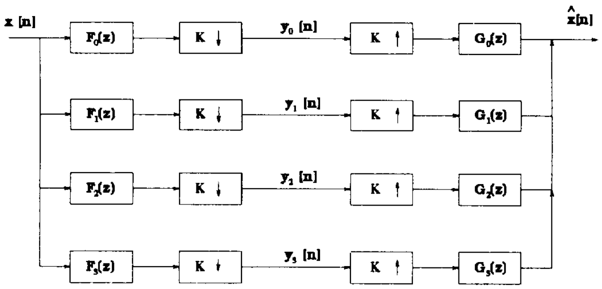

FIGURE 1
DIAGRAM OF A NONSEPARABLE HEXAGONAL QMF FILTER BANK.

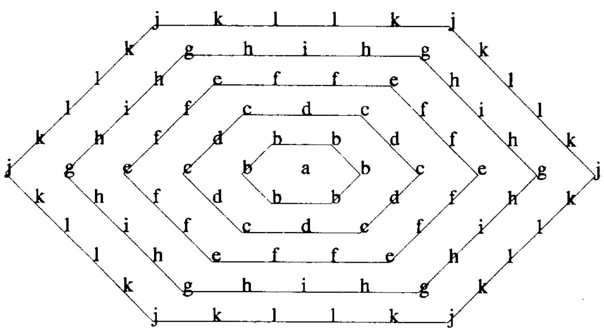

FIGURE 2
COEFFICIENTS OF A HEXAGONAL FILTER WITH SYMMETRY.

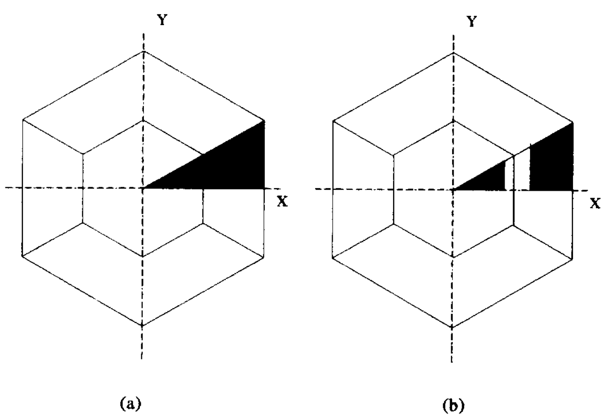

FIGURE 3
SAMPLING GRIDS IN FORMING THE OBJECTIVE FUNCTION.

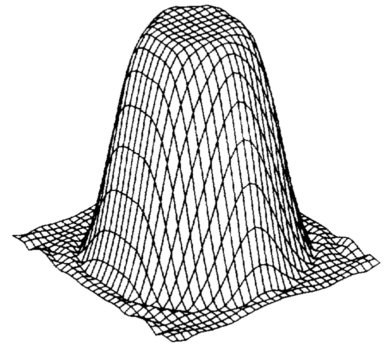

FIGURE 4
AMPLITUDE RESPONSE OF A 4-RING FILTER BY PROPOSED METHOD.

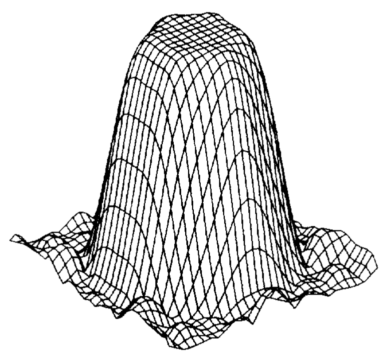

FIGURE 5
AMPLITUDE RESPONSE OF A 4-RING FILTER BY [2].