Multiuser Detectors for Synchronous DS-CDMA Systems Based on a Recursive p-Norm Convex Relaxation Approach

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Abstract

Multiuser detectors for direct-sequence code-division multiple-access (DS-CDMA) systems based on a recursive convex programming (RCP) relaxation approach are proposed. In these detectors, maximum likelihood (ML) detection is carried out in two steps: first, the combinatorial problem associated with ML detection is relaxed into a convex programming problem and then a recursive approach is applied to get an approximate solution for ML detection. Computer simulations demonstrate that the bit-error rate performance offered by the new detectors is near-optimal and superior to that offered by many existing suboptimal detectors including some recently proposed semidefinite-programming relaxation (SDPR) detectors. In addition, the amount of computation required by the RCP detectors is much less than that required by SDPR detectors.

Keywords Multiuser detection, maximum likelihood detection, relaxation method, convex programming, semidefinite programming.

I. INTRODUCTION

Multiuser detection in direct-sequence code-division multiple-access (DS-CDMA) systems has received a great deal of attention since Verdú proposed the maximum likelihood (ML) multiuser detection method in [1]. In an ML detector, detection is carried out by solving a combinatorial optimization problem that involves a quadratic objective function of a binary variable vector. The worst-case computational complexity in ML detection increases exponentially with the number of users although in some special applications [2][3], ML detection may be carried out with an average complexity of order between K^3 and K^4 (designated as $O(K^3)$ and $O(K^4)$, respectively, hereafter) where K is the number of users. To mitigate this problem, many suboptimal detectors offering reduced computational complexity have been proposed during the past two decades [4].

In recent years, several suboptimal multiuser detectors have been proposed based on optimization concepts and methods. In these detectors, the combinatorial problem is relaxed into some other type of optimization problem that can be solved more efficiently. The earliest multiuser detector of this category can be traced back to the decorrelating detectors described in [5][6] which were obtained by removing the binary constraints of the ML detection problem. Another example is the bound-constrained (BC) detector where the ML detection problem is relaxed into a quadratic minimization problem with simple bound constraints [7][8]. Yet another example is the semidefinite-programming relaxation (SDPR) detector [9-12] where the ML detection problem is relaxed into a semidefinite programming (SDP) problem. It has been shown that the bit error rate (BER) performance associated with the SDPR detector is close to that of Verdú's ML detector and is superior to that offered by several suboptimal detectors. In the present context, Verdú's ML detector is a detector implementing the detection method presented in [1]. The computational complexity associated with the SDPR detector is considerably lower than that of an ML detector but, as shown in [9-12], it is much higher than that required by the above mentioned suboptimal detectors. In [13], a multiuser detector is described based on a probabilistic data association (PDA) approach for synchronous CDMA systems. By treating the multiuser interference as Gaussian noise with matched mean and covariance, this detector offers a performance which is close to that of an ML detector but requiring a worst-case computational complexity of $O(K^3)$ in each stage.

In this paper, new multiuser detectors for synchronous DS-CDMA systems based on a *recursive convex programming* (RCP) approach are proposed. In these detectors, the combinatorial problem associated with ML detection is relaxed into a convex programming (CP) problem and then a recursive approach is used to obtain an approximate solution for ML detection. Computer simulations are presented which demonstrate that the proposed detectors offer near-optimal BER performance and, as a consequence, they outperform many existing suboptimal detectors. Specifically, in DS-CDMA systems with strong multiuser interference (MUI), the proposed detectors offer better BER performance than the SDPR detector and, in addition, they require less computation than the ML and SDPR detectors.

The paper is organized as follows. In Sec. II, the model for DS-CDMA systems and the ML detection problem are described. In Sec. III, three suboptimal detectors, namely, the SDPR, GMMSE, and BC detectors, are briefly reviewed. The application of CP relaxation to the ML detection problem and the derivation of the RCP detectors are discussed in Secs. IV and V, respectively. In Sec. VI, the proposed detectors are compared with other multiuser detectors in terms of BER performance and computational complexity based on computer simulations. Conclusions are drawn in Sec. VII.

II. Model For DS-CDMA Systems

We consider a synchronous DS-CDMA system where K users transmit data packets through an additive white Gaussian noise (AWGN) channel. The bit interval of each user is T_b seconds and each information bit belongs to the set $\{1, -1\}$. The kth user signal is assigned a normalized signature waveform $s_k(t)$ and the received baseband signal is given by

$$y(t) = \sum_{k=1}^{K} A_k b_k s_k(t) + n(t) \quad \text{for} \;\; t \in [0,\ T_b] \tag{1}$$

where b_k is the information bit and A_k is the amplitude of the kth user signal, and $n(t)$ is an additive white Gaussian noise process with zero mean and variance σ^2.

The demodulation begins by filtering the received signal $y(t)$ in (1) with a bank of matched filters comprising K filters each matched to each signature waveform, and the filtered signals are sampled at the end of each symbol interval. The outputs of the matched filters are given by

$$y_k = A_k b_k + \sum_{j \neq k} A_j b_j \rho_{jk} + n_k \qquad \text{for} \;\; k = 1,\ 2,\ \ldots,\ K \tag{2}$$

where ρ_{jk} denotes the crosscorrelation between $s_j(t)$ and $s_k(t)$, and n_k is a zero-mean Gaussian random variable with variance σ^2. The discrete-time baseband signal in (2) can be expressed in matrix form as

$$\mathbf{y} = \mathbf{RAb} + \mathbf{n} \tag{3}$$

where $\mathbf{y} = [y_1\ y_2\ \cdots\ y_K]^T$, $\mathbf{A} = \text{diag}\{A_1,\ A_2,\ \ldots,\ A_K\}$, $\mathbf{b} = [b_1\ b_2\ \cdots\ b_K]^T$, $\mathbf{R}$ is the crosscorrelation matrix whose (i, j)th component is $r_{ij} = \rho_{ij}$, and $\mathbf{n} = [n_1\ n_2\ \cdots\ n_K]^T$ is a vector of zero-mean Gaussian variables whose covariance matrix is given by $\sigma^2\mathbf{R}$. Note that $\mathbf{R}$ can be expressed as $\mathbf{R} = \mathbf{S}^T\mathbf{S}$ where $\mathbf{S} = [\mathbf{s}_1\ \mathbf{s}_2\ \cdots\ \mathbf{s}_K]$ denotes the signature matrix of K users. Hence, (3) can be expressed as

$$\mathbf{y} = \mathbf{S}^T\mathbf{r} = \mathbf{S}^T(\mathbf{SAb} + \mathbf{n}_c) \tag{4}$$

where $\mathbf{r} = \mathbf{SAb} + \mathbf{n}_c$ denotes the discrete-time signal obtained after a chip-rate matched filter, $\mathbf{n}_c$ is a vector of zero-mean Gaussian variables whose covariance matrix is given by $\sigma^2\mathbf{I}$.

The objective of multiuser detection is to identify the information vector $\mathbf{b}$ embedded in $y(t)$ in (1). This can be done by solving the optimization problem [1][4]

$$\text{minimize} \quad \mathbf{x}^T\mathbf{Hx} + \mathbf{x}^T\mathbf{p} \tag{5a}$$

$$\text{subject to}: \quad x_i \in \{1, -1\} \qquad \text{for} \ \ i = 1, 2, \ldots, K \tag{5b}$$

where $\mathbf{x} = [x_1\ x_2\ \cdots\ x_K]^T$ stands for the information-bit vector $\mathbf{b}$, $\mathbf{H} = \mathbf{ARA}$, and $\mathbf{p} = -2\mathbf{Ay} = -2\mathbf{AS}^T\mathbf{r}$. Because of the binary constraints in (5b), the optimization problem in (5) is a combinatorial problem whose solution can in principle be obtained only by exhaustive evaluation of the objective function over the 2^K possible values of $\mathbf{x}$.

An algorithm or system that can solve the problem in (5) is said to be an ML detector and an implementation based on Verdú's method in [1] will be referred to as the ML detector hereafter for the sake of convenience. Verdú's ML detector offers optimal demodulation performance and is often used as a comparison baseline for other detectors.

III. Review of Suboptimal Detectors

In this section, three suboptimal detectors, namely, the SDPR, GMMSE, and BC detectors are briefly reviewed in an optimization framework. This will help us develop the relationships between these popular detectors and the proposed detectors. These relationships will enable us to compare the performance of the proposed detectors with that of the suboptimal detectors.

A. SDPR Detector

If we let

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}\mathbf{x}^T & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix} \tag{6}$$

and

$$\mathbf{C} = \begin{bmatrix} \mathbf{H} & \mathbf{p}/2 \\ \mathbf{p}^T/2 & 1 \end{bmatrix} \tag{7}$$

the problem in (5) can be formulated as

$$\text{minimize} \;\; \text{tr}(\mathbf{C}\mathbf{X}) \tag{8a}$$

$$\text{subject to:} \quad \mathbf{X} \succeq 0, \text{rank}(\mathbf{X}) = 1 \tag{8b}$$

$$x_{ii} = 1 \quad \text{for} \;\; i = 1, 2, \cdots, K+1 \tag{8c}$$

where the notation $\mathbf{X} \succeq 0$ denotes that matrix $\mathbf{X}$ is positive semidefinite. If we neglect the rank condition in (8b), then the problem in (5) can be relaxed into the SDP problem given by

$$\text{minimize} \;\; \text{tr}(\mathbf{C}\mathbf{X}) \tag{9a}$$

$$\text{subject to:} \quad \mathbf{X} \succeq \mathbf{0} \tag{9b}$$

$$x_{ii} = 1 \quad \text{for} \;\; i = 1, 2, \cdots, K+1 \tag{9c}$$

where tr$(\cdot)$ represents the trace of $(\cdot)$. Once the solution of the problem in (9), $\mathbf{X}^*$, is obtained, the binary information-bearing vector, $\mathbf{x}$, can be determined by using the rank-one approximation approach described in [12] or the randomization approach described in [10]. An algorithm or system that can solve the SDP problem in (9) is referred to as an SDPR detector. Specific SDPR detectors are described in [9-12]. Efficient interior-point algorithms have been proposed in [16-20] for solving this problem. The computational complexity of interior-point algorithms is of polynomial order with respect to the number of users. In [12], we have shown that a more efficient SDPR detector can be obtained by reformulating the optimization problem in (9) as a dual SDP problem which can be solved with reduced computational effort by using the *projective method* [20].

B. Generalized MMSE Detector

The constraints in (5b) imply that $\mathbf{x}^T\mathbf{x} = K$. This equation defines the surface of a K-dimensional ball centered at the origin with radius $\sqrt{K}$. If we extend the feasible region to the entire ball, then the problem in (5) is relaxed to the problem

$$\text{minimize} \quad \mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{p} \tag{10a}$$

$$\text{subject to :} \quad \mathbf{x}^T\mathbf{x} \leq K \tag{10b}$$

Note that the problem in (10) involves minimizing a convex function over a convex region and, therefore, it is a convex programming problem that has a unique global solution. The dual problem of (10) is given by

$$\text{maximize} \quad -\frac{1}{4}\mathbf{p}^T(\mathbf{H} + \lambda \cdot \mathbf{I})^{-1}\mathbf{p} - \lambda \cdot K \tag{11a}$$

$$\text{subject to :} \quad \lambda \geq 0 \tag{11b}$$

where λ is the *Lagrange multiplier* associated with the constraint in (10b). Since the variable of the problem in (11), λ, is a scalar, the solution of this problem can be obtained very efficiently by using gradient-based algorithms. If we denote the solution of (11) as λ^*, then the solution of the problem in (10) is given by

$$\mathbf{x}_G^* = (\mathbf{H} + \lambda^*\mathbf{I})^{-1}\mathbf{A}\mathbf{y} \tag{12}$$

An algorithm or system that can solve the optimization problem in (10) is referred to as a *generalized MMSE* (GMMSE) detector. A specific GMMSE detector has been described in [7][8]. It can be shown that when λ^* assumes the values of 0, σ^2, and a large positive scalar, the GMMSE detector in [7][8] becomes the decorrelating detector in [5][6], the linear MMSE detector in [14][15], and conventional matched-filter detectors, respectively. The performance of the GMMSE detector in [7][8] has been found to be comparable with that of the linear MMSE detector [7].

C. Bound-Constrained Detector

The constraints in (5b) imply that $-1 \leq x_i \leq 1$. Hence the problem in (5) can be relaxed into the bound-constrained optimization problem

$$\text{minimize} \quad \mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{p} \tag{13a}$$

$$\text{subject to :} \quad -1 \leq x_i \leq 1 \quad \text{for} \;\; i = 1, 2, \ldots, K \tag{13b}$$

where $\mathbf{p} = -2\mathbf{A}\mathbf{S}^T\mathbf{r}$. The feasible region defined by (13b) is the unit hypercube centered at the origin. Since the constraints in (13b) are linear, the problem in (13) is a convex quadratic programming (CQP) problem which can be solved by using efficient QP algorithms [21-23]. It has been pointed out in [7][8] that this problem can be solved by using the *nonlinear Gaussian-Seidel* and *nonlinear Jacobi* algorithms which can be implemented in an interference cancellation framework.

Because the feasible region in (13b) is a subset of the feasible region in (10), a BC detector is expected to offer superior performance relative to that of a GMMSE detector. The tightness of the relaxation in the BC and SDPR detection methods is studied in [10] where it is shown that the relaxation made in the SDPR detection method is tighter than that made in the BC detection method. Hence, a BC detector is expected to offer inferior performance relative to that of an SDPR detector.

The dual problem associated with the problem in (13) is given by

$$\text{maximize} \quad -\frac{1}{4}\mathbf{p}^T[\mathbf{H} + \text{diag}(\boldsymbol{\lambda})]^{-1}\mathbf{p} - \mathbf{e}^T\boldsymbol{\lambda} \tag{14a}$$

$$\text{subject to : } \boldsymbol{\lambda} \geq 0 \tag{14b}$$

where $\mathbf{e} = [1\ 1\ \cdots\ 1]^T$, $\boldsymbol{\lambda} = [\lambda_1\ \lambda_2\ \cdots\ \lambda_K]^T$ with λ_i being the Lagrange multiplier associated with the ith constraint in (13b), and diag$(\boldsymbol{\lambda})$ is the diagonal matrix with λ_i being its ith diagonal component. Note that the solutions of the problems in (13) and (14), namely, $\mathbf{x}^*$ and λ^*, respectively, can be related to each other by

$$\mathbf{x}^* = -\frac{1}{2}[\mathbf{H} + \text{diag}(\boldsymbol{\lambda}^*)]^{-1}\mathbf{p} \tag{15}$$

IV. Application of Convex Relaxation to ML Detection

The relaxation for the GMMSE and BC detectors is carried out by expanding the discrete feasible set defined in (5b) to the continuous and convex feasible regions defined in (10b) and

(13b), respectively. This motivates us to consider the optimization problem

$$\text{minimize} \quad \mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{p} \tag{16a}$$

$$\text{subject to} : c(\mathbf{x}, p) \leq 0 \tag{16b}$$

where the feasible region, denoted as $\mathcal{R}_p$, depends on a scalar parameter $p > 0$ and is given by

$$c(\mathbf{x}, p) = |x_1|^p + |x_2|^p + \cdots + |x_K|^p - K \leq 0 \tag{17}$$

In what follows, the optimization problem

$$\text{minimize} \quad \mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{p} \tag{18a}$$

$$\text{subject to} : |x_1|^p + |x_2|^p + \cdots + |x_K|^p - K \leq 0 \tag{18b}$$

is referred to as Problem $\mathcal{U}_p$. The following two propositions describe some properties of the feasible region defined in (17).

Proposition 1: The feasible region $\mathcal{R}_p$ includes all feasible points of the problem in (5). In particular, $\mathcal{R}_p$ is convex for $p \geq 1$.

Proposition 1 implies that (a) for any fixed scalar $p \geq 1$, the problem in (18) can be regarded as a relaxation of the problem in (5), and (b) for $p \geq 1$ the problem in (18) is a convex programming problem that has a unique global solution. Note that if K in (18b) is replaced by other positive scalars less than K, then all feasible points of the problem in (5) will be located outside $\mathcal{R}_p$ and, therefore, K is the smallest scalar required for the validity of the proposition.

Proposition 2: If $\mathcal{R}_m$ and $\mathcal{R}_n$ are the feasible regions defined by $c(\mathbf{x}, m) \leq 0$ and $c(\mathbf{x}, n) \leq 0$, respectively, and $m \geq n \geq 1$, then $\mathcal{R}_m \subseteq \mathcal{R}_n$.

Several observations can be made based on Proposition 2:

Observation 1: If $m \geq n \geq 1$, the relaxation made in problem $\mathcal{U}_m$ is tighter than the relaxation made in problem $\mathcal{U}_n$. For this reason, the demodulation performance of a detector based on problem $\mathcal{U}_m$ is expected to be superior to that of a detector based on problem $\mathcal{U}_n$.

Observation 2: The tightest CP relaxation problem that can be obtained from (18) is given by

$$\mathcal{U}_\infty = \lim_{p \to +\infty} \mathcal{U}_p \tag{19}$$

Following the above reasoning, the demodulation performance of a detector based on problem $\mathcal{U}_\infty$ is expected to be superior to that of a detector based on any other CP relaxation problem generated by (18).

Observation 3: The decorrelating, GMMSE, or BC detector can be considered as special cases of the CP detector with p equal to 0, 2, or ∞, respectively. Hence, the demodulation performance of the GMMSE detector is expected to be better than that of the decorrelating detector, and the demodulation performance of the BC detector is expected to be better than that of the GMMSE detector.

To see the equivalence between the BC detector and a detector based on problem $\mathcal{U}_\infty$, we write

$$c(\mathbf{x},\infty) = \lim_{p\to\infty} c(\mathbf{x},p) = \lim_{p\to\infty} [(||\mathbf{x}||_p)^p - K] \tag{20}$$

Hence, the feasible region defined by $c(\mathbf{x},\infty) \leq 0$ can be expressed as

$$\left\{\mathbf{x} : ||\mathbf{x}||_\infty \leq \lim_{p\to\infty} (K^{1/p}) = 1\right\} \tag{21}$$

Note that the feasible region defined by (21) is equivalent to the region defined by

$$\{\mathbf{x} : -1 \leq x_i \leq 1 \quad \text{for} \ \ i = 1,\ 2,\ \ \ldots,\ K\} \tag{22}$$

which defines the feasible region of the problem in (13). The feasible regions defined by (5b) and $\mathcal{R}_p$ for $K = 2$, $p = 1, 2, 3$, and 20 are illustrated in Fig. 1.

V. Recursive CP-based Multiuser Detectors

Although the performance of the $\mathcal{U}_p$-based detector can be improved by increasing the value of p, the improvement is limited by that of the $\mathcal{U}_\infty$-based detector, which has been shown to offer inferior performance to that of the SDPR detector. In this section, a recursive approach is proposed for the CP relaxation to improve the performance.

In the proposed approach, decisions are made in each iteration only for those information bits that can be detected with high accuracy. These bits are fixed in subsequent iterations, which leads to a CP problem of reduced size for the undetected information bits. This process is continued until the detection of all information bits is completed. In what follows, we first describe a recursive approach to ML detection and on the basis of these principles, we then introduce the recursive CP approach for multiuser detection.

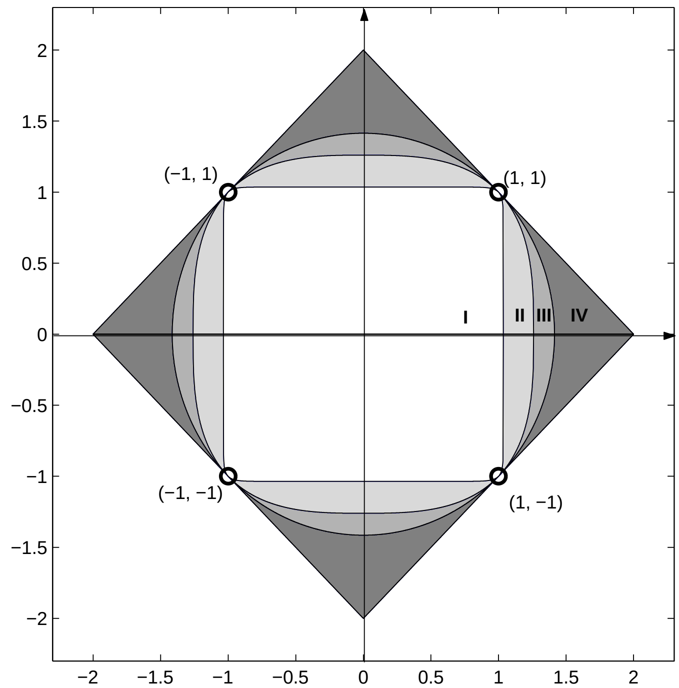

Fig. 1. Feasible regions defined by (5b) (labeled with small circles) and by (18b) for $p = 20$ (denoted as I), $p = 3$ (I+II), $p = 2$ (I+II+III), and $p = 1$ (I+II+III+IV).

A. Recursive ML Approach for Multiuser Detection

Denote the set of indices associated with the information bits that have been detected before the jth iteration as Ω_j and let $\bar{b}_i$ for $i \in \Omega_j$ be the value of the ith information bit detected (i.e., *hard decision*). By using a recursive approach, the combinatorial optimization problem in (5) can be modified into the problem

$$\text{minimize} \quad \mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{p} \tag{23a}$$

$$\text{subject to :} \quad x_i \in \{1, -1\} \quad \text{for} \;\; i \notin \Omega_j \tag{23b}$$

$$x_i = \bar{b}_i \quad \text{for} \;\; i \in \Omega_j \tag{23c}$$

Once the solution of the minimization problem is obtained, Ω_j is updated to Ω_{j+1} by including the indices of the information bits that are detected in the jth iteration, and a similar problem can be formulated in the next iteration for detecting the rest of the information bits.

Obviously, the above approach attempts to get an approximate solution for ML detection with the help of determined information bits (i.e., hard decisions) obtained in previous iterations. This motivation is similar to what has been considered for decision-aided multiuser detectors such as multistage detectors, interference cancellation detectors, and decision-feedback detectors (see [24][4] for more details). To see the relationship between the recursive approach and these decision-aided detection approaches, the problem in (23) will now be expressed in a different but equivalent form.

Proposition 3: The problem in (23) is equivalent to

$$\text{minimize} \quad \mathbf{x}_j^T \mathbf{A}_j \mathbf{S}_j^T \mathbf{S}_j \mathbf{A}_j \mathbf{x}_j - 2\mathbf{x}_j^T \mathbf{A}_j \mathbf{S}_j^T (\mathbf{r} - \bar{\mathbf{S}}_j \bar{\mathbf{A}}_j \bar{\mathbf{b}}_j) \tag{24a}$$

$$\text{subject to :} \quad x_i \in \{1, -1\} \quad \text{for} \quad i \notin \Omega_j \tag{24b}$$

where $\mathbf{x}_j = \{x_i, i \notin \Omega_j\}$ denotes the variable vector obtained by removing the components of $\mathbf{x}$ whose indices are in Ω_j, $\bar{\mathbf{b}}_j = \{\bar{b}_i, i \in \Omega_j\}$ denotes the vector of binary bits that have been determined *before* the jth iteration, $\mathbf{r}$ is defined in (4), $\mathbf{A}_j$ and $\bar{\mathbf{A}}_j$ are the submatrices of $\mathbf{A}$ obtained by removing the columns and rows of $\mathbf{A}$ whose indices are in Ω_j and not in Ω_j, respectively, $\mathbf{S}_j$ and $\bar{\mathbf{S}}_j$ are the submatrices of $\mathbf{S}$ obtained by removing the columns of $\mathbf{S}$ whose indices are in Ω_j and not in Ω_j, respectively.

Proof: By substituting (23c) into (23a), the objective function in (23a) can be expressed as

$$\mathbf{x}^T \mathbf{H} \mathbf{x} + \mathbf{x}^T \mathbf{p} = \mathbf{x}_j^T \mathbf{H}_j \mathbf{x}_j + \mathbf{x}_j^T (\mathbf{p}_j + 2\tilde{\mathbf{H}}_j \bar{\mathbf{b}}_j) + const \tag{25}$$

where $\mathbf{p}_j$ is obtained by removing the components of $\mathbf{p}$ whose indices are in Ω_j, $\mathbf{H}_j$ and $\tilde{\mathbf{H}}_j$ denote submatrices of $\mathbf{H}$ where $\mathbf{H}_j$ is obtained by removing those rows and columns of $\mathbf{H}$ whose indices are in Ω_j, and $\tilde{\mathbf{H}}_j$ is obtained by removing the rows of $\mathbf{H}$ whose indices are in Ω_j and columns of $\mathbf{H}$ whose indices are not in Ω_j. The term *const* in (25) is a constant associated with (23a), which depends on all quantities that are not related to $\mathbf{x}_j$. Note that $\mathbf{H}_j$, $\tilde{\mathbf{H}}_j$, and $\mathbf{p}_j$ can be expressed as

$$\mathbf{H}_j = \mathbf{A}_j \mathbf{S}_j^T \mathbf{S}_j \mathbf{A}_j \tag{26a}$$

$$\tilde{\mathbf{H}}_j = \mathbf{A}_j \mathbf{S}_j^T \bar{\mathbf{S}}_j \bar{\mathbf{A}}_j \tag{26b}$$

$$\mathbf{p}_j = -2\mathbf{A}_j \mathbf{S}_j^T \mathbf{r} \tag{26c}$$

Substituting (26) into (25), the objective function in (23a) can be expressed as

$$\mathbf{x}^T \mathbf{H} \mathbf{x} + \mathbf{x}^T \mathbf{p} = \mathbf{x}_j^T \mathbf{A}_j \mathbf{S}_j^T \mathbf{S}_j \mathbf{A}_j \mathbf{x}_j - 2\mathbf{x}_j^T \mathbf{A}_j \mathbf{S}_j^T (\mathbf{r} - \bar{\mathbf{S}}_j \bar{\mathbf{A}}_j \bar{\mathbf{b}}_j) + const \tag{27}$$

Obviously, the problem in (23) is equivalent to the problem in (24).

It can be seen in Proposition 3 that, in each iteration, the recursive ML approach involves two steps. First, the MUI corresponding to the determined information bits is expressed as $\bar{\mathbf{S}}_j\bar{\mathbf{A}}_j\bar{\mathbf{b}}_j$ and then subtracted from the received signal $\mathbf{r}$. Second, an ML detection problem is formulated to detect the rest of the information bits based on the updated received signal $\mathbf{r}_j = \mathbf{r} - \bar{\mathbf{S}}_j\bar{\mathbf{A}}_j\bar{\mathbf{b}}_j$. Obviously, if the bits in $\bar{\mathbf{b}}_j$ are correct, then the MUI of these bits can be represented accurately, which helps reduce the interference observed in $\mathbf{r}_j$ and thus improve the performance in detecting the other information bits. However, if the bits are incorrect, the interference in $\mathbf{r}_j$ will be doubled and this may lead to significant performance degradation.

B. RCP-Based Multiuser Detectors

Due to the binary constraints in (23b), the problem in (23) is still a combinatorial problem although the number of variables involved is less relative to that in problem (5). In this section, we consider relaxing this problem by using the CP relaxation approach proposed in Sec. IV. On comparing the problems in (5), (16), and (23), we note that the problem in (23) can be relaxed into the problem

$$\text{minimize} \quad \mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{p} \tag{28a}$$

$$\text{subject to}: \quad c(\mathbf{x}, p) \leq 0 \tag{28b}$$

$$x_i = \bar{b}_i \quad \text{for} \quad i \in \Omega_j \tag{28c}$$

By virtue of Proposition 3, this problem can be expressed in the equivalent form

$$\text{minimize} \quad \mathbf{x}_j^T\mathbf{H}_j\mathbf{x}_j + \mathbf{x}_j^T(\mathbf{p}_j + 2\tilde{\mathbf{H}}_j\bar{\mathbf{b}}_j) \tag{29a}$$

$$\text{subject to}: \quad c_j(\mathbf{x}_j, p) \leq 0 \tag{29b}$$

where $\mathbf{x}_j$, $\bar{\mathbf{b}}_j$ are defined in (24), $\mathbf{H}_j$, $\tilde{\mathbf{H}}$, and $\mathbf{p}_j$ are defined in (26), and

$$c_j(\mathbf{x}_j, p) = \sum_{i \notin \Omega_j} |x_{j,i}|^p - k_j \tag{30}$$

In (30), $x_{j,i}$ is the ith component of $\mathbf{x}_j$ and k_j is the number of variables not in Ω_j.

Once the problem in (29) is solved, some information bits can be determined through a threshold process as follows: if the magnitude of a solution component is greater than a prescribed threshold, then a decision for this information bit is made as the sign of the component; otherwise, the information bit is detected in a subsequent iteration.

The RCP detection approach described can be implemented in terms of the algorithm summarized in Table 1, which will be referred to as the RCP detectors hereafter.

TABLE 1

RECURSIVE CP DETECTOR

Parameters: n_j is the number of bits detected before the jth iteration.

Ω_j is the index set of bits detected before the jth iteration.

ξ_j is the threshold used in the jth iteration.

p is the order of the polynomial used in defining the feasible region.

Initialization: $j = 1$, $n_1 = 0$, $\Omega_1 = null$, $\mathbf{H}_1 = \mathbf{H}$, $\mathbf{p}_1 = \mathbf{p}$, $\bar{\mathbf{b}}_1 = \mathbf{0}$, and $\tilde{\mathbf{H}}_1 = \mathbf{0}$

Step 1: Solve the CP problem

$$\text{minimize:} \quad \mathbf{x}_j^T \mathbf{H}_j \mathbf{x}_j + \mathbf{x}_j^T (\mathbf{p}_j + 2\tilde{\mathbf{H}}_j \bar{\mathbf{b}}_j) \tag{31a}$$

$$\text{subject to:} \quad c_j(\mathbf{x}_j, p) \leq 0 \tag{31b}$$

and denote the solution as $\mathbf{x}_j^*$.

Step 2: Let $\xi_j \in [0, \max(|\mathbf{x}_j^*|)]$. For $i = 1, 2, \ldots, K - n_j$,

if $|x_{ji}^*| \geq \xi_j$, determine sign(x_{ji}^*) as the binary decision for the information bit corresponding to x_{ji}^*.

Step 3: Denote Ω_{j+1} as the index set which includes all indices of Ω_j and those of the information bits detected in Step 2. Update n_{j+1}. If $n_{j+1} = K$, stop and output $\bar{\mathbf{b}}_j$ as the vector of all the information bits detected.

Step 4: Update $\bar{\mathbf{b}}_{j+1}$ according to (24), $\mathbf{H}_{j+1}$, $\tilde{\mathbf{H}}_{j+1}$, $\mathbf{p}_{j+1}$ according to (26), and $c_{j+1}(\mathbf{x}_{j+1}, p)$ according to (30). Set $j = j + 1$ and repeat from Step 1.

In step 2, x_{ji}^* denotes the ith component of $\mathbf{x}_j^*$ and $\max(|\mathbf{x}_j^*|)$ denotes the largest magnitude in $\mathbf{x}_j^*$. The threshold ξ_j is assigned a value less than $\max(|\mathbf{x}_j^*|)$ so that at least one information bit is determined in each iteration. Consequently, all bits can be detected in at most K iterations. It can be seen that if we let $p \to \infty$ in (31) in Table 1, the magnitude of each solution component belongs to $[0, 1]$. Hence, the magnitude of the solution can be interpreted as the likelihood that the corresponding information bit can be detected correctly as the component sign, thus a decision is made for an information bit only if the associated likelihood for correct detection is higher than a specified value. A similar threshold-based interference cancellation (TIC) scheme is used in the PDA detector [13] primarily for speeding up the convergence. In other words, the threshold process used in the PDA detector does not affect the performance of the PDA detector significantly but reduces the number of stages required by the PDA detector to converge. On the other hand, although the TIC scheme is proposed primarily for improving the performance of the CP based detector, it also increases the computational complexity of the detector owing to an increased number of CP problems that need to be solved. In general, smaller threshold values reduce the number of iterations but may degrade the detection performance and larger threshold values tend to improve the detection performance but would require more iterations. The effect of the threshold values on the performance of the RCP detectors will be investigated through computer simulations described in Sec. VI.

The RCP detectors described in Table 1 assume hard-decision schemes where the decoder operates on demodulated binary data. Soft-decision versions of multiuser detectors based on sphere decoding and SDP relaxation have recently been proposed by several authors [26]-[28] and a soft-decision version of the RCP detectors may be possible based on these techniques.

C. Improved RCP Detectors

The computation used by the proposed RCP detectors is largely required in solving the CP problem, and it is more intensive than that required by linear multiuser detectors. In this section, the efficiency of the RCP detectors are improved by reducing the amount of computation required in solving the CP problem. The amount of computation is critically dependent on the initial point and a good initial point usually leads to a considerable reduction in computation.

If the constraint in (31b) is removed, the CP problem in (31) in Table 1 becomes the unconstrained optimization problem

$$\text{minimize :} \quad \mathbf{x}_j^T \mathbf{H}_j \mathbf{x}_j + \mathbf{x}_j^T (\mathbf{p}_j + 2\tilde{\mathbf{H}}_j \bar{\mathbf{b}}_j) \tag{32}$$

whose solution can be computed in closed-form as

$$\mathbf{x}_j^\dagger = -\frac{1}{2}\mathbf{H}_j^{-1}(\mathbf{p}_j + 2\tilde{\mathbf{H}}_j \bar{\mathbf{b}}_j) \tag{33}$$

If the solution in (33) happens to satisfy the condition

$$c_j(\mathbf{x}_j^{\dagger},\ p) \leq 0 \tag{34}$$

then obviously $\mathbf{x}_j^{\dagger}$ is the solution for the CP problem in (31) in Table 1. Note that the amount of computation required to obtain (33) is comparable to that for linear detectors, which in general is significantly less than that for solving the CP problem in (31) in Table 1. Motivated by this observation, the RCP detectors in Table 1 can be modified as follows. In the first step of each iteration, $\mathbf{x}_j^{\dagger}$ is first computed according to (33) and then the constraint in (34) is checked for $\mathbf{x}_j^*$. If this constraint is satisfied for $\mathbf{x}_j^{\dagger}$, then obtain $\mathbf{x}_j^{\dagger}$ as the solution of the CP problem in (31) and continue with the subsequent steps; otherwise, solve the CP problem in (31) using

$$\mathbf{x}_j^0 = \text{sign}(\mathbf{x}_j^{\dagger}) \tag{35}$$

as the initial solution. Since the components of $\mathbf{x}_j^0$ assume only binary values, they satisfy the constraint in (34). The improved RCP detector is described in Table 2.

The following proposition will be useful in the investigation of the efficiency of the improved RCP detector.

Proposition 4: Denote the two CP problems associated with the jth and $(j+1)$th iterations of the algorithm in Table 2 as $\mathcal{G}_j$ and $\mathcal{G}_{j+1}$, i.e.,

$$\mathcal{G}_j : \quad \begin{array}{c} \text{minimize}: \quad \mathbf{x}_j^T \mathbf{H}_j \mathbf{x}_j + \mathbf{x}_j^T \left(\mathbf{p}_j + 2\tilde{\mathbf{H}}_j \bar{\mathbf{b}}_j\right) \\ \text{subject to}: \quad c_j(\mathbf{x}_j, p) \leq 0 \end{array} \tag{37}$$

$$\mathcal{G}_{j+1} : \quad \begin{array}{c} \text{minimize}: \quad \mathbf{x}_{j+1}^T \mathbf{H}_{j+1} \mathbf{x}_{j+1} + \mathbf{x}_{j+1}^T \left(\mathbf{p}_{j+1} + 2\tilde{\mathbf{H}}_{j+1} \bar{\mathbf{b}}_{j+1}\right) \\ \text{subject to}: \quad c_{j+1}(\mathbf{x}_{j+1}, p) \leq 0 \end{array} \tag{38}$$

and let the solutions of problems $\mathcal{G}_j$ and $\mathcal{G}_{j+1}$ be $\mathbf{x}_j^*$ and $\mathbf{x}_{j+1}^*$, respectively. If the detected information bits b_i for $i \in \Omega_{j+1}$ are all correct and, in addition, if $c_j(\mathbf{x}_i^*, p) < 0$ and $\xi_j < 1$, then the probability that

$$\mathbf{x}_{j+1}^* = -0.5\mathbf{H}_{j+1}^{-1}(\mathbf{p}_{j+1} + 2\tilde{\mathbf{H}}_{j+1}\bar{\mathbf{b}}_{j+1}) \tag{39}$$

is bounded from below by

$$\text{P}\left(\mathbf{x}_{j+1}^* = -0.5\mathbf{H}_{j+1}^{-1}(\mathbf{p}_{j+1} + 2\tilde{\mathbf{H}}_{j+1}\bar{\mathbf{b}}_{j+1})\right) \geq \prod_{i \notin \Omega_{j+1}} \text{erf}\left(\frac{1-\xi_j}{\sqrt{2}\sigma\rho_i}\right) \tag{40}$$

TABLE 2

IMPROVED RECURSIVE CP DETECTOR

Parameters: n_j is the number of bits detected before the jth iteration.

Ω_j is the index set of bits detected before the jth iteration.

ξ_j is the threshold used in the jth iteration.

p is the order of polynomial used in defining the feasible region.

Initialization: $j = 1$, $n_1 = 0$, $\Omega_1 = null$, $\mathbf{H}_1 = \mathbf{H}$, $\mathbf{p}_1 = \mathbf{p}$, $\bar{\mathbf{b}}_1 = \mathbf{0}$, and $\tilde{\mathbf{H}}_1 = \mathbf{0}$.

Step 1: Compute $\mathbf{x}_j^\dagger = -0.5\mathbf{H}_j^{-1}(\mathbf{p}_j + 2\tilde{\mathbf{H}}_j\bar{\mathbf{b}}_j)$ and check the constraint in (34).

If the constraint in (34) is valid, let $\mathbf{x}_j^* = \mathbf{x}_j^\dagger$ and go to Step 2. Otherwise, solve the CP problem

$$\text{minimize:} \quad \mathbf{x}_j^T\mathbf{H}_j\mathbf{x}_j + \mathbf{x}_j^T(\mathbf{p}_j + 2\tilde{\mathbf{H}}_j\bar{\mathbf{b}}_j) \qquad \text{(36a)}$$

$$\text{subject to:} \quad c_j(\mathbf{x}_j, p) \le 0 \qquad \text{(36b)}$$

using $\mathbf{x}_j^0 = \text{sign}(\mathbf{x}_j^\dagger)$ as an initial solution.

Step 2: Let $0 \le \xi_j \le \min(1, \max(|\mathbf{x}_j^*|))$. For $i = 1, 2, \ldots, K - n_j$, if $|x_{ji}^*| \ge \xi_j$, determine sign(x_{ji}^*) as the binary decision for the information bit corresponding to x_{ji}^*.

Step 3: Denote Ω_{j+1} as the index set which includes all indices of Ω_j and those of the information bits detected in Step 2. Update n_{j+1}. If $n_{j+1} = K$, stop and output $\bar{\mathbf{b}}_j$ as the vector of all the information bits detected.

Step 4: Update $\bar{\mathbf{b}}_{j+1}$ according to (24), $\mathbf{H}_{j+1}$, $\mathbf{p}_{j+1}$, $\tilde{\mathbf{H}}_{j+1}$ according to (26), and $c_{j+1}(\mathbf{x}_{j+1}, p)$ according to (30). Set $j = j + 1$ and repeat from Step 1.

In (40), erf$(\cdot)$ denotes the *error function* given by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \qquad (41)$$

and ρ_i is defined as

$$\rho_i = \sqrt{\left[\left(\mathbf{H}_{j+1} - \hat{\mathbf{H}}_j(\check{\mathbf{H}}_j)^{-1}(\hat{\mathbf{H}}_j)^T\right)^{-1} - \mathbf{H}_{j+1}^{-1}\right]_{i,i}} \qquad (42)$$

In (41), $\check{\mathbf{H}}_j$ is the submatrix of $\mathbf{H}_j$ obtained by removing the rows and columns of $\mathbf{H}_j$ whose indices are not in Ω_{j+1} and $\hat{\mathbf{H}}_j$ is the submatrix of $\mathbf{H}_j$ obtained by removing the rows of $\mathbf{H}_j$

whose indices are in Ω_{j+1} and the columns of $\mathbf{H}_j$ whose indices are not in Ω_{j+1}.

Proof: : See Appendix for proof.

As can be seen from (40) and (42), the lower bound of the probability for which (39) is valid is related to the signature waveforms, the SNR of the received signals, and the threshold used in each iteration of the proposed approach. In particular, based on (40) and (42) we observe that if near orthogonal signature waveforms are used for all users so that the crosscorrelation matrix $\hat{\mathbf{H}}_j$ is close to zero, then the value of ρ_i is close to zero, and if the threshold ξ_j are small, then in high SNR channels the probability bound in (40) is close to one.

Based on Proposition 4 and the above discussion, we conclude that if the constraint of the CP problem in the jth iteration of the RCP approach is inactive, then the probability that $\mathbf{x}^*_{j+1}$ can be correctly computed using (39) is close to one. In such a case, the amount of computation required for solving the CP problem in (31) in Table 1 can be reduced to that required to evaluate $\mathbf{x}^\dagger_j$ in (33).

VI. Simulation Results

A. Average Number of Iterations and Worst-Case Computational Complexity

Table 2 shows that the reduction in computational complexity achieved with the RCP detectors depends on when the constraint in (36b) starts to be inactive. In order to investigate this feature of the RCP detectors, Monte Carlo simulations were conducted. In the simulations, we considered an equal user-power system. Normalized random sequences of length 31 were used which were regenerated after the transmission of every 10^3 information bits. The SNR of the received signal was set to 10 dB for all users and the results were averaged over 10^5 runs. The thresholds used were assumed to be

$$\xi_j = \alpha \cdot \min\big(1, \max(|\mathbf{x}^*_j|)\big) \qquad \text{for } j = 1,\ 2,\ \ldots \tag{43}$$

where α is a positive scalar. Note that when $p = 2$, the CP problem in (36) becomes equivalent to the problem in (10) for the GMMSE detector and when $p = \infty$ it becomes equivalent to the problem in (13) for the BC detector. In the following discussion, we will focus on the RCP detectors with $p = 2$ and $p = \infty$ (denoted as the RCP-2 and the RCP-∞ detectors, respectively). As can be seen in Fig. 2, the constraint in (36b) in the CP problem of (36) becomes inactive in two to three iterations for both the RCP-2 and RCP-∞ detectors and, according to Table 2, the solutions of the subsequent CP problems can be efficiently computed using (33).

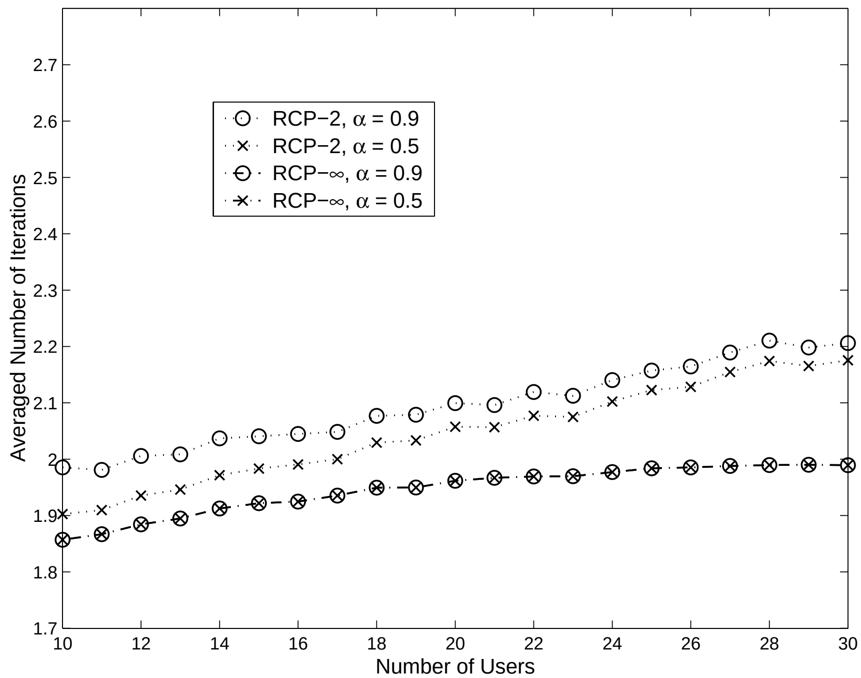

Fig. 2. Averaged number of iterations for the constraint in (36b) to become inactive.

Next we studied the worst-case computational complexity of the RCP-2 and RCP-∞ detectors. According to Table 2, the computation involved consists of two parts: the first part is for solving the CP problem in (36) with active constraints and the second part is for computing (33) in subsequent iterations. When $p = 2$, the CP problem in (36) in Table 2 is equivalent to the problem in (10) which can be solved by solving its dual problem in (11). Since the computational complexity for solving the problem in (11) is $O(K^3)$, the worst-case computational complexity for each iteration of the RCP-2 detector is $O(K^3)$. On the other hand, when $p = \infty$, the CP problem in (36) is equivalent to the problem in (13). The solution of the problem can be obtained by using either the *Gauss-Seidel (GS)* or *Jacobi* approach [7]. Since these approaches entail a worst-case computational complexity $O(K^3)$, the worst-case computational complexity for each iteration of the RCP-∞ detector is also $O(K^3)$. Note that a large tolerance can be used in solving the problems in (10) and (13) since a threshold process is applied to their solutions and thus a reduced computational complexity can be achieved in both the RCP-2 and RCP-∞ detectors.

B. BER Performance and Average Computational Complexity

Computer simulations were conducted to evaluate the performance of the RCP-2 and RCP-∞ detectors in terms of BER and average computational complexity and to compare them with that of the ML detector and several other suboptimal detectors.

For a more realistic comparison of the demodulation performance of the SDPR and RCP detectors, we considered a DS-CDMA system subjected to mild as well as strong MUI. The thresholds used in the RCP-2 and RCP-∞ detectors were determined using (43). The BERs were determined through the detection of 10^5 symbols in each simulation. The SDPR detector was implemented using an efficient algorithm proposed in [12] where the solution of the SDP problem is obtained by solving a dual SDP problem and the binary solution is obtained by using the rank-one approximation approach in [12].

In the first example, we evaluated the BERs of the RCP-2, RCP-∞, and ML detectors as well as the those of the GMMSE detector in [7][8], the BC detector in [7][8], the SDPR detector in [12], and the PDA detector in [13] for a 10-user equal-power system. Gold sequences of length 31 were assumed for all users. The thresholds used for the two RCP detectors were determined using $\alpha = 0.8$. The difference between the BER of the various detectors and that of the ML detector, designated as ΔBER, is plotted in Fig. 3. As can be seen, the BERs of the RCP-2, RCP-∞, SDPR, and PDA detectors are very close to each other and all are superior relative to the BERs of the GMMSE and BC detectors.

In the second example, we considered a 14-user equal-power system. The signature sequences were 31-chip normalized random sequences which were regenerated after the transmission of every 10^3 information bits. The thresholds used for the RCP-2 and RCP-∞ detectors were determined as before using the values 0.8, 0.5, 0.3 for α. In this example, we also compared the proposed detectors with the successive interference cancellation (SIC) detector described in [29]. In this detector, the information bit corresponding to the user with the strongest power is detected first and the MUI corresponding to the detected information bit is then recreated and subtracted from the received signal. The BERs of the RCP-2, RCP-∞, and ML detectors as well as those of the SIC detector in [29], the GMMSE detector in [7][8], the BC detector in [7][8], the SDPR detector in [12], the PDA detector in [13], and the recursive SDPR (R-SDPR) detector are plotted in Figs. 4-6. The recursive SDPR detector was implemented using the same recursive

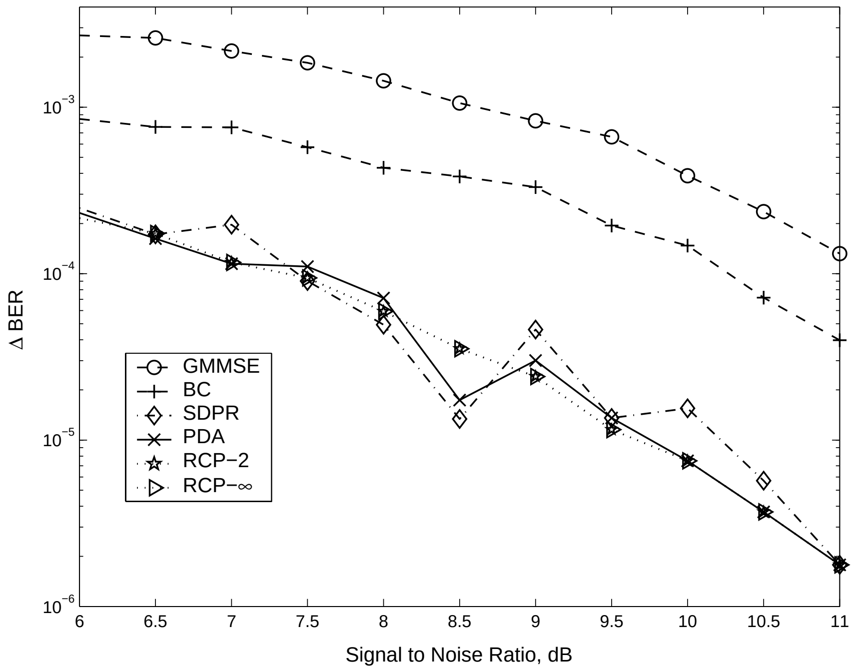

Fig. 3. Difference between the BER of various detectors and that of the ML detector for the 10-user system with $\alpha = 0.80$.

scheme as in the RCP detectors. In Figs. 4 to 6, we observe that for $\alpha = 0.8$, the BERs of the RCP-2 and RCP-∞, R-SDPR, and ML detectors are very close to each other and all are superior to the BERs offered by the SIC, GMMSE, BC, SDPR, and PDA detectors. However, a reduction in α leads to degradation in the BER performance in both RCP detectors but the degradation is not severe until α is reduced to a value of 0.5 or less. In such a case, the RCP-∞ detector offers better BER performance than all the other suboptimal detectors considered.

In the third example, we considered an 8-user system with the received signal power of the eight users set to 2.0, 1.3, 1.8, 0.8, 0.95, 0.8, 0.7, and 1.0. The crosscorrelation matrix for the

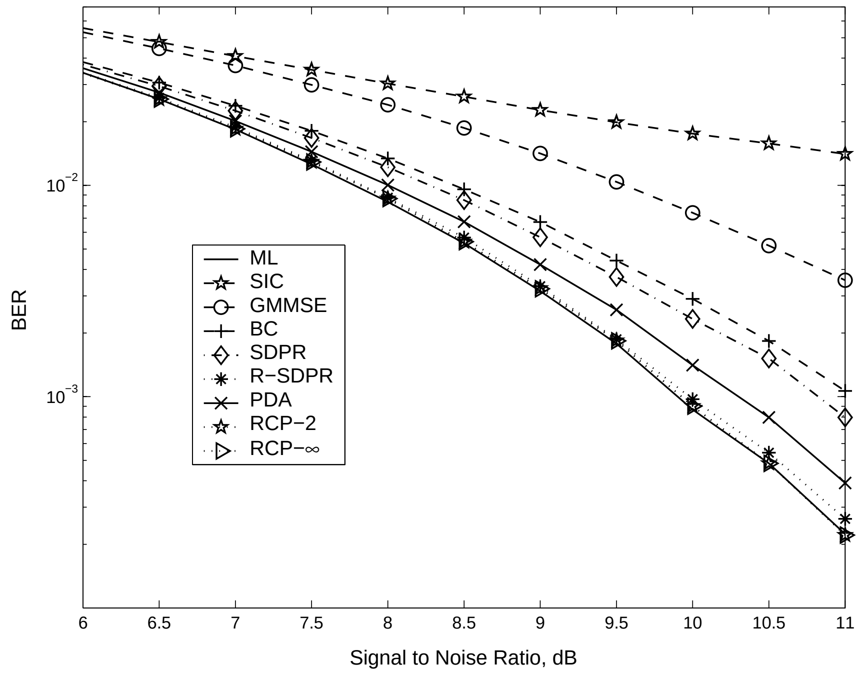

Fig. 4. BERs for the 14-user system with $\alpha = 0.80$.

user signatures is given by

$$\mathbf{R} = \begin{bmatrix} 1 & 0.01 & 0.18 & 0.05 & -0.38 & -0.07 & -0.12 & -0.13 \\ 0.01 & 1 & 0.30 & -0.01 & 0.39 & -0.09 & -0.24 & 0.28 \\ 0.18 & 0.30 & 1 & -0.27 & 0.27 & -0.05 & -0.03 & 0.20 \\ 0.05 & -0.01 & -0.27 & 1 & -0.03 & 0.16 & 0.08 & -0.11 \\ -0.38 & 0.39 & 0.27 & -0.03 & 1 & -0.03 & 0.21 & 0.14 \\ -0.07 & -0.09 & -0.05 & 0.16 & -0.03 & 1 & 0.15 & 0.25 \\ -0.12 & -0.24 & -0.03 & 0.08 & 0.21 & 0.15 & 1 & -0.41 \\ -0.13 & 0.28 & 0.20 & -0.11 & 0.14 & 0.25 & -0.41 & 1 \end{bmatrix} \tag{44}$$

The thresholds used for the RCP-2 and RCP-∞ detectors were determined as before with α set to 0.8, 0.5, 0.3. The BERs of various detectors are plotted in Figs. 7 to 9. In these plots, we observe that the BER of the RCP-2, RCP-∞, R-SDPR, and PDA detectors are close to each other and all are superior relative to BERs of the SIC, GMMSE, BC, and SDPR detectors.

The average computational complexity of the SDPR, R-SDPR, PDA, RCP-2, and RCP-∞ detectors was evaluated in terms of the CPU time[1] required for the detection of one information

[1] The simulation was conducted on a Sun Blade 2000 workstation using MATLAB 6.5. The CPU time required by each

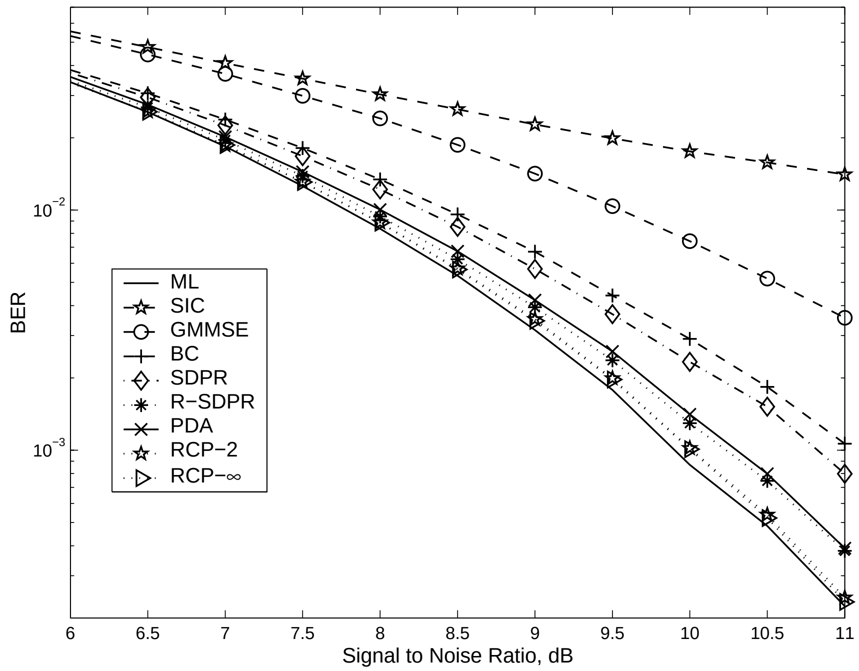

Fig. 5. BERs for the 14-user system with $\alpha = 0.50$.

bit. The number of users was varied from 11 to 33 and the CPU time was averaged based on the detection of 10^4 information bits. The results obtained are plotted in Fig. 10. As can be seen, the average computational complexity of the RCP-2 and RCP-∞ detectors is much lower than those of the R-SPDR and SDPR detectors. The average computational complexity of the RCP-2 detector is slightly higher than that of the PDA detector and that of the RCP-∞ detector is slightly lower.

VII. Conclusions

New multiuser detectors have been proposed based on the RCP approach. In this approach, the combinatorial problem associated with ML detection is relaxed into a convex programming problem and it is then implemented in terms of a recursive approach. On the basis of this approach, efficient RCP detectors have been developed and a theoretical analysis of their efficiency has been presented. Computer simulations have been performed which show that the RCP detectors offer near-optimal performance and are superior to many existing suboptimal detectors whereas the computation complexity associated with these detectors is much lower than that

algorithm was obtained by using MATLAB command *CPUTIME*.

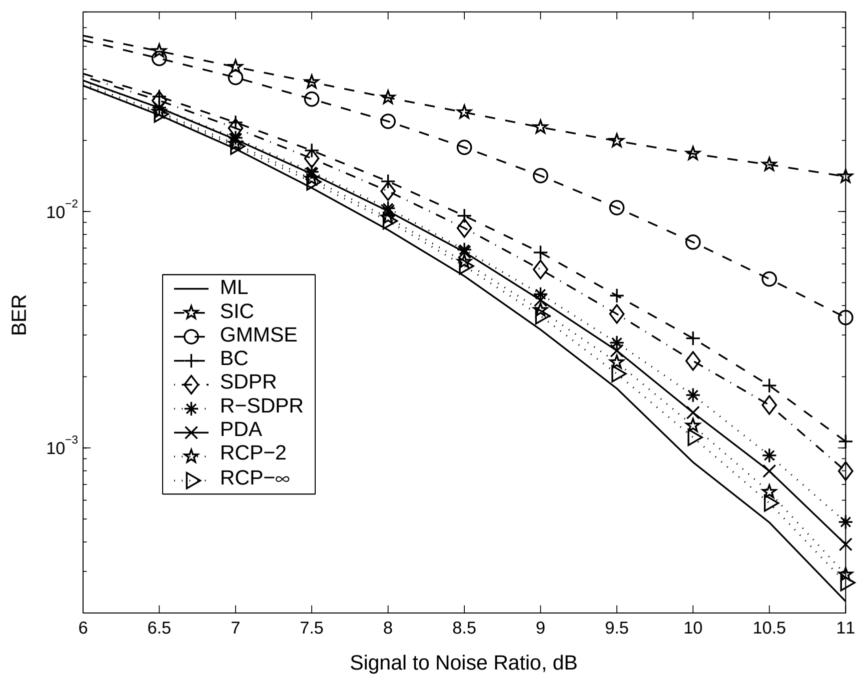

Fig. 6. BERs for the 14-user system with $\alpha = 0.30$.

associated with the ML or SDPR detectors.

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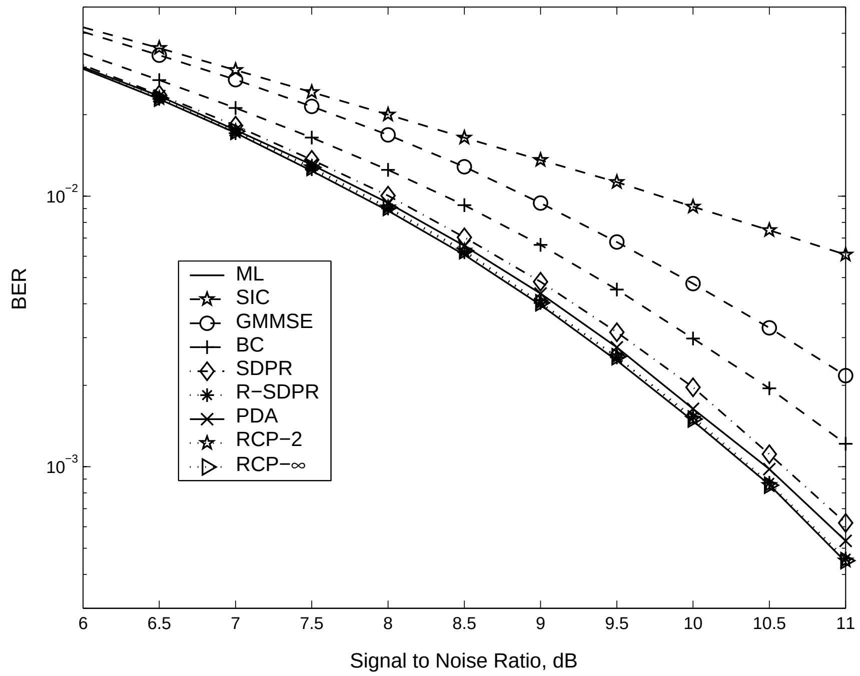

Fig. 7. BERs for the 8-user system with $\alpha = 0.80$.

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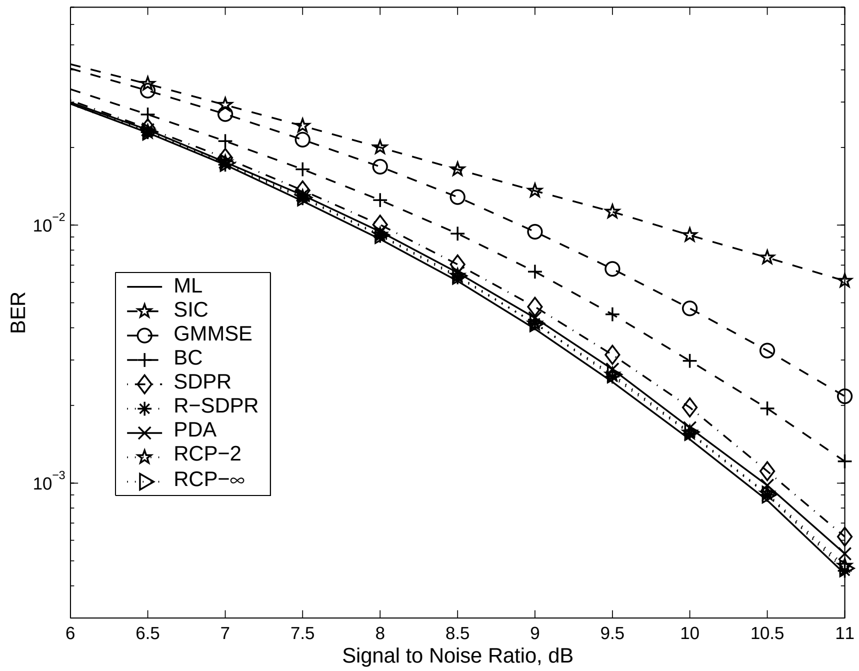

Fig. 8. BERs for the 8-user system with $\alpha = 0.50$.

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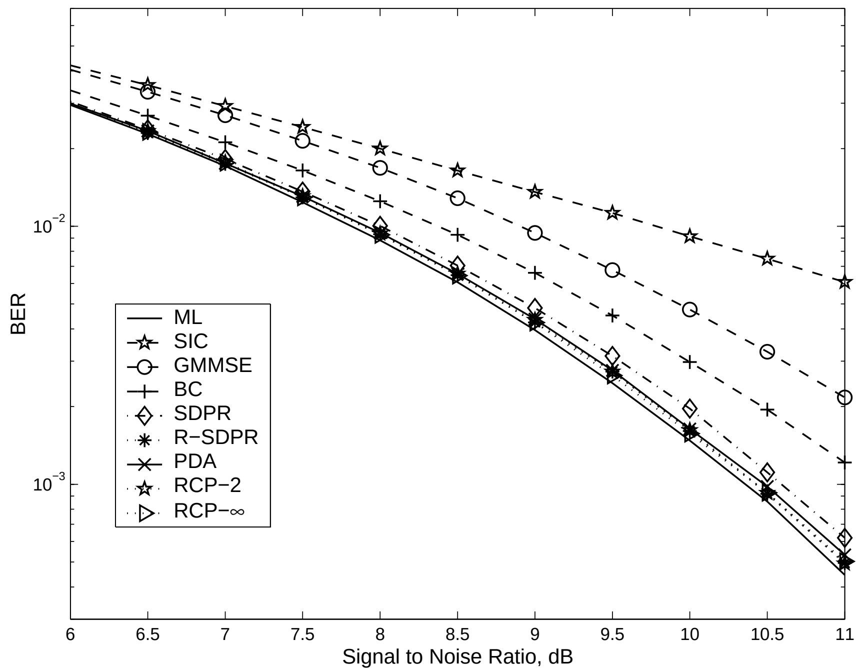

Fig. 9. BERs for the 8-user system with $\alpha = 0.30$.

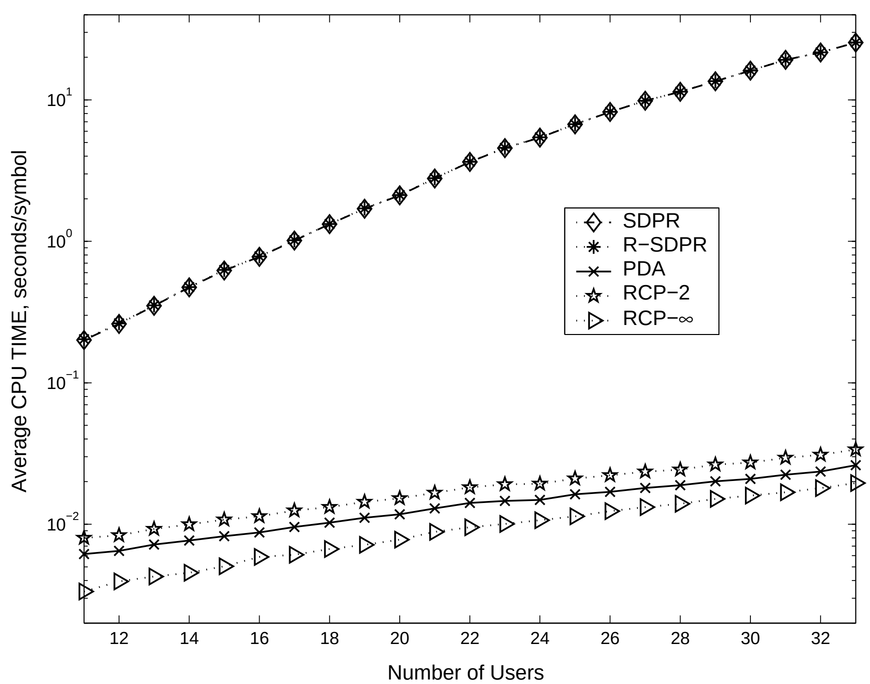

Fig. 10. Computational complexity of various detectors.

Appendix

Proof of Proposition 4

The following lemma will be used in the proof.

Lemma 1: If $\mathbf{A}$ is an $n \times n$ matrix composed of four block matrices as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{11} \end{bmatrix} \tag{45}$$

where $\mathbf{A}_{11} \in \mathcal{C}^{k\times k}$, $\mathbf{A}_{12} \in \mathcal{C}^{k\times(n-k)}$, $\mathbf{A}_{21} \in \mathcal{C}^{(n-k)\times k}$, and $\mathbf{A}_{11} \in \mathcal{C}^{(n-k)\times(n-k)}$, then the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{B} & -\mathbf{B}\cdot\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ -\mathbf{C}\cdot\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{C} \end{bmatrix} \tag{46}$$

where

$$\mathbf{B} = (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})^{-1}$$
$$\mathbf{C} = (\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1}$$

This lemma can be easily verified by showing that $\mathbf{A}^{-1}\cdot\mathbf{A} = \mathbf{I}$.

To simplify the notation, the solution of the problem in (36) in Table 2, $\mathbf{x}_j^*$, is expressed as

$$\mathbf{x}_j^* = \begin{bmatrix} \hat{\mathbf{x}}_j^* \\ \check{\mathbf{x}}_j^* \end{bmatrix} \tag{47}$$

where $\check{\mathbf{x}}_j^*$ is composed of the variables corresponding to the information bits that are detected in the jth iteration and $\hat{\mathbf{x}}_j^*$ is composed of the variables corresponding to the other information bits. According to the RCP algorithm in Table 1, we have

$$|\hat{\mathbf{x}}_j^*(i)| < \xi_j \quad \text{and} \quad |\check{\mathbf{x}}_j^*(i)| \geq \xi_j \tag{48}$$

where $\mathbf{x}(i)$ denotes the ith component of $\mathbf{x}$. Note that according to the assumption in (47), $\mathbf{H}_j$ in (37) can be decomposed into submatrices as

$$\mathbf{H}_j = \begin{bmatrix} \mathbf{H}_{j+1} & \hat{\mathbf{H}}_j \\ \hat{\mathbf{H}}_j^T & \check{\mathbf{H}}_j \end{bmatrix} \tag{49}$$

where $\mathbf{H}_{j+1}$ is defined in (26) and $\hat{\mathbf{H}}_j$ and $\check{\mathbf{H}}_j$ are defined in (41).

Note that the constraint $c_j(\mathbf{x}_j^*, p) < 0$ implies that the constraint of this problem is inactive; hence, the solution of the problem in (37) can be obtained as

$$\mathbf{x}_j^* = \begin{bmatrix} \hat{\mathbf{x}}_j^* \\ \check{\mathbf{x}}_j^* \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} \mathbf{H}_{j+1} & \hat{\mathbf{H}}_j \\ \hat{\mathbf{H}}_j^T & \check{\mathbf{H}}_j \end{bmatrix}^{-1} \left(\mathbf{p}_j + 2\tilde{\mathbf{H}}_j \bar{\mathbf{b}}_j \right) \tag{50}$$

According to Lemma 1, $\hat{\mathbf{x}}_j^*$ in (50) can be expressed as

$$\hat{\mathbf{x}}_j^* = \mathbf{\Delta}_j \cdot (\mathbf{r} - \bar{\mathbf{S}}_j \bar{\mathbf{A}}_j \bar{\mathbf{b}}_j) = \mathbf{b}_{j+1} + \mathbf{\Delta}_j \cdot (\bar{\mathbf{S}}_j \bar{\mathbf{A}}_j \bar{\delta}_{\mathbf{b},j} + \mathbf{n}_c) \tag{51}$$

where $\mathbf{\Delta}_j$ is defined by

$$\mathbf{\Delta}_j = \left(\mathbf{H}_{j+1} - \hat{\mathbf{H}}_j \check{\mathbf{H}}_j^{-1} \hat{\mathbf{H}}_j^T \right)^{-1} \cdot \left(\mathbf{A}_{j+1} \mathbf{S}_{j+1}^T - \hat{\mathbf{H}}_j \check{\mathbf{H}}_j^{-1} \check{\mathbf{A}}_j \check{\mathbf{S}}_j^T \right) \tag{52}$$

$\bar{\mathbf{S}}_j$, $\bar{\mathbf{A}}_j$, $\mathbf{S}_{j+1}$, and $\mathbf{A}_{j+1}$ are defined in (24), $\check{\mathbf{S}}_j$ denotes the signature matrix obtained by removing the columns of $\mathbf{S}_j$ whose indices are not in Ω_{j+1}, $\check{\mathbf{A}}_j$ denotes the amplitude matrix obtained by removing the rows and columns of $\mathbf{A}_j$ whose indices are not in Ω_{j+1}, and $\bar{\delta}_{\mathbf{b},j}$ denotes the vector composed of the decision errors of the information bits that are detected before the jth iteration.

Next we define vector $\mathbf{x}_{j+1}^\dagger$ as

$$\begin{aligned} \mathbf{x}_{j+1}^\dagger &= -\frac{1}{2} \mathbf{H}_{j+1}^{-1} (\mathbf{p}_{j+1} + 2\tilde{\mathbf{H}}_{j+1} \bar{\mathbf{b}}_{j+1}) = \mathbf{H}_{j+1}^{-1} \mathbf{A}_{j+1} \mathbf{S}_{j+1}^T (\mathbf{r} - \bar{\mathbf{S}}_{j+1} \bar{\mathbf{A}}_{j+1} \bar{\mathbf{b}}_{j+1}) \\ &= \mathbf{b}_{j+1} + \mathbf{H}_{j+1}^{-1} \mathbf{A}_{j+1} \mathbf{S}_{j+1} \left(\bar{\mathbf{S}}_j \bar{\mathbf{A}}_j \bar{\delta}_{\mathbf{b},j} + \check{\mathbf{S}}_j \check{\mathbf{A}}_j \check{\delta}_{\mathbf{b},j} + \mathbf{n}_c \right) \end{aligned} \tag{53}$$

where $\check{\delta}_{\mathbf{b},j}$ denotes the vector composed by the decision errors of the information bits that are detected in the jth iteration. It is easy to verify that

$$\delta_{\mathbf{x},j} = \hat{\mathbf{x}}_j^* - \mathbf{x}_{j+1}^\dagger = \mathbf{\Sigma}_j \cdot \mathbf{n}_c + \mathbf{\Sigma}_j \cdot \bar{\mathbf{S}}_j \ \bar{\mathbf{A}}_j \bar{\delta}_{\mathbf{b},\ j} - \mathbf{H}_{j+1}^{-1} \ \hat{\mathbf{H}}_j \ \check{\delta}_{\mathbf{b},\ j} \tag{54}$$

where $\mathbf{\Sigma}$ is defined as

$$\mathbf{\Sigma}_j = \mathbf{\Delta}_j - \mathbf{H}_{j+1}^{-1} \mathbf{A}_{j+1} \mathbf{S}_{j+1}^T \tag{55}$$

When the decisions about the ith information bits for $i \in \Omega_{i+1}$ are made correctly, $\bar{\delta}_{\mathbf{b},\ j}$ and $\check{\delta}_{\mathbf{b},\ j}$ become zero vectors, and (54) can be simplified to

$$\delta_{\mathbf{x},j} = \hat{\mathbf{x}}_j^* - \mathbf{x}_{j+1}^\dagger = \mathbf{\Sigma}_j \cdot \mathbf{n}_c \tag{56}$$

Hence

$$\mathrm{P}\left[\mathbf{x}_{j+1}^{*}=\mathbf{x}_{j+1}^{\dagger}\right] \geq \prod_{i\notin\Omega_{j+1}} \mathrm{P}\left(|\mathbf{x}_{j+1}^{\dagger}(i)|\leq 1\right) \tag{57}$$

$$\geq \prod_{i\notin\Omega_{j+1}} \mathrm{P}\left(|\mathbf{n}_{\Sigma,j}(i)|<1-\xi_j\right) \tag{58}$$

where $\mathbf{n}_{\Sigma,j}=\mathbf{\Sigma}_j\mathbf{n}_c$ and $\mathbf{n}_{\Sigma,j}(i)$ denotes the ith component of $\mathbf{n}_{\Sigma,j}$. Note that the inequality in (57) is obtained by letting $p\rightarrow\infty$ and thus it is satisfied for any scalar $p\geq 1$. Since $\mathbf{n}_{\Sigma,j}(i)$ is a zero-mean Gaussian variable whose variance is $\sigma^2\rho_i^2$ with ρ_i being defined in (41), we obtain

$$\mathrm{P}\left(|\mathbf{n}_{\Sigma}(i)|<1-\xi_j\right)=\frac{1}{\sqrt{2\pi}\sigma\rho_i}\int_{-(1-\xi_j)}^{1-\xi_j}\exp\left(-\frac{t^2}{2\sigma^2\rho_i^2}\right)\,dt=\mathrm{erf}\left(\frac{1-\xi_j}{\sqrt{2}\sigma\rho_i}\right) \tag{59}$$

Substituting (59) into (58), we have

$$\mathrm{P}\left[\mathbf{x}_{j+1}^{*}=-\frac{1}{2}\mathbf{H}_{j+1}^{-1}(\mathbf{p}_{j+1}+2\tilde{\mathbf{H}}_{j+1}\bar{\mathbf{b}}_{j+1})\right]\geq\prod_{i\notin\Omega_{j+1}}\mathrm{erf}\left(\frac{1-\xi_j}{\sqrt{2}\sigma\rho_i}\right) \tag{60}$$