Towards global design of orthogonal filter banks and wavelets

Conception orientée vers les bancs de filtres orthogonaux et à ondelettes

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This paper investigates several design issues concerning two-channel conjugate quadrature (CQ) filter banks and orthogonal wavelets. It is well known that optimal designs of CQ filters and wavelets in the least squares (LS) or minimax sense are nonconvex problems and to date only local solutions can be claimed. By virtue of recent progress in global polynomial optimization and a direct design technique for CQ filters, we in this paper present a design strategy that may be viewed as our endeavors towards global solutions for CQ filters. Two design scenarios are considered, namely the least squares designs with vanishing moment (VM) requirement, and equiripple (i.e. minimax) designs with VM requirement. Simulation studies are presented to verify our design concept for both LS and minimax designs of low-order CQ filters, and to evaluate and compare the proposed algorithms with existing design algorithms for high-order CQ filters.

Ce document examine plusieurs publications de conception concernant la quadrature conjuguée à deux canaux (Conjugate Quadrature - CQ) des banques de filtre et des ondelettes orthogonales. Il est bien connu que les conceptions optimales de filtres de CQ et des ondelettes au plus petit carré (Least Square - LS) et où le sens du minimax sont des problèmes non convexes où jusqu'à présent, seules des solutions locales peuvent être utilisées. Grâce au récent progrès dans l'optimisation de polynômes globaux et d'une technique de conception directe pour les filtres CQ, nous présentons dans ce document notre stratégie de conception qui une solution globale pour les filtres CQ. Nous considérons deux scénarios de conception : le plus petit carré avec la condition du moment de disparition (Vanishing Moment - VM) et l'equiripple (c'est-à-dire le minimax) avec la condition VM. Les études de simulation sont illustrées pour valider notre conception du LS et pour la conception de minimax d'ordre faible de CQ des filtres. Il est également possible de comparer les algorithmes proposés avec des algorithmes de conception existants pour l'ordre élevé de CQ.

Keywords: orthogonal filter banks; wavelets; global optimization

I Introduction

The class of two-channel conjugate quadrature (CQ) filter banks, also known as power-symmetric filter banks [14], is one of the most well-known building blocks for multirate systems and wavelet-based coding systems, as it offers perfect reconstruction (PR) and other desirable properties. Despite the fact that many algorithms for the design of CQ filters have been proposed since the 1980s, see example references [1], [2], [10]-[14] and the work cited therein, to date only locally optimal designs can be claimed. From a mathematical point of view, this is primarily because the design problems are inherently nonconvex, admitting many local solutions. In this regard, this paper is an attempt to develop feasible a strategy towards global designs of CQ filters.

The design method proposed in this paper is made possible by virtue of recent progress in global polynomial optimization [6], [9] and a direct design technique for the CQ filters [1] in conjunction with our observations on a common pattern shared among globally optimal impulse responses of low-order CQ filters and a progressive design procedure in terms of filter length. Two design scenarios are considered: equiripple (i.e. minimax) designs with vanishing moment (VM) requirement and least squares (LS) designs with VM requirement. Concerning the first, in digital filters the magnitude of the largest amplitude-response error is usually required to be as small as possible, thus minimax solutions are generally preferred [15]. On the other hand, in several applications — especially telecommunications — digital filters are required to have minimal stopband energy; hence LS solutions are of importance in these applications. Simulation results for both LS and minimax CQ filters are presented to verify our design concept for low-order CQ filters, and to evaluate and compare the performance of the proposed algorithms with existing design algorithms for high-order CQ filters. This paper extends in a significant way the work reported in [16].

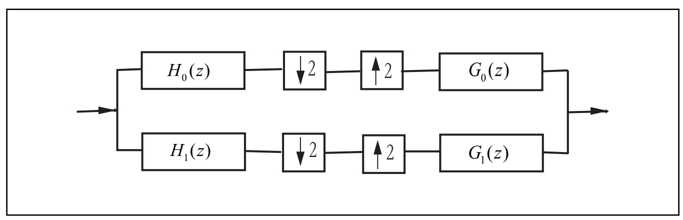

Figure 1: *A two-channel CQ filter bank.*

II Notation and Background

II.A Two-channel orthogonal filter banks

A two-channel causal FIR CQ filter bank consists of a pair of analysis filters H_0, H_1 and a pair of synthesis filters G_0 and G_1 as shown in Fig. 1, where the four filters are related by [2]

$$\begin{aligned} H_1(z) &= -z^{-(N-1)}H_0(-z^{-1}) \\ G_0(z) &= H_1(-z) \\ G_1(z) &= -H_0(-z) \end{aligned} \tag{1}$$

where $H_0(z) = \sum_{n=0}^{N-1} h_n z^{-n}$ is a lowpass FIR transfer function of length N with N even. With (1), the aliasing is eliminated, and the PR is achieved if $H_0(z)$ satisfies

$$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2 \tag{2}$$

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Eq. (2) is equivalent to a set of $N/2$ equality constraints as

$$\sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta_m \quad \text{for} \quad m = 0,1,...,(N-2)/2 \tag{3}$$

where δ_m is the Dirac sequence with $\delta_0 = 1$ and $\delta_m = 0$ for nonzero m. Eq. (3) is known as the double shift orthogonality in the wavelet literature. In addition to the PR conditions, CQ filters may be required to meet other constraints such as possessing a certain number of VMs for constructing wavelets [10]-[13]. It is known that the number of VMs of a CQ filter bank is equal to the number of zeros of $H_0(z)$ at $\omega = \pi$. Because

$$\frac{d^l H_0(e^{j\omega})}{d\omega^l}\Big|_{\omega=\pi} = (-j)^l \sum_{n=0}^{N-1} (-1)^n \cdot n^l \cdot h_n$$

a CQ filter has L vanishing moments if

$$\sum_{n=0}^{N-1} (-1)^n \cdot n^l \cdot h_n = 0 \quad \text{for} \quad l = 0,1,...,L-1 \tag{4}$$

Thus an LS design of CQ lowpass filter $H_0(z)$ having L VMs can be cast as

$$\text{minimize} \quad \int_{\omega_a}^{\pi} |H_0(e^{j\omega})|^2 \, d\omega \tag{5}$$

$$\text{subject to:} \quad \text{constraints (3) and (4)} \tag{6}$$

where ω_a is the normalized stopband edge of $H_0(z)$.

In this paper, we also consider the minimization of maximum instantaneous power of lowpass filter $H_0(z)$ over its stopband, subject to PR and VM constraints. Thus the minimax design can be formulated as

$$\text{minimize} \quad \max_{\omega_a \le \omega \le \pi} |H_0(e^{j\omega})| \tag{7}$$

$$\text{subject to:} \quad \text{constraints(3) and (4)} \tag{8}$$

II.B Polynomial optimization problems

1) ***Polynomial Optimization Problems***: A real-valued polynomial $f(\boldsymbol{x})$ in n-dimensional space R^n can be expressed as

$$f(\boldsymbol{x}) = \sum_{\alpha \in \mathcal{F}} c(\boldsymbol{\alpha}) \boldsymbol{x}^{\alpha} \tag{9}$$

where $c(\boldsymbol{\alpha}) \in R$, $\boldsymbol{x} = [x_1\, x_2 \ldots x_n]$, $\boldsymbol{\alpha} = [\alpha_1\, \alpha_2 \ldots \alpha_n] \in \mathcal{F} \subset \mathcal{Z}_+^n$ —the set of all vectors in R^n whose components are nonnegative integers, and $\boldsymbol{x}^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$. The order (degree) of $f(\boldsymbol{x})$ is defined as the largest $\sum_i \alpha_i$.

A general polynomial optimization problem (POP) has the form

$$\text{minimize} \quad f_0(\boldsymbol{x}) \tag{10}$$

$$\text{subject to:} \quad f_k(\boldsymbol{x}) \ge 0 \quad \text{for} \quad k = 1,...,l \tag{11}$$

$$f_k(\boldsymbol{x}) = 0 \quad \text{for} \quad k = L+1,...,K \tag{12}$$

where $f_k(\boldsymbol{x})$ for $k = 0,1,...,K$ are real-valued polynomials. POPs include the following problems as its special cases: linear programming (LP), convex quadratic programming (QP), semidefinite programming (SDP) and second-order cone programming (SOCP). More importantly, POPs stand for a substantially broader class that covers many nonconvex optimization problems [6].

2) ***Global Optimization of Problem:*** A recent breakthrough in the field is made by Lasserre [6] in which it is proved that when the feasible region in (10)–(12) is compact (not necessarily convex), the global solution of (10)–(12) can be approximated as closely as desired (and often can be obtained exactly) by solving a finite sequence of SDP problems. A technical difficulty with the method of [6] is that the size of the SDP problems involved in a POP usually grows very quickly. This may cause numerical difficulties even for POPs of moderate scales.

More recently, sparse SDP relaxation [7] is proposed for global solutions of POPs of relatively larger scales with improved efficiency. The method is supported by MATLAB toolbox SparsePOP2000 [4] [5]. Another MATLAB toolbox for POPs is GloptiPoly 3.4 [8], which is intended to solve the generalized problems of moments (GPM) that can be viewed as an extension of the classical problem of moments [9].

Because of the SDP relaxation approach taken by [6][7], to date the size of POP problems that GloptiPoly and SparsePOP can solve is quite limited. Nevertheless, the globally optimal solutions provided by these toolboxes for low-order CQ filter banks and wavelets form one of the key ingredients in the proposed design technique. In addition, the availability of the global designs for low-order filter banks helps demonstrate the validity of our method for designing globally optimal high-order filter banks and wavelets. We shall illustrate these points with details in the next two sections.

III Least square designs

III.A Brief review of a direct method for local LS designs

A technique for direct design of LS CQ filter banks and wavelets is reported in [1]. The design technique is simple and gives local solutions of good quality. Since it is one of the ingredients of our design method, below we sketch its main steps.

The design formulation in (5),(6) and (7),(8) can be expressed as

$$\text{minimize} \quad \mathbf{h}^T\mathbf{Q}\mathbf{h} \tag{13}$$

$$\text{subject to:} \quad \sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta_m \tag{14}$$

$$\sum_{n=0}^{N-1} (-1)^n \cdot n^l \cdot h_n = 0 \tag{15}$$

where $m = 0,1,...,(N-2)/2$, $l = 0,1,...,L-1$, $\mathbf{h} = [h_0\, h_1 \ldots h_{N-1}]^T$ and $\mathbf{Q}$ is a constant symmetric positive definite Toeplitz matrix characterized by its first row $[\pi - \omega_a \; -\sin\omega_a \ldots -\sin(N-1)\omega_a/(N-1)]$. Suppose we are in the kth iteration to update the coefficient vector from $\mathbf{h}^{(k)}$ to $\mathbf{h}^{(k+1)} = \mathbf{h}^{(k)} + \mathbf{d}$, and we write (3) at $\mathbf{h}^{(k+1)}$ as

$$\sum_n h_n^{(k)} h_{n+2m}^{(k)} + \sum_n h_n^{(k)} d_{n+2m} + \sum_n d_n h_{n+2m}^{(k)} + \sum_n d_n d_{n+2m} = \delta_m \tag{16}$$

By assuming $\mathbf{d}$ is small in magnitude and neglecting the second-order term in (16), we obtain

$$\sum_n h_n^{(k)} d_{n+2m} + \sum_n d_n h_{n+2m}^{(k)} \approx \delta_m - \sum_n h_n^{(k)} h_{n+2m}^{(k)} \equiv u_m^{(k)}$$

which can be put in the form of

$$\mathbf{C}^{(k)}\mathbf{d} = \mathbf{u}^{(k)}$$

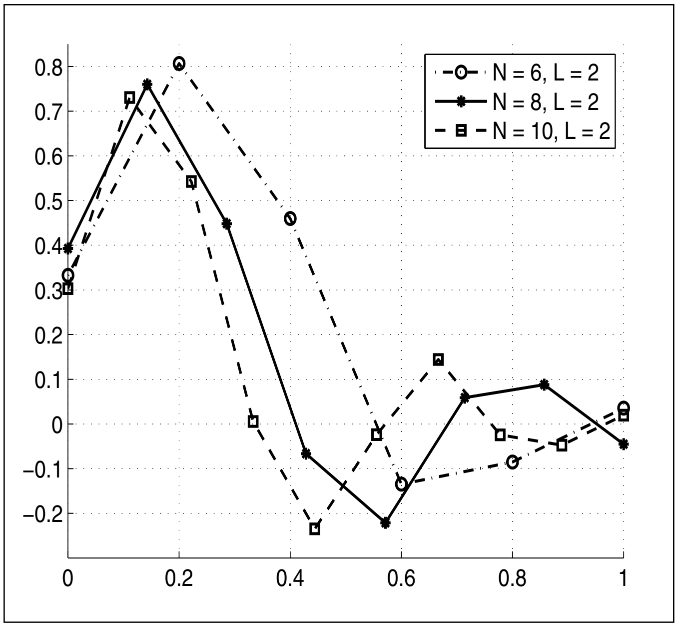

Figure 2: *Pattern of LS impulse responses with different filter length* N.

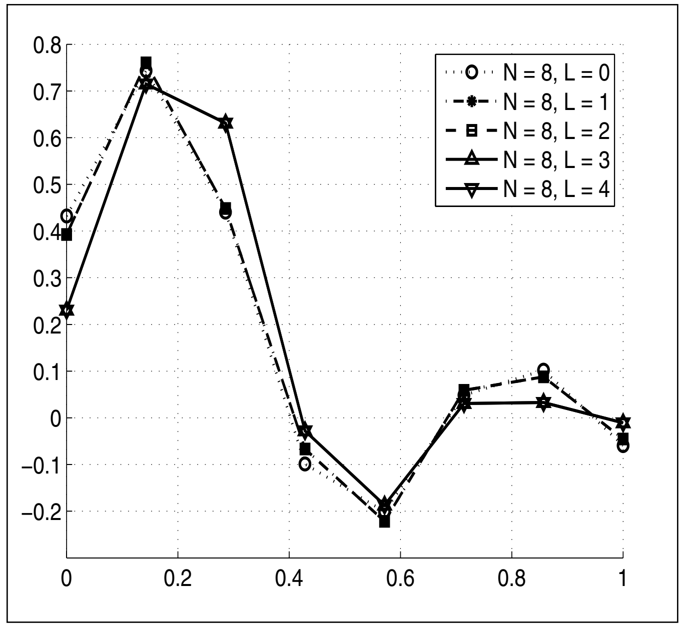

Figure 3: *Pattern of LS impulse responses with various number of VMs* L.

The constraints (4) on VMs at $\mathbf{h}^{(k+1)}$ can be expressed as

$$\mathbf{Dd} = \mathbf{v}^{(k)}$$

where $\mathbf{v}^{(k)} = -\mathbf{Dh}^{(k)}$. The smallness of $\mathbf{d}$ can be characterized as $|d_i| \le \beta$ for $i = 1, 2, ..., N$, which can be put together as

$$\mathbf{Ad} \le \mathbf{b}$$

with $\mathbf{A} = \begin{bmatrix}\mathbf{I}_N & -\mathbf{I}_N\end{bmatrix}^T$ and $\mathbf{b} = \beta \cdot \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$. In this way, the LS problem can be formulated as

$$\text{minimize} \quad \mathbf{d}^T\mathbf{Qd} + 2\mathbf{d}^T\mathbf{q}^{(k)} + \kappa \tag{17}$$

$$\text{subject to:} \quad \mathbf{Ad} \le \mathbf{b} \tag{18}$$

$$\begin{bmatrix}\mathbf{C}^{(k)} \\ \mathbf{D}\end{bmatrix}\mathbf{d} = \begin{bmatrix}\mathbf{u}^{(k)} \\ \mathbf{v}^{(k)}\end{bmatrix} \tag{19}$$

with $\mathbf{q}^{(k)} = \mathbf{Qh}^{(k)}$ and κ a constant. Since $\mathbf{Q}$ is positive definite, (17)–(19) is a convex quadratic programming (QP) problem.

III.B Global LS design of low-order filter banks

Evidently, the problem in (13)–(15) is a POP with $N/2+L$ constraints, and the maximum order of all the polynomials involved is two.

For low-order filter banks, toolbox GloptiPoly 3.4 was found to work well. For example, with $N = 6$, $L = 2$ and $\omega_a = 0.56\pi$, the software produces four globally optimal impulse responses as

$$\mathbf{h}_{\text{LS}}^{(6,2)} = \begin{bmatrix} 0.33268098788629 \\ 0.80689591454849 \\ 0.45986215652386 \\ -0.13501431772967 \\ -0.08543638600240 \\ 0.03522516035714 \end{bmatrix}$$

$-\mathbf{h}_{\text{LS}}^{(6,2)}$, $\text{flipud}(\mathbf{h}_{\text{LS}}^{(6,2)})$ and $-\text{flipud}(\mathbf{h}_{\text{LS}}^{(6,2)})$ where $\text{flipud}(\mathbf{h})$ denotes a vector generated by flipping vector $\mathbf{h}$ upside down. We remark that the above four impulse responses satisfy constraints (14) and (15) and yield the same minimum objective function value as $\mathbf{h}_{\text{LS}}^{(6,2)^T}\mathbf{Q}\mathbf{h}_{\text{LS}}^{(6,2)} = 0.173458$. Also note that $\mathbf{h}_{\text{LS}}^{(6,2)}$ (as well as $-\mathbf{h}_{\text{LS}}^{(6,2)}$) possess minimum-phase as no zeros of their corresponding transfer functions are outside the unit circle. For a minimum-phase filter, both the system function and its inverse are causal and stable so that the filter can be of practical use. Unfortunately, the software fails to work as long as the filter length N is greater than or equal to 10. On the other hand, toolbox SparsePOP2000 was found to work for global design of filter banks up to $N = 16$. A technical problem with SparsePOP2000 is that, unlike GloptiPoly 3.4's ability to produce multiple global solutions, it requires setting a lower bound and an upper bound for the impulse response, and only one global solution that falls within the bounds will be generated. Our design experiences suggest that the following bounds work well:

$$\mathbf{h_d} - 0.5\mathbf{e} \le \mathbf{h} \le \mathbf{h_d} + 0.5\mathbf{e} \tag{20}$$

where $\mathbf{h_d}$ is the impulse response of the length$-N$ Daubechies filter [14] and $\mathbf{e}$ is an $N \times 1$ all-one vector.

III.C Potentially global LS design of high-order filter banks

In this section, we propose a method for potentially global LS design of CQ filters with length N that is too high for the above mentioned software to handle.

1) ***Pattern of impulse responses of globally optimal filter banks:*** Although the current versions of the software examined earlier are of limited use, it turns out that observations made on the pattern of the impulse responses of low-order designs do provide useful clues for tackling the design of high-order CQ filters. Our observations are illustrated in Figs. 2 and 3. Shown in Fig. 2 are the impulse responses of globally optimal minimum-phase lowpass CQ filters of lengths $N = 6, 8$ and 10 (all with $L = 2$) obtained using SparsePOP2000 where the impulse responses are plotted over normalized interval $[0,1]$ for better comparison. From the figure, it is clear that these impulse responses are distinctly different from each other. Nevertheless, it is equally clear that they exhibit a similar pattern: it starts with a short uphill to a peak, then goes down to components of small values. In addition, viewing each impulse response as a curve (function), we see that the nearest neighbor to a given curve associated with filter length N is the curve associated with length $N+2$. Furthermore, for a fixed filter length N, the impulse responses of globally optimal CQ filters with various VMs are clustered and exhibit a pattern similar to that in Fig. 2. As an example, Fig. 3 shows the impulse responses of lowpass CQ filters with $N = 8$ and $L = 0, 1, 2, 3, 4$ obtained using either GloptiPoly 3.4 or SparsePOP2000.

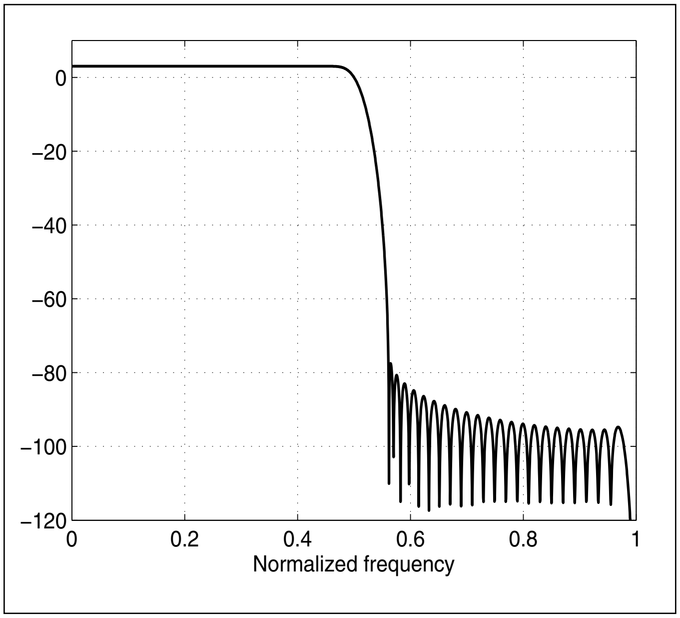

Figure 4: *Magnitude response of LS* $H_0(z)$ *with* $N=96$, $L=3$ *and* $\omega_a=0.56\pi$.

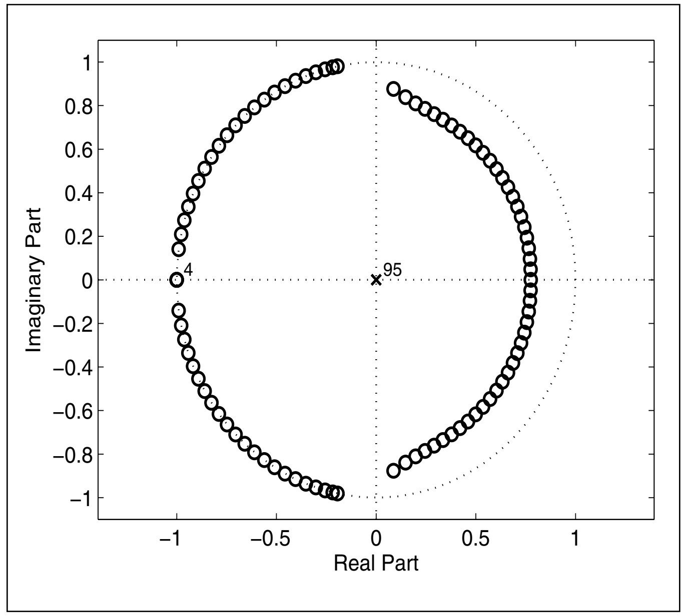

Figure 5: *Zero-pole plot of LS* $H_0(z)$ *with* $N=96$, $L=3$ *and* $\omega_a=0.56\pi$.

2) ***A design strategy:*** Both the LS and minimax designs of CQ filters as formulated in (5),(6) and (7),(8) are nonconvex problems that possess multiple local solutions, and several (local) design techniques for CQ filters are available in the literature [2], [10]-[14]. A recent addition to this rich field of research is a direct design method [1] that deals with problems (5),(6) and (7),(8) by local convex approximations in a sequential manner, and the method is shown to produce satisfactory results. Brief reviews of the method for local LS and local minimax designs are sketched in III.A and IV.A.

Taking the above analysis into account, the situation facing the designer may be summarized as follows: (i) global designs of CQ filters are possible by using the methods of [6][7], but only for short filter lengths; (ii) a common pattern exists among the impulse responses of globally optimal lowpass minimum-phase CQ filters of short lengths and, the optimal impulse response of length $N+2$ falls within a small vicinity of the optimal impulse response of length N; and, (iii) a sequential design method that requires a reasonable initial design for producing a locally optimal design is within reach.

A strategy for the design of minimum-phase CQ filters of a long (even) length N is developed based on the above observations, and can be described in steps as follows:

a. Set an initial working filter length N_w, say, to 4, and design a globally optimal, minimum-phase, CQ filter of length N_w using, for example, GloptiPoly 3.4. Denote the impulse response obtained by $\mathbf{h}_w$.

b. Generate a length (N_w+2) interpolated version of $\mathbf{h}_w$ by, for example, linear interpolation. Denote the interpolated vector by $\mathbf{h}_{wi}$.

c. Apply the method of [1] with $\mathbf{h}_{wi}$ as its initial point (impulse response) to design an optimal CQ filter of length N_w+2. Denote the impulse response obtained by $\mathbf{h}_w$.

d. If $N=N_w+2$, output $\mathbf{h}_w$ as the optimal design and terminate; otherwise, set $N_w=N_w+2$ and repeat from Step b.

Although no theoretical claim about the global optimality of the above design methodology can be made for large N, we speculate that the designs obtained by this approach are quite likely to be globally optimal. This is because in each round of iteration the initial point is sufficiently close to the global minimizer and the algorithm in [1] is known to converge to a nearby minimizer. In the next section, we provide experimental evidence that supports our speculation.

III.D Design examples and performance evaluation

1) ***Performance of the proposed method for low-order designs:*** The method just described was applied to design LS lowpass CQ filters of length $N=6,8,\ldots,16$. For all designs, the normalized stopband edge was set to $\omega_a=0.56\pi$ and the number of VMs was set to $L=1$. In each design, toolbox GloptiPoly 3.4 was applied only once to (13)–15) to generate a globally optimal minimum-phase impulse response with $N=4$ and $L=1$, denoted by $\mathbf{h}_{\text{LS}}^{(4,1)}$. In the case of $N=6$, $\mathbf{h}_{\text{LS}}^{(4,1)}$ was linearly interpolated to length 6 and then used as the initial point to run the LS design algorithm in [1], and the impulse response obtained is denoted by $\hat{\mathbf{h}}_{\text{LS}}^{(6,1)}$. In the case of $N=8$, we first obtain impulse response $\hat{\mathbf{h}}_{\text{LS}}^{(6,1)}$ as above; then linearly interpolate $\hat{\mathbf{h}}_{\text{LS}}^{(6,1)}$ to length 8 and use it as the initial point to run the LS algorithm in [1] to generate impulse response $\hat{\mathbf{h}}_{\text{LS}}^{(8,1)}$. The designs for $N=10,\ldots,16$ were carried out in a similar manner to produce impulse responses $\hat{\mathbf{h}}_{\text{LS}}^{(N,1)}$. For comparison purposes, globally optimal impulse responses $\mathbf{h}_{\text{LS}}^{(N,1)}$ for $N=6,8,\ldots,16$ were obtained by using GloptiPoly 3.4 or SparsePOP2000. It was found that $\hat{\mathbf{h}}_{\text{LS}}^{(N,1)}$ and $\mathbf{h}_{\text{LS}}^{(N,1)}$ are practically identical for all even N from 6 to 16. We also remark that with the starting impulse response $\mathbf{h}_{\text{LS}}^{(4,1)}$ having minimum-phase, the CQ filters so designed all have minimum-phase, a desirable property for digital filters to be of practical use.

2) ***Performance of the proposed method for high-order designs:*** Supported by the verification of our design concept as just illustrated, we now proceed to apply the proposed method to design high-order lowpass CQ filters with length N up to 96. As an example, Fig. 4 shows the magnitude response of the CQ lowpass filter designed by the proposed method with $N=96$, $L=3$, and $\omega_a=0.56\pi$. The energy of the filter over stopband, i.e., the value of the objective function $\mathbf{h}^T\mathbf{Q}\mathbf{h}$ in (13), was found to be $E_{\text{LS}}^{(96,3)}=1.185993\times10^{-9}$. For comparison, a CQ filter with the same design specifications, i.e., $N=96$, $L=3$, and $\omega_a=0.56\pi$ was designed using the LS algorithm of [1]. The initial point used in the design was a linear-phase lowpass filter obtained by the conventional window-based technique. The stopband energy of the CQ filter obtained was found to be $\tilde{E}_{\text{LS}}^{(96,3)}=1.309040\times10^{-9}$, which represents a 10% increase compared with $E_{\text{LS}}^{(96,3)}$. Like the low-order designs, it was found that all high-order CQ filters possess minimum-phase. As an example, Fig. 5 shows the zero-pole plot of the LS lowpass CQ filter of length 96 obtained by the proposed method. We observe that no zeros are outside the unit circle of the complex plane.

IV Minimax designs

IV.A Brief review of a direct method for local minimax designs

This section sketches the method for direct design of minimax CQ filters. The technique is addressed in [1] and turns out to be of critical importance towards global minimax designs.

The problem in (7),(8) can be formulated as

$$\text{minimize} \quad \eta \tag{21}$$

$$\text{subject to:} \quad \eta^2 - |H_0(e^{j\omega})|^2 \ge 0 \quad \text{for } \omega \in \Omega \tag{22}$$

$$\sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta_m \tag{23}$$

$$\sum_{n=0}^{N-1} (-1)^n \cdot n^l \cdot h_n = 0 \quad \text{,where} \tag{24}$$

$m=0,1,...,(N-2)/2$, $l=0,1,...,L-1$ and $\Omega=\{\omega_a=\omega_1,\omega_2,...,\omega_K=\pi\}$ is a set of K uniformly distributed frequency grids over stopband $[\omega_a,\pi]$. Define $\mathbf{c}(\omega)=[1\cos\omega ... \cos(N-1)\omega]^T$ and $\mathbf{s}(\omega)=[0\sin\omega ... \sin(N-1)\omega]^T$, we can write $|H_0(e^{j\omega})|^2$ in (22) as

$$|H_0(e^{j\omega})|^2 = [\mathbf{h}^T\mathbf{c}(\omega)]^2 + [\mathbf{h}^T\mathbf{s}(\omega)]^2$$

$$= \left\| \begin{bmatrix} \mathbf{c}(\omega)^T \\ \mathbf{s}(\omega)^T \end{bmatrix} \cdot \mathbf{h} \right\|^2 \equiv \|\mathbf{T}(\omega)\cdot\mathbf{h}\|^2$$

With an analysis similar to that outlined in III.A, the problem in (21)–(24) can be converted into

$$\text{minimize} \quad \eta \tag{25}$$

$$\text{subject to:} \quad \|\mathbf{T}(\omega)(\mathbf{h}^{(k)}+\mathbf{d})\| \le \eta \quad \text{for } \omega \in \Omega \tag{26}$$

$$\mathbf{Ad} \le \mathbf{b} \tag{27}$$

$$\begin{bmatrix} \mathbf{C}^{(k)} \\ \mathbf{D} \end{bmatrix} \mathbf{d} = \begin{bmatrix} \mathbf{u}^{(k)} \\ \mathbf{v}^{(k)} \end{bmatrix} \tag{28}$$

which can be solved as a second-order cone programming (SOCP) problem.

IV.B Global minimax design of low-order filter banks

Obviously the objective function as well as the constraint functions in (21)–(24) are all polynomials. Therefore, (21)–(24) is a POP with $K+N/2+L$ constraints and the maximum order of all the polynomials involved is two.

Toolbox GloptiPoly 3.4 was found to work for CQ filter of order 4. With $N=4$, $L=1$, $\omega_a=0.56\pi$, and $\Omega=\{\omega_a,\omega_a+0.025\pi,\omega_a+0.05\pi,...,\pi\}$ (which gives $K=18$), the toolbox was able to produce four globally optimal impulse responses as

$$\mathbf{h}_{\text{minimax}}^{(4,1)} = \begin{bmatrix} 0.48296282173531 \\ 0.83651623138234 \\ 0.22414405492402 \\ -0.12940935473280 \end{bmatrix}$$

$-\mathbf{h}_{\text{minimax}}^{(4,1)}$, flipud $(\mathbf{h}_{\text{minimax}}^{(4,1)})$ and $-$ flipud $(\mathbf{h}_{\text{minimax}}^{(4,1)})$. The maximum instantaneous energy over stopband for the above four impulse responses was found to be the same value $\eta^2=0.722218$. It was also observed that $\mathbf{h}_{\text{minimax}}^{(4,1)}$ (as well as $-\mathbf{h}_{\text{minimax}}^{(4,1)}$) possess minimum-phase. However, GloptiPoly 3.4 failed to work for N as small as 6 because of the relatively large number of constraints in the minimax design. On the other hand, with the same bounds set for $\mathbf{h}$ as in (20) and $0 \le \eta \le 2$, SparsePOP2000 was able to produce global minimax designs for $N=4$ and 6.

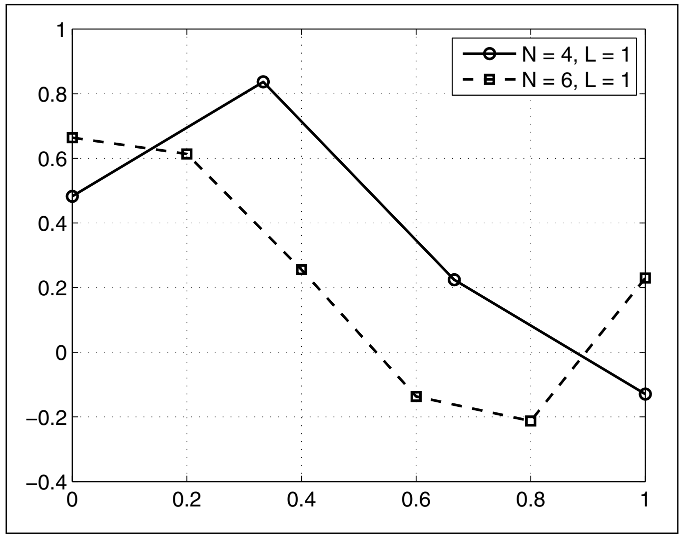

Figure 6: *Pattern of minimax impulse responses with different length* N.

IV.C Potentially global minimax design of high-order filter banks

Like the observations made on the LS designs (see III.C.1), globally optimal (in minimax sense) impulse responses obtained in IV.B appear to exhibit a pattern similar to that in the LS case, as can be seen in Fig. 6. It is therefore natural to follow the strategy described in III.C.2 for minimax design of high-order filter banks.

IV. D Design examples and performance evaluation

1) ***Performance of proposed method for a low-order design:*** The design strategy in III.C.2 for potentially global minimax design was applied to design minimax lowpass CQ filters. We set $\omega_a=0.56\pi$, $L=1$, and Ω contains $K=110$ frequency grids. GloptiPoly 3.4 was applied to (21)–(24) to generate a globally optimal minimum-phase impulse response with $N=4$ and $L=1$, denoted by $\mathbf{h}_{\text{minimax}}^{(4,1)}$. Impulse response $\mathbf{h}_{\text{minimax}}^{(4,1)}$ was then linearly interpolated to length 6 and used as the initial point to run the minimax algorithm of [1]. The impulse response obtained is denoted by $\hat{\mathbf{h}}_{\text{minimax}}^{(6,1)}$. For comparison, SparsePOP2000 was applied to problem (21)–(24) to generate globally optimal impulse response $\mathbf{h}_{\text{minimax}}^{(6,1)}$. The two impulse responses, $\hat{\mathbf{h}}_{\text{minimax}}^{(6,1)}$ and $\mathbf{h}_{\text{minimax}}^{(6,1)}$ were found to be practically identical, giving support to our design concept for the minimax designs. We also note that with the starting impulse response $\mathbf{h}_{\text{minimax}}^{(4,1)}$ having minimum-phase, the CQ filter $\hat{\mathbf{h}}_{\text{minimax}}^{(6,1)}$ obtained possesses minimum-phase as well.

2) ***Performance of proposed method for high-order designs:*** Following the design approach outlined above, $\hat{\mathbf{h}}_{\text{minimax}}^{(6,1)}$ was interpolated to length 8 and used as the initial point for the minimax algorithm of [1] to generate $\hat{\mathbf{h}}_{\text{minimax}}^{(8,1)}$. This process was repeated and high-order lowpass minimax CQ filters with length N up to 96 were designed. The filters obtained tend to be equiripple. As an example, Fig. 7 shows the magnitude response of the minimax lowpass CQ filter designed by the proposed method with $N=96$, $L=3$, $\omega_a=0.56\pi$ and Ω containing $K=110$ uniformly distributed frequency grids. The maximum instantaneous energy η^2 over stopband was found to be $E_{\text{minimax}}^{(96,3)}=6.362729\times10^{-9}$. For comparison, a minimax CQ filter with the same specifications was designed using the minimax algorithm of [1]. The initial point used in the design was a linear-phase lowpass filter obtained by the conventional window-based technique. The maximum instantaneous energy over stopband of the CQ filter was found to be $\tilde{E}_{\text{minimax}}^{(96,3)}=7.265100\times10^{-9}$, which represents a 14% increase compared with $E_{\text{minimax}}^{(96,3)}$. As in the low-order designs,

Table 1

Coefficients of $H_0(z)$ of [14] and from global design

$H_0(z)$ of [14]	$H_0(z)$ from global design
0.1605476	0.151132584528507
0.4156381	0.406751138104326
0.4591917	0.465073716955923
0.1487153	0.164100745264147
– 0.1642893	– 0.159230874305372
– 0.1245206	– 0.132446162371893
0.08252419	0.077632712518187
0.08875733	0.092962310257929
– 0.05080163	– 0.047219604152222
– 0.06084593	– 0.062990546331313
0.03518087	0.032739512761500
0.03989182	0.040781157734971
– 0.02561513	– 0.023979520144301
– 0.02440664	– 0.024517125714218
0.01860065	0.017475746917452
0.01354778	0.013173547986633
– 0.01308061	– 0.012191078582176
– 0.007449561	– 0.006651596447548
0.01293440	0.011254662805676
– 0.004995356	– 0.004181786154910

Table 2

Filter performance comparison

	$H_0(z)$ of [14]	$H_0(z)$ of global design
η^2	0.954568e-3	0.709881e-3
Largest eq. error	5.591345e-7	<1e-15

we also found that all high-order minimax CQ filters produced by the proposed method possess minimum-phase. As an example, Fig. 8 depicts the zero-pole plot of the minimax lowpass CQ filter of length 96 designed from the proposed method

3) ***Comparisons with other existing methods:*** In this section, we compare the global minimax design of filter banks with two existing methods that are well established in the literature.

a) *Comparison with a half-band filter based method in [14]:* A simple and effective method for two-channel CQ filter banks is described in Sec. 5.3.6 of [14]. The method first designs a zero-phase lowpass FIR half-band filter $W(z)$ of order $2(N-1)$ by, for example, the Parks-McClellan algorithm. One then defines $Y(z)=W(z)+\delta$ with δ the peak stopband ripple of $W(z)$ to ensure $Y(e^{j\omega})\geq 0$ for all ω. The following design specifications were chosen for the comparison: length $N=20$, stopband edge $\omega_a=0.6\pi$ and number of VMs $L=0$ (example 5.3.2 of [14]). The resulting coefficients of the CQ filter [14] and optimized lowpass CQ filter (algorithm proposed in IV, number of frequency grids K set to 30) are listed in Table 1, columns 1 and 2 respectively. The maximum instantaneous energy η^2 in stopband and the largest magnitude error among all $N/2$ equations in (23) for the two designs are given in Table 2. The magnitude responses of the two filters are depicted in Fig. 9. It is apparent that the proposed design technique is able to produce CQ filters with reduced instantaneous stopband energy and more accurate satisfaction of the PR conditions.

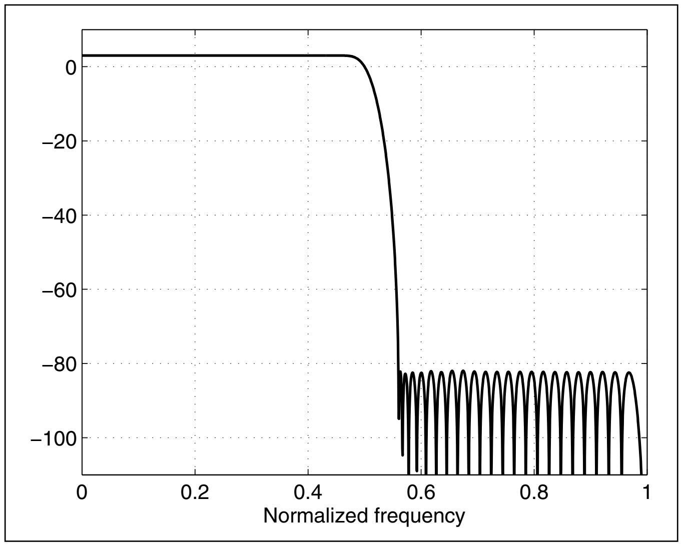

Figure 7: *Magnitude response of minimax $H_0(z)$ with $N=96$, $L=3$ and $\omega_a=0.56\pi$.*

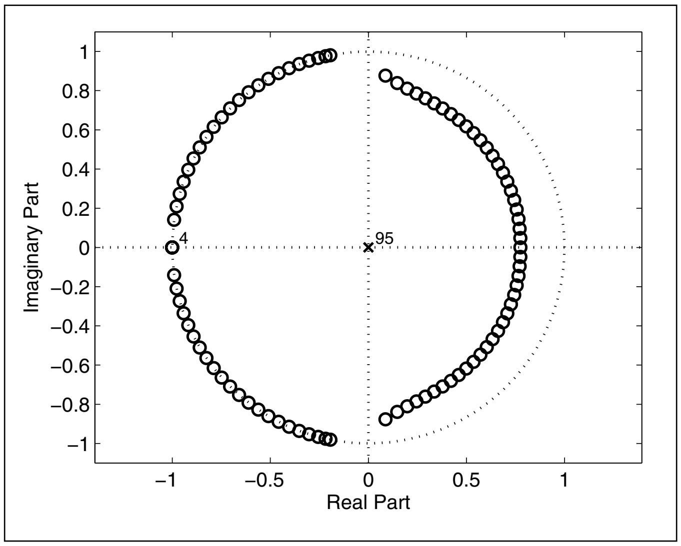

Figure 8: *Zero-pole plot of minimax $H_0(z)$ with $N=96$, $L=3$ and $\omega_a=0.56\pi$.*

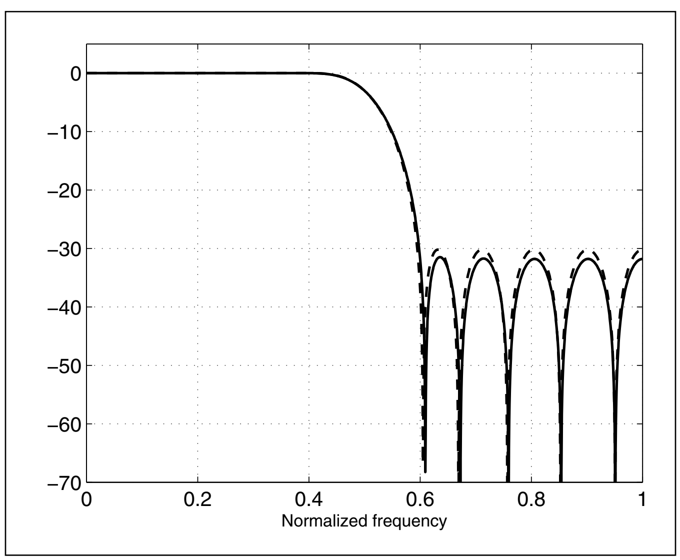

Figure 9: *Magnitude responses of minimax $H_0(z)$ with $N=20$ of the global design (solid line) versus that of $H_0(z)$ in [14] (dashed line).*

Table 3

Filter length $N = 8$

	$H_0(z)$ of [2]	$H_0(z)$ of global design
η^2	0.101105e-3	0.100221e-3
Largest eq. error	8.316794e-8	<1e-15

Table 4

Filter length $N = 16$

	$H_0(z)$ of [2]	$H_0(z)$ of global design
η^2	0.101276e-3	0.099122e-3
Largest eq. error	2.635550e-6	<1e-15

Table 5

Filter length $N = 32$

	$H_0(z)$ of [2]	$H_0(z)$ of global design
η^2	0.101805e-3	0.098748-3
Largest eq. error	2.162285e-6	<1e-15

b) *Comparison with the method of Smith-Barnwell:* The technique is developed in [2] for tree-structured analysis/reconstruction systems and has since been a popular benchmark for performance evaluation and comparison as it also provides accurate numerical results of CQ filters of high quality. For comparison, we applied the algorithm proposed in IV to design minimax lowpass CQ filters of length 8, 16 and 32 to meet the requirement of 40 dB minimum stopband attenuation; minimax designs of CQ filters with the same filter lengths and stopband attenuation are also reported in [2]. The numbers of frequency grids for $N = 8, 16$ and 32 were set to $K = 25, 28$ and 50 respectively, and the number of vanishing moments was set to zero in all three designs. The design results in terms of maximum instantaneous stopband energy η^2 and largest equation error in magnitude are shown in Tables 3, 4 and 5 for the CQ filters of length $N = 8, 16$ and 32, respectively. It is observed that in all three instances the proposed method produces designs with improved performance.

IV.E Remarks on complexity of the proposed algorithms

As made clear in III.C and IV.C, both the LS and minimax algorithms are progressive in terms of filter order. As such, more computations relative to conventional design algorithms are required especially for high order CQ filters because the number of iterations is nearly $N/2$. On the other hand, we note that because of the way the initial point is constructed, each iteration is done with high efficiency and, as a result, the total CPU time for designing a high-order LS CQ filter remains fairly reasonable. For example, it took 202 seconds on a PC laptop with a 1.66 GHz dual core processor to design an LS CQ filter of length $N = 96$. The time taken to design a minimax CQ filter of length $N = 96$ is relatively longer but still bearable (4857 seconds). Based on the evaluations of a large number of designs we carried out to date, the complexity of both the LS and minimax designs was found to have an approximately linear growth versus filter length. This means for example that a minimax design of a CQ filter of length $2N$ will take twice as long as that for a minimax CQ filter of length N. Finally, we stress that designing filters and filter banks is typically an off-line (non real-time) task and to get a design done very quickly is of secondary importance when compared to the performance the CQ filter can achieve.

V Conclusion

A new method for the design of two-channel orthogonal filter banks and wavelets has been proposed. Attempting to develop a methodology for global design of CQ filters, the proposed method is built on some recent progress in global polynomial optimization and a direct design technique for CQ filters, in conjunction with several critical observations on the globally optimal impulse responses and a progressive design procedure in terms of filter length. Several design examples have been presented to verify the design concept and demonstrate the performance of the proposed algorithms in comparison with several existing methods.

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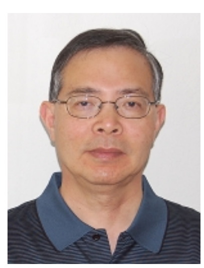

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