An Improved Weighted Least-Squares Design for Variable Fractional Delay FIR Filters

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***Abstract*— Digital filters capable of changing their frequency response characteristics are often referred to as variable digital filters (VDF's) and have been found useful in a number of digital signal processing applications. An important class of VDF's is the class of digital filters with variable fractional delay. This paper describes an enhanced weighted least-squares design for variable-fractional-delay finite-impulse response filters, which offers improved performance of the filters obtained with considerably reduced computational complexity compared to a recently proposed weighted least-squares (WLS) design method. The design enhancement is achieved by deriving a closed-form formula for evaluating the WLS objective function. The formula facilitates accurate and efficient function evaluations as compared to summing up a large number of discrete terms, which would be time consuming and inevitably introduce additional errors into the design.**

***Index Terms*— Digital filters, variable fractional delay filters, weighted least-squares design.**

I. Introduction

Digital filters capable of changing their frequency response characteristics, such as group delay, magnitude response, and resonance frequency, etc., are often referred to as *variable digital filters* (VDF's). Typically, the transfer function of a VDF contains a number of *parameters* that can be used to tune the frequency response of the VDF. Thus, the main objective in the design of a VDF is to find a *parameterized* transfer function which, in a certain sense, best approximates a given set of frequency response characteristics that vary with the parameters in a desired manner. Applications of VDF's in image processing, two- and three-dimensional signal migration in seismic data processing, digital telecommunications, and modeling of music instruments have been reported in, for example, [1]–[12]. A detailed account of the basic theory, design, and implementation of various VDF's can be found in the survey papers [13], [14].

In this paper, we focus our attention on the design of finite-impulse response (FIR) VDF's with variable fractional delay. Digital filters with fractional delay represent an important class of digital filters as they find many applications, and several algorithms for the design of such filters have recently been proposed [14]–[18]. In [17], a weighted least-squares (WLS) method is proposed to design a single-parameter FIR VDF, and the method was applied to design a variable fractional delay filter. This paper describes an enhanced WLS design for single-parameter FIR VDF's. The proposed algorithm is applied to design a variable fractional delay filter that demonstrates improved performance with reduced computational complexity compared to that of [17]. Essentially, the improved performance and design efficiency is achieved by deriving a closed-form formula for evaluating the WLS objective function in which the weighting function is assumed to be separable and piecewise constant. It avoids using large number of frequency and parameter grids and allows one to carry out the needed function evaluations accurately and quickly. A design example is included to illustrate the proposed method.

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II. Problem Formulation

We adopt the notation used in [17] to denote the transfer function of the fractional delay FIR filter by

$$H(z, p) = \sum_{n=0}^{N} a_n(p) z^{-n} \tag{1}$$

where p is the parameter representing the fractional delay, and $a_n(p)$ $(n = 0, 1, \cdots, N)$ are polynomials of degree K, i.e.,

$$a_n(p) = \sum_{k=0}^{K} a_{nk} p^k. \tag{2}$$

In matrix notation, the frequency response of the filter can be expressed as

$$H(\omega, p) = \boldsymbol{\omega}^T \boldsymbol{A} \boldsymbol{p} \tag{3}$$

where

$$\begin{aligned} \boldsymbol{\omega} &= [1 \quad e^{-j\omega} \quad e^{-j2\omega} \quad \cdots \quad e^{-jN\omega}]^T \\ \boldsymbol{p} &= [1 \quad p \quad p^2 \quad \cdots \quad p^K]^T \end{aligned}$$

and

$$\boldsymbol{A} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0K} \\ a_{10} & a_{11} & \cdots & a_{1K} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N0} & a_{N1} & \cdots & a_{NK} \end{bmatrix}.$$

For the design of fractional delay filters, the desired (variable) frequency response is given by

$$H_d(\omega, p) = e^{-j\omega(D+p)} \tag{4}$$

where $p \in [0, 1]$ is the parameter encountered in (1)–(3), and D is an integer usually chosen to be $(N-1)/2$ if N is odd or $N/2$ if N is even. For a WLS design, the objective function is set to be

$$J(\boldsymbol{A}) = \int_0^{\pi} \int_0^1 W(\omega, p)|H(\omega, p) - H_d(\omega, p)|^2 \, dp \, d\omega \quad (5)$$

and one seeks to find coefficient matrix $\boldsymbol{A}$ that minimizes $J(\boldsymbol{A})$ in (5).

III. Closed-Form Formulation for Evaluating $J(\boldsymbol{A})$

Throughout we assume a separable and piecewise constant weighting function $W(\omega, p)$, namely

$$W(\omega, p) = W_1(\omega)W_2(p). \quad (6)$$

In (6), $W_1(\omega)$ is assumed to be a constant w_{1l} on interval $[\omega_{l-1}, \omega_l)$ with $l = 1, \cdots, L$, and $W_2(p)$ is assumed to be a constant w_{2m} on interval $[p_{m-1}, p_m)$ with $m = 1, \cdots, M$, where $\{[\omega_{l-1}, \omega_l), l = 1, \cdots, L\}$ and $\{[p_{m-1}, p_m), m = 1, \cdots, M\}$ are partitions of frequency interval $[0, \pi]$ and parameter interval [0, 1], respectively. The objective function $J(\boldsymbol{A})$ in (5) can be evaluated as follows.

First, we write $J(\boldsymbol{A})$ as

$$\begin{aligned} J(\boldsymbol{A}) = \int_0^{\pi} \int_0^1 W(\omega, p) \cdot \Big\{ & \boldsymbol{\omega}^T \boldsymbol{A}\boldsymbol{p}\boldsymbol{p}^T \boldsymbol{A}^T \overline{\boldsymbol{\omega}} \\ & -2 \text{ Re } \cdot \left[\boldsymbol{\omega}^T \boldsymbol{A}\boldsymbol{p} e^{j\omega(D+p)}\right] + 1 \Big\} dp \, d\omega \\ = J_1 + J_2 + \text{const.} & \end{aligned} \quad (7)$$

By using the property of matrix trace that $\mathrm{tr}(\boldsymbol{AB}) = \mathrm{tr}(\boldsymbol{BA})$, where tr$(\cdot)$ denotes matrix trace, a bilinear form $\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{y}$ can be written as $\mathrm{tr}(\boldsymbol{M}\boldsymbol{y}\boldsymbol{x}^T)$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors and $\boldsymbol{M}$ is a matrix. It then follows that term J_1 in (7) can be calculated as

$$\begin{aligned} J_1 &= \int_0^{\pi} \int_0^1 W_1(\omega)W_2(p)\boldsymbol{\omega}^T \boldsymbol{A}\boldsymbol{p}\boldsymbol{p}^T \boldsymbol{A}^T \overline{\boldsymbol{\omega}} \, dp \, d\omega \\ &= \mathrm{tr}\Big\{ \int_0^1 W_2(p)\boldsymbol{p}\boldsymbol{p}^T \, dp \cdot \boldsymbol{A}^T \\ &\qquad \cdot \mathrm{Re}\left[\int_0^{\pi} W_1(\omega)\overline{\boldsymbol{\omega}}\boldsymbol{\omega}^T \, d\omega\right] \cdot \boldsymbol{A} \Big\} \\ &= \mathrm{tr}(\boldsymbol{P}\boldsymbol{A}^T \boldsymbol{\Omega}\boldsymbol{A}) \end{aligned} \quad (8)$$

where $\overline{\boldsymbol{\omega}}$ is the complex-conjugate of $\boldsymbol{\omega}$

$$\boldsymbol{P} = \int_0^1 W_2(p)\boldsymbol{p}\boldsymbol{p}^T \, dp = \sum_{m=1}^{M} w_{2m}\boldsymbol{P}_m \quad (9)$$

with the (i, j)th entry of $\boldsymbol{P}_m$ given by

$$P_m(i, j) = \frac{p_m^{i+j-1} - p_{m-1}^{i+j-1}}{i+j-1}, \quad 1 \le i, j \le K+1 \quad (10)$$

$$\boldsymbol{\Omega} = \mathrm{Re}\left[\int_0^{\pi} W_1(\omega)\overline{\boldsymbol{\omega}}\boldsymbol{\omega}^T \, d\omega\right] = \sum_{l=1}^{L} w_{1l}\boldsymbol{\Omega}_l \quad (11)$$

with the (i, j)th entry of $\boldsymbol{\Omega}_l$, given by

$$\begin{aligned} \boldsymbol{\Omega}_l(i, j) = \omega_l \mathrm{sinc}[(i-j)\omega_l/\pi] - \omega_{l-1}\mathrm{sinc}[(i-j)\omega_{l-1}/\pi] \\ 1 \le i, j \le N+1 \end{aligned} \quad (12)$$

and

$$\mathrm{sinc}(x) = \begin{cases} 1, & \text{for } x = 0 \\ \dfrac{\sin \pi x}{\pi x}, & \text{elsewhere.} \end{cases}$$

From (11) and (12), we see that $\boldsymbol{\Omega}_l$ and, hence, $\boldsymbol{\Omega}$ are Toeplitz matrices.

By using the same trace property, the second term in (7) can be written as

$$\begin{aligned} J_2 &= -2\mathrm{Re}\left[\int_0^{\pi} \int_0^1 W_1(\omega)W_2(p)\boldsymbol{\omega}^T \boldsymbol{A}\boldsymbol{p} e^{j\omega(D+p)} \, dp \, d\omega\right] \\ &= -2\mathrm{tr}\left[\int_0^1 W_2(p)\boldsymbol{p}\boldsymbol{\omega}_p^T \, dp \cdot \boldsymbol{A}\right] \end{aligned} \quad (13)$$

where

$$\boldsymbol{\omega}_p^T = \mathrm{Re}\left[\int_0^{\pi} W_1(\omega)\boldsymbol{\omega}^T e^{j\omega(D+p)} \, d\omega\right] = \sum_{l=1}^{L} w_{1l}\boldsymbol{\omega}_{lp}^T \quad (14)$$

and (14a), as shown at the bottom of the page. Equation (13) leads to

$$J_2 = -2 \text{ tr}(\boldsymbol{S}\boldsymbol{A}) \quad (15)$$

where

$$\boldsymbol{S} = \int_0^1 W_2(p)\boldsymbol{p}\boldsymbol{\omega}_p^T \, dp = \sum_{l=1}^{L} \sum_{m=1}^{M} w_{1l}w_{2m}\boldsymbol{S}_{lm} \quad (16)$$

with

$$\boldsymbol{S}_{lm} = \int_{p_{m-1}}^{p_m} \boldsymbol{p}\boldsymbol{\omega}_{lp}^T \, dp.$$

It follows that the (i, j)th entry of $\boldsymbol{S}_{lm}$ is given by

$$\begin{aligned} S_{lm}&(i, j) \\ &= \int_{p_{m-1}}^{p_m} p^{i-1} q_{l,j}(p) \, dp \\ &= \int_{p_{m-1}}^{p_m} p^{i-1} \cdot \left[\int_{\omega_{l-1}}^{\omega_l} \cos(D+p-j+1)\omega \, d\omega\right] dp. \end{aligned} \quad (17)$$

Using (7), (8), and (15), we obtain

$$J(\boldsymbol{A}) = \mathrm{tr}(\boldsymbol{P}\boldsymbol{A}^T\boldsymbol{\Omega}\boldsymbol{A}) - 2 \text{ tr}(\boldsymbol{S}\boldsymbol{A}) + \text{const.} \quad (18)$$

$$\begin{aligned} \boldsymbol{\omega}_{lp}^T &= \int_{\omega_{l-1}}^{\omega_l} [\cos(D+p)\omega \;\; \cos(D+p-1)\omega \;\; \cdots \;\; \cos(D+p-N)\omega] \, d\omega \\ &= [q_{l,1}(p) \;\; q_{l,2}(p) \;\; \cdots \;\; q_{l,N+1}(p)] \\ q_{l,j}(p) &= \omega_l \mathrm{sinc}[(D+p-j+1)\omega_l/\pi] - \omega_{l-1}\mathrm{sinc}[(D+p-j+1)\omega_{l-1}/\pi] \end{aligned} \quad (14a)$$

TABLE I
CONDITION NUMBER OF $\boldsymbol{P}$ VERSUS K

K	1	2	3	4	5
cond($\boldsymbol{P}$)	19.28	524.06	1.55e04	4.77e05	1.50e07
K	6	7	8	9	10
cond($\boldsymbol{P}$)	4.75e08	1.53e10	4.93e11	1.60e13	5.22e14

The gradient of $J(\boldsymbol{A})$ with respect to $\boldsymbol{A}$ is given by

$$\frac{\partial J(\boldsymbol{A})}{\partial \boldsymbol{A}} = 2\boldsymbol{\Omega}\boldsymbol{A}\boldsymbol{P} - 2\boldsymbol{S}^T \tag{19}$$

[see Appendix for a proof of (19)]. By setting $\partial J(\boldsymbol{A})/\partial \boldsymbol{A} = 0$, the optimal coefficient matrix $\boldsymbol{A}$ is obtained as

$$\boldsymbol{A} = \boldsymbol{\Omega}^{-1}\boldsymbol{S}^T\boldsymbol{P}^{-1}. \tag{20}$$

We now conclude this section with several remarks on the derivation. First, (17) can be expressed as

$$S_{lm}(i, j) = \int_{p_{m-1}}^{p_m} p^{i-1}\{\omega_l \text{sinc}[(D+p-j+1)\omega_l/\pi] - \omega_{l-1}\text{sinc}[(D+p-j+1)\omega_{l-1}/\pi]\}\, dp \tag{21}$$

which can be evaluated using fast and reliable numerical integration methods, such as adaptive Simpson's rule or adaptive Newton–Cotes rule [19]. Second, (20) indicates that the design is determined by three matrices, i.e., $\boldsymbol{\Omega}$, $\boldsymbol{S}$, and $\boldsymbol{P}$. It follows from (9)–(12) that $\boldsymbol{P}$ and $\boldsymbol{\Omega}$ are symmetric and positive definite and are entirely determined by N, K, and the weighting function $W(\omega, p)$, and matrix $\boldsymbol{S}$ is the only entity that depends on the desired $H_d(\omega, p)$. This suggests that the same $\boldsymbol{P}$ and $\boldsymbol{\Omega}$ can be used in the different variable filter design as long as the same N, K, and $W(\omega, p)$ are employed. Third, although $\boldsymbol{P}$ and $\boldsymbol{\Omega}$ are positive definite and, hence, their inverses do exist, computer simulations have indicated that these matrices (especially matrix $\boldsymbol{P}$) may become ill-conditioned even for moderate filter order N and polynomial degree K. Take matrix $\boldsymbol{P}$ in (9) as an example and assume $W_2(p) \equiv 1$. In this case, $\boldsymbol{P}$ is the $(K+1)\times(K+1)$ symmetric and positive-definite Hankel matrix given by

$$\boldsymbol{P} = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{K+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{K+2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{K+1} & \frac{1}{K+2} & \cdots & \frac{1}{2K+1} \end{bmatrix}.$$

As can be seen from Table I, the condition number of $\boldsymbol{P}$, denoted by cond($\boldsymbol{P}$), is fairly large, even for small values of K. This indicates that numerical difficulties may be encountered when (20) is used to compute the solution. We shall address this issue in the next section when presenting a design example using the proposed method.

TABLE II
COMPARISONS OF THE PROPOSED METHOD WITH THE METHOD OF [17]

	Method of [17]	Proposed Method
e_{max} (dB)	-100.0088	-100.7215
e_2	1.9375×10^{-4}	1.7975×10^{-4}
Flops ($\times 10^6$)	55.66	1.70
CPU time (Sec.)	22.69	15.91

IV. AN ILLUSTRATIVE EXAMPLE

In this section, we illustrate the proposed algorithm by applying it to design a variable fractional delay filter with the same specifications as adopted in [17]: $N = 67$, $K = 7$, and the cutoff frequency $\omega_c = 0.9\pi$. The design of variable fractional delay filters were considered in [14] using the Lagrange interpolation method and other methods [15]. It appears that the best result achieved so far was that reported in [17]: by using a set carefully selected weights, the design obtained in [17] was able to keep the frequency-domain error

$$e(\omega, p) = 20\log_{10}|H(\omega, p) - H_d(\omega, p)| \tag{22}$$

below -100 dB in the entire region $0 \le \omega \le 0.9\pi$ and $0 \le p \le 1$.

The comparisons of the proposed method with that of [17] were made in terms of the maximum error defined by

$$e_{\max} = \max\{e(\omega, p),\ 0 \le \omega \le 0.9\pi,\ 0 \le p \le 1\}$$

with the L_2-error defined by

$$e_2 = \left[\int_0^{0.9\pi}\int_0^1 |H(\omega, p) - H_d(\omega, p)|^2\, dp\, d\omega\right]^{1/2}$$

the number of floating point operations (flops), and the central processing unit (CPU) time used.

With $N = 67$, $K = 7$, $W_2(p) \equiv 1$ for $p \in [0, 1]$, and

$$W_1(\omega) = \begin{cases} 1, & \text{for } \omega \in [0, \omega_1) \\ 3, & \text{for } \omega \in [\omega_1, \omega_2) \\ 0, & \text{for } \omega \in [\omega_2, \pi] \end{cases}$$

where $\omega_1 = 0.88\pi$ and $\omega_2 = 0.8994\pi$, both the method [17] and the proposed method were implemented using MATLAB 5 on a SUN Ultrasparc I. The results are summarized in Table II. Fig. 1 shows the error function $e(\omega, p)$ defined in (22). The group delays of the filter in frequency range $[0, 0.9\pi]$ for different p are depicted in Fig. 2, and the error in group delay for $0 \le \omega \le 0.9\pi$ and $0 \le p \le 1$ is shown in Fig. 3.

It is observed that the proposed design leads to reduced $e_{\max}$ and e_2, and needs only a small fraction of computations required by the method of [17]. This is mainly attributed to the closed-form evaluation of the matrices that are involved in (20). The integrals in (21) were evaluated numerically using MATLAB command quad8, which makes use of an adaptive Newton–Cotes 8-panel rule [19]. As mentioned in the preceding section, the condition numbers of matrices $\boldsymbol{\Omega}$ and $\boldsymbol{P}$ are fairly large ($\sim 10^8$ for $\boldsymbol{\Omega}$ and $\sim 10^{10}$ for $\boldsymbol{P}$ in the present design). An effective remedy for dealing with ill-conditioned

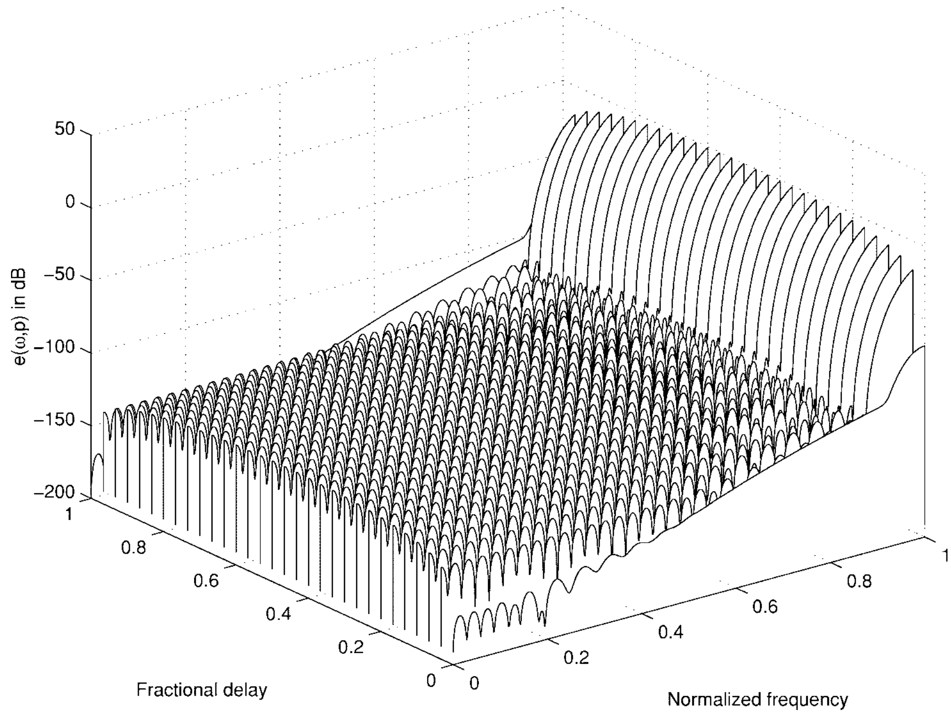

Fig. 1. Error function $e(\omega,\, p)$.

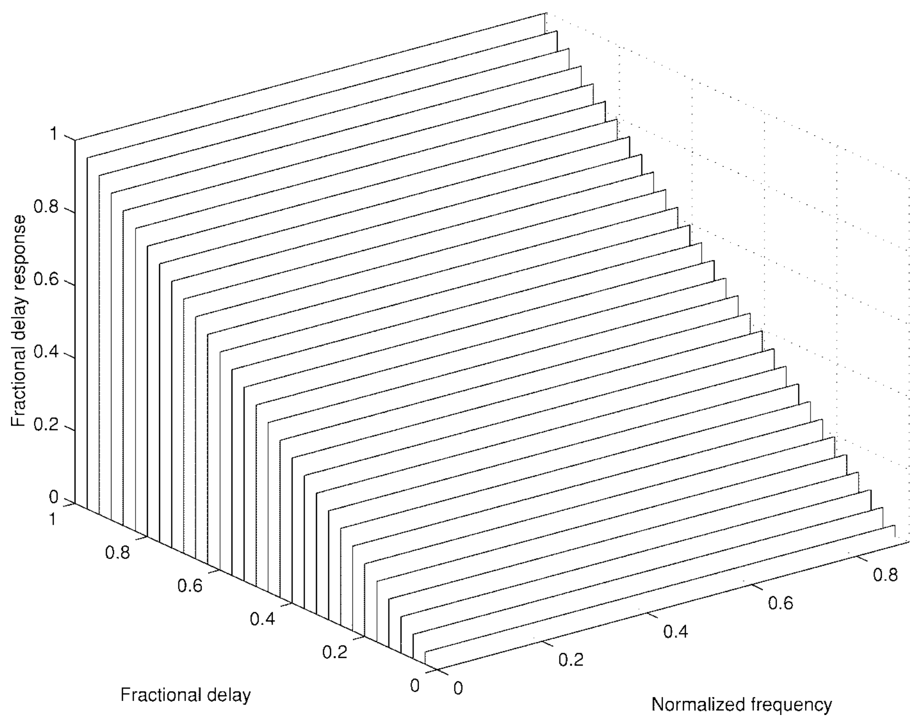

Fig. 2. Fractional delay response of $H(\omega,\, p)$.

matrices is to perform the Cholesky decomposition [20] of matrices $\boldsymbol{\Omega}$ and $\boldsymbol{P}$, i.e.,

$$\boldsymbol{P} = \boldsymbol{P}_1^T \boldsymbol{P}_1; \qquad \boldsymbol{\Omega} = \boldsymbol{\Omega}_1^T \boldsymbol{\Omega}_1 \tag{23}$$

where $\boldsymbol{P}_1$ and $\boldsymbol{\Omega}_1$ are upper triangular matrices, and then multiplying the matrices involved in (20) in a right order

$$\boldsymbol{A} = \boldsymbol{\Omega}_1^{-1}\left[\boldsymbol{\Omega}_1^{-T}(\boldsymbol{S}^T \boldsymbol{P}_1^{-1})\boldsymbol{P}_1^{-T}\right] \tag{24}$$

where grouping $\boldsymbol{S}^T$ with $\boldsymbol{P}_1^{-1}$ turns out critical in obtaining a numerically stable solution. The role of the Cholesky decomposition in (23) is to obtain matrices $\boldsymbol{P}_1$ and $\boldsymbol{\Omega}_1$, whose condition numbers are significantly reduced to $\sim 10^5$ and $\sim 10^4$, respectively. This, in conjunction with the fact that most entries in $\boldsymbol{S}$ are small in magnitude ($\sim 10^{-2}$), suggests that the multiplication of $\boldsymbol{S}^T$ with $\boldsymbol{P}_1^{-1}$ would largely "cancel out" the large-magnitude entries in $\boldsymbol{P}_1^{-1}$, which in turn

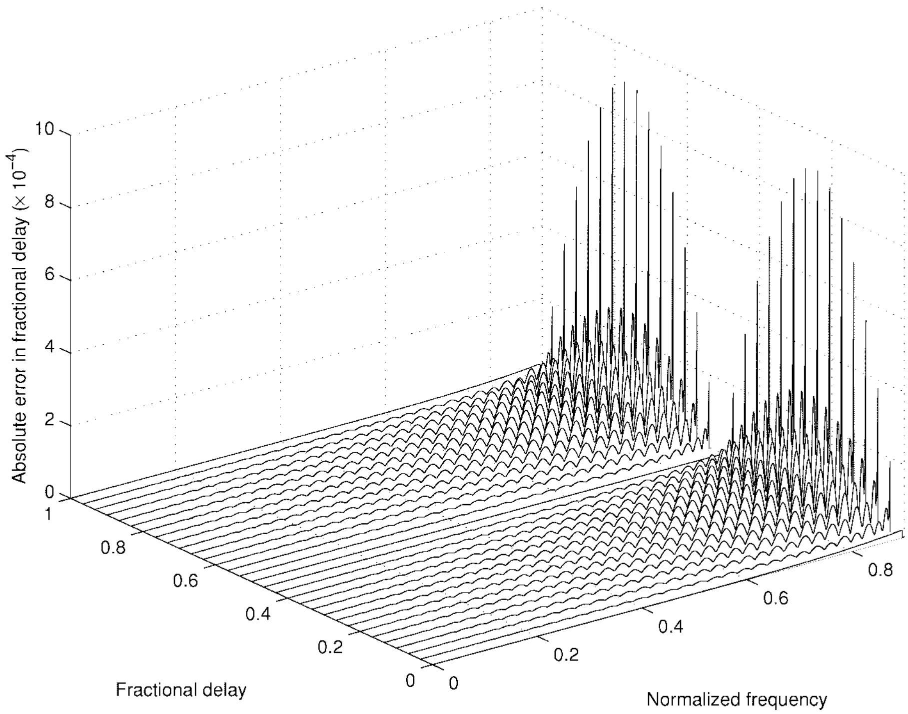

Fig. 3. Absolute error in passband fractional delay.

considerably eases off the numerical instability. Alternatively, other matrix decompositions, such as the orthogonal-upper-triangular (known as QR) decomposition and the singular value decomposition, have also been tried with similar design results. But in the present case, the Cholesky decomposition offers a slightly reduced computational complexity compared to other matrix decompositions.

We now conclude this section with some remarks on the choice of the weighting functions. First, the separability of $W(\omega, p)$ as assumed in (6) is primarily for the sake of computational feasibility: a nonseparable $W(\omega, p)$ would lead to a far more complex solution procedure, in which the nice structure as seen in (20) would not exist. In addition, it seems quite hard to explicitly specify a nonseparable $W(\omega, p)$ in the design so as to obtain a considerably better design compared to the one which utilizes a separable $W(\omega, p)$. Second, if one agrees to employ a separable $W(\omega, p)$ as in (6), then which types of $W_1(\omega)$ and $W_2(p)$ should one use? Our numerical experiences indicated that the use of piecewise constant $W_1(\omega)$ and a constant $W_2(p)$ is a reasonable point to start. As one may notice from the above example, the function $W_1(\omega)$ used there has zero value for $[\omega_2, \pi]$ as this is the do-not-care frequency region, and assumes value 1 for most part of the frequency region of interest. There is only a small interval, $[\omega_1, \omega_2)$, in which $W_1(\omega)$ assumes a larger value in order to handle the frequency boundary at $\omega = 0.9\pi$. We had also tried a number of more sophisticated piecewise constant weights, which inevitably led to more computations, yet with little performance improvement. As for $W_2(p)$, since no "do-not-care" region is specified, all values of p between zero and one are considered equally important. This leads to a $W_2(p) \equiv 1$. As can be observed from Fig. 1, the frequency response appears to be pretty flat across the entire region $0 \leq p \leq 1$. Of course, one is always in a position to make a further improvement if $W_2(p)$ is modified from constant to piecewise constant based on the current design result, but the minor improvement is obtained at the cost of increased computational complexity. It is therefore a tradeoff the designer has to make to generate a satisfactory design with a minimum amount of computations.

V. Conclusion

An algorithm for the weighted-least-squares design of variable fractional delay FIR filters has been proposed. The design is accomplished by developing a closed-form formula that can be used to evaluate the WLS error function accurately and quickly, which leads to improved filter performance with reduced computational complexity.

Appendix

Proposition: Let $\boldsymbol{P} \in R^{M\times M}$, $\boldsymbol{A} \in R^{N\times M}$, $\boldsymbol{\Omega} \in R^{N\times N}$, and $\boldsymbol{S} \in R^{M\times N}$ with $\boldsymbol{P}$ and $\boldsymbol{\Omega}$ symmetric, then

$$\frac{\partial}{\partial \boldsymbol{A}}\left[\mathrm{tr}\left(\boldsymbol{P}\boldsymbol{A}^T\boldsymbol{\Omega}\boldsymbol{A}\right) - 2\ \mathrm{tr}(\boldsymbol{S}\boldsymbol{A})\right] = 2\boldsymbol{\Omega}\boldsymbol{A}\boldsymbol{P} - 2\boldsymbol{S}^T. \tag{25}$$

Proof: Denote $\boldsymbol{A} = [a_{ij}]$ and let δ be a small perturbation of a_{ij}. Denote by $\tilde{\mathbf{A}}$ the matrix $\boldsymbol{A}$ with its entry a_{ij} perturbed by δ. We can then write

$$\tilde{\mathbf{A}} = \boldsymbol{A} + \delta \boldsymbol{e}_i \boldsymbol{e}_j^T$$

where $\boldsymbol{e}_i$ is the ith coordinate vector, and

$$\begin{aligned}
&\mathrm{tr}(\boldsymbol{P}\tilde{\mathbf{A}}^T\boldsymbol{\Omega}\tilde{\mathbf{A}}) - \mathrm{tr}(\boldsymbol{P}\boldsymbol{A}^T\boldsymbol{\Omega}\boldsymbol{A})\\
&\quad = \delta\ \mathrm{tr}(\boldsymbol{P}\boldsymbol{e}_j\boldsymbol{e}_i^T\boldsymbol{\Omega}\boldsymbol{A} + \boldsymbol{P}\boldsymbol{A}^T\boldsymbol{\Omega}\boldsymbol{e}_i\boldsymbol{e}_j^T) + O(\delta^2)\\
&\quad = \delta\boldsymbol{e}_i^T\boldsymbol{\Omega}\boldsymbol{A}\boldsymbol{P}\boldsymbol{e}_j + \delta\boldsymbol{e}_j^T\boldsymbol{P}\boldsymbol{A}^T\boldsymbol{\Omega}\boldsymbol{e}_i + O(\delta^2)\\
&\quad = 2\delta\boldsymbol{e}_i^T\boldsymbol{\Omega}\boldsymbol{A}\boldsymbol{P}\boldsymbol{e}_j + O(\delta^2).
\end{aligned}$$

Hence

$$\lim_{\delta \to 0} \frac{\operatorname{tr}(\boldsymbol{P}\tilde{\mathbf{A}}^T \boldsymbol{\Omega} \tilde{\mathbf{A}}) - \operatorname{tr}(\boldsymbol{P}\boldsymbol{A}^T \boldsymbol{\Omega} \boldsymbol{A})}{\delta} = 2\boldsymbol{e}_i^T \boldsymbol{\Omega} \boldsymbol{A} \boldsymbol{P} \boldsymbol{e}_j$$

i.e.,

$$\frac{\partial \operatorname{tr}(\boldsymbol{P}\boldsymbol{A}^T \boldsymbol{\Omega} \boldsymbol{A})}{\partial a_{ij}} = \boldsymbol{e}_i^T (2\boldsymbol{\Omega} \boldsymbol{A} \boldsymbol{P}) \boldsymbol{e}_j$$

which implies that

$$\frac{\partial}{\partial \boldsymbol{A}} \operatorname{tr}(\boldsymbol{P}\boldsymbol{A}^T \boldsymbol{\Omega} \boldsymbol{A}) = 2\boldsymbol{\Omega} \boldsymbol{A} \boldsymbol{P}. \tag{26}$$

Similarly, one can show that

$$\frac{\partial}{\partial \boldsymbol{A}} \operatorname{tr}(\boldsymbol{S}\boldsymbol{A}) = \boldsymbol{S}^T. \tag{27}$$

See also [21, ch. 6] for (27). Together, (26) and (27) lead to (25). □

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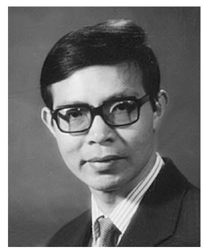

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