Improved Iterative Methods for the Design of Quadrature Mirror-Image Filter Banks

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***Abstract*— An iterative method for the design of two-band quadrature mirror-image filter (QMF) banks proposed by Chen and Lee is examined and improved by deriving simple, explicit, and precise expressions for the evaluation of two integrals involved in the computation of the objective function. The method is then extended to the design of QMF banks with low reconstruction delays and to the design of filter banks with equiripple complex reconstruction error. The paper concludes with design examples which illustrate the efficiency and performance of the algorithms developed and the quality of the filter banks that can be designed.**

I. Introduction

THE IMPORTANCE of quadrature mirror-image filter (QMF) banks in subband coding has been widely recognized and various analysis, design, and implementation issues pertaining to these filters have been intensively studied since the mid-1970's [1]–[18]. The quadrature mirror structure of QMF banks leads to the complete cancellation of interband aliasing due to the overlapping filter responses. If the filters involved have symmetrical impulse responses of finite duration (FIR), no phase distortion will occur and, therefore, the design can be focused on selecting filter parameters so as to minimize the system's amplitude distortion.

Like the design of conventional digital filters, the design of QMF banks can be accomplished by using least-squares and minimax methods. In [5], the reasoning of using the least-squares criterion for telecommunication applications has been clarified. On the other hand, in [15], [17], and [18] it has been shown that minimax design can be accomplished if an adequately updated weighting function is included in a least-squares objective function. In either the nonweighted or weighted case, the least-squares objective function is a fourth-order function of the design parameters. Hence the design is a typical unconstrained, *nonquadratic* optimization problem. In [17], Chen and Lee have proposed an iterative design method in which the least-squares objective function is modified by assigning a set of appropriate values, which form a vector $\mathbf{h}$, to a part of the design parameters in such a way as to obtain a *quadratic* and globally convex objective function. Upon obtaining the global minimum point, say $\mathbf{f}$, the value of $\mathbf{h}$ is updated accordingly and is then re-assigned to the same part of the design parameters. This procedure is repeated until $\mathbf{f}$ and $\mathbf{h}$ become identical. Since in each iteration $\mathbf{f}$ can be formulated in closed form and only a few iterations are needed for convergence when a good initial $\mathbf{h}$ is used, the algorithm is efficient and good performance is achieved in the filter bank. In the method described, the objective function depends on two integrals which are evaluated by discretization. This gives rise to two problems. First, the solution obtained actually minimizes the discretized version of the objective function rather than the objective function itself. This can degrade the performance of the QMF bank designed. Second, in order to reduce the performance degradation the density of sample points needs to be high, which leads to increased computational complexity. In this paper, we describe an improved version of the above algorithm in which the integral discretization is avoided by deriving a simple and explicit formula for the precise evaluation of the two integrals. As a result, filter banks with better performance can be designed with considerably reduced computational complexity.

For real-time or quasireal-time applications, filter banks with low reconstruction delay are highly desired. Unfortunately, in conventional QMF banks the filter length N is required to be fairly high in order to achieve satisfactory performance. Consequently, the reconstruction delay, which is $(N-1)$ sampling periods, can become too high. A time-domain approach to the design of analysis/synthesis systems with low reconstruction delays was proposed in [16]. In the present paper, the iterative algorithm in [17] is extended to the design of low-delay QMF banks. As above, simple and explicit expressions are derived for the precise evaluation of the objective function, which increase the computational efficiency and improve the quality of the filter banks designed.

The method is also extended to the case where a reconstruction error is defined in terms of the difference between the frequency response of the filter bank and that of an ideal allpass channel, and the magnitude of the complex reconstruction error so obtained is forced to become equiripple. These designs will be referred to as QMF banks with equiripple complex reconstruction error.

The paper concludes with design examples which illustrate the algorithms developed and demonstrate their efficiency and performance, and the quality of the filter banks that can be designed.

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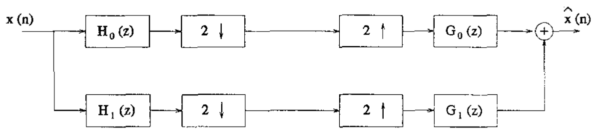


Fig. 1. A two-band filter bank.

II. Design of Two-Channel QMF Banks

A. General Objective Function

Consider the two-band filter bank depicted in Fig. 1. The input-output relation of the filter bank and the basic constraints for perfect reconstruction of input signals are well known [3], [13]. By assuming that $H_1(z) = H_0(-z)$, $G_0(z) = H_0(z)$, $G_1(z) = -H_0(-z)$ in order to cancel the aliasing, a general objective function for the design of several types of QMF banks can be formulated as

$$E = E_1 + \alpha E_2 \tag{1a}$$

where

$$E_1 = \int_0^{\pi} W(\omega)\left|H_0^2(e^{j\omega}) - H_0^2(e^{j(\omega+\pi)}) - e^{-jk_d\omega}\right|^2 d\omega \tag{1b}$$

and

$$E_2 = \int_{\omega_s}^{\pi} |H_0(e^{j\omega})|^2 d\omega \tag{1c}$$

The parameter α in (1a) is a positive weight that can be used to control stopband attenuation for filter H_0 and ω_s is the stopband edge; k_d is the desired reconstruction delay and is assumed to be an odd integer. The objective function in (1a), which is essentially a generalization of an objective function due to Johnston [3], can be used to design a variety of filter banks. For example, if $k_d = N-1$ and $W(\omega) = 1$, QMF banks with linear phase response can be designed; if $k_d < N-1$ and $W(\omega) = 1$, banks with low reconstruction delays can be obtained; and if $k_d \le N-1$ and $W(\omega)$ is a nontrivial weighting function, equiripple complex reconstruction error can be achieved.

B. Design of Conventional QMF Banks

Let us first consider designing a conventional QMF bank with linear phase response, i.e., $k_d = N-1$ and $W(\omega) = 1$. The basic idea in the iterative algorithm of [17] is that instead of directly minimizing the objective function in (1a), which is a fourth-order function of parameter vector $\mathbf{h}$, one minimizes the modified objective function

$$E' = E_1' + \alpha E_2' \tag{2a}$$

where

$$E_1' = \int_0^{\pi} [T'(\omega) - 1]^2 d\omega \tag{2b}$$

$$E_2' = \int_{\omega_s}^{\pi} M_f^2(\omega) d\omega \tag{2c}$$

$$T'(\omega) = M_h(\omega)M_f(\omega) + M_h(\omega+\pi)M_f(\omega+\pi) \tag{2d}$$

$$M_h(\omega) = 2\mathbf{h}^T\mathbf{c}(\omega) \tag{2e}$$

$$M_f(\omega) = 2\mathbf{f}^T\mathbf{c}(\omega) \tag{2f}$$

$$\mathbf{h} = [h_0\ h_1\ \cdots\ h_{N/2-1}]^T \tag{2g}$$

$$\mathbf{f} = [f_0\ f_1\ \cdots\ f_{N/2-1}]^T \tag{2h}$$

$$\mathbf{c}(\omega) = [\cos(N-1)\omega/2\ \cdots\ \cos\omega/2]^T \tag{2i}$$

under the assumption that coefficient vector $\mathbf{h}$ is *fixed* during the minimization. Obviously, as a function of $\mathbf{f}$, the modified objective function E' is *quadratic* and globally convex and, therefore, its minimum can be easily obtained. Having obtained the minimum point of E', say $\mathbf{f}$, vector $\mathbf{h}$ is updated using a linear combination of $\mathbf{f}$ and $\mathbf{h}$ as

$$\mathbf{h} := (1-\tau)\mathbf{h} + \tau\mathbf{f} \tag{3}$$

where τ is a smoothing parameter between 0 and 1. With an appropriate choice of τ, which is found to be in a vicinity of 0.5, $\mathbf{f}$ quickly converges to $\mathbf{h}$ and an optimal design parameter vector results. In the approach in [17], discretization is used to evaluate E_1' and E_2'. An alternative approach is to derive a closed-form and explicit formulation of E' as a quadratic function of $\mathbf{f}$ without discretizing E_1' and E_2'. As will be shown, this formulation in conjunction with the iterative procedure described above leads to improved computation efficiency and performance in the designed filter bank.

From (2b) and (2d), E_1' can be written as

$$E_1' = 4\mathbf{f}^T\mathbf{U}\mathbf{f} - 8\pi\mathbf{h}^T\mathbf{f} + \pi \tag{4}$$

where $\mathbf{U} = \{u_{ij}\}$ with u_{ij} given by

$$u_{ij} = \pi \sum_{n=0}^{N/2-1}\sum_{m=0}^{N/2-1} h_n h_m \left\{\sum_{l=1}^{8}\left[1+(-1)^{k_l'}\right]\delta(k_l)\right\}, \quad 1 \le i,j \le N/2 \tag{5}$$

where

$$\delta(k_l) = \begin{cases} 1, & k_l = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$k_1 = \beta+\gamma+\xi+\eta, \quad k_2 = \beta+\gamma+\xi-\eta$$
$$k_3 = \beta+\gamma-\xi+\eta, \quad k_4 = \beta+\gamma-\xi-\eta$$
$$k_5 = \beta-\gamma+\xi+\eta, \quad k_6 = \beta-\gamma+\xi-\eta$$
$$k_7 = \beta-\gamma-\xi+\eta, \quad k_8 = \beta-\gamma-\xi-\eta$$
$$\beta = n - \frac{N-1}{2}, \quad \gamma = m - \frac{N-1}{2}$$
$$\xi = \frac{N+1}{2} - i, \quad \eta = \frac{N+1}{2} - j$$
$$k_1' = m-j+1, \quad k_2' = m+j-N$$
$$k_3' = k_1', \quad k_4' = k_2'$$
$$k_5' = -k_2', \quad k_6' = -k_1'$$
$$k_7' = -k_2', \quad k_8' = -k_1'$$

Now from (2c),

$$E_2' = 4\mathbf{f}^T\mathbf{U}_s\mathbf{f} \tag{6}$$

where the (i,j)th entry of $\mathbf{U}_s$ is given by (7) shown at the bottom of the next page. Equation (7) was also used in [7] in a time-domain formulation of the design problem. Note that matrix $\mathbf{U}_s$ is *independent* of $\mathbf{h}$ and can be pre-calculated as long as the filter length N and stopband edge ω_s are specified.

The objective function E' can now be expressed as

$$E' = 4\mathbf{f}^T\mathbf{Q}\mathbf{f} - 8\pi\mathbf{h}^T\mathbf{f} + \pi \tag{8}$$

where

$$\mathbf{Q} = \mathbf{U} + \alpha\mathbf{U}_s \tag{9}$$

From (5) and (6), it follows that matrix $\mathbf{Q}$ is positive definite and, therefore, with a fixed $\mathbf{h}$ the global minimum of E' is given by

$$\mathbf{f} = \pi\mathbf{Q}^{-1}\mathbf{h} \tag{10}$$

On the basis of the preceding analysis, an iterative algorithm can now be constructed as follows:

Algorithm 1

Step 1 Use a conventional method (e.g., the window method) to design a linear-phase, low-pass FIR filter of length N with stopband edge ω_s, and use the coefficient vector of the filter obtained as the initial $\mathbf{h}$.

Step 2 Use (5) to compute matrix $\mathbf{U}$.

Step 3 Form matrix $\mathbf{Q}$ using (9) and compute $\mathbf{f}$ in (10).

Step 4 If $||\mathbf{h} - \mathbf{f}|| < \epsilon$, where ϵ is a prescribed tolerance, output $\mathbf{f}$ as the design result and stop. Otherwise, update $\mathbf{h}$ using (3) with a τ close to 0.5 and repeat from Step 2.

A remark on the evaluation of matrix $\mathbf{U}$ is appropriate at this point. Since $\mathbf{U}$ is symmetric, only $N(N+2)/8$ entries of $\mathbf{U}$ need to be calculated. For each entry of $\mathbf{U}$, $\sum_{l=1}^{8}[1+(-1)^{k_l'}]\delta(k_l)$ is a simple combination of eight unit-impulse functions, each of which is very sparse and easy to determine. Consequently, designs using Algorithm 1 can be accomplished very quickly. Experiments have shown that Algorithm 1 can reduce the amount of computation by up to 75% over that in [17].

C. Design of QMF Banks with Low Reconstruction Delays

In this section, the iterative algorithm described in the Section II-B will be extended to the design of QMF banks with low reconstruction delays. It is well known that the reconstruction delay of a QMF bank with linear phase response is $(N-1)$ sampling periods. If the filter order $N-1$ is high but a relatively low reconstruction delay $k_d < N-1$ is required, the design problem amounts to finding the filter parameters in $H_0(e^{j\omega})$ that minimize (1a) with $W(\omega) = 1$ [19]. Note that in this case filter H_0 does not have a symmetrical impulse response in general and, therefore, the frequency response should be expressed as

$$H_0(e^{j\omega}) = \mathbf{h}_L^T\mathbf{c}_L(\omega) \tag{11a}$$

where

$$\mathbf{h}_L = [h_0\ h_1\ \cdots\ h_{N-1}]^T \tag{11b}$$

and

$$\mathbf{c}_L(\omega) = \left[1\ e^{-j\omega}\ \cdots\ e^{-j(N-1)\omega}\right]^T \tag{11c}$$

As in the normal case treated in Section II-B, the iterative algorithm is based on a modified objective function

$$E_L' = E_{L1}' + \alpha E_{L2}' \tag{12a}$$

where

$$E_{L1}' = \int_0^\pi \left|H_0(e^{j\omega})F_0(e^{j\omega}) - H_0\left(e^{j(\omega+\pi)}\right) \times F_0\left(e^{j(\omega+\pi)}\right) - e^{-jk_d\omega}\right|^2 d\omega \tag{12b}$$

$$E_{L2}' = \int_{\omega_s}^\pi |F_0(e^{j\omega})|^2 d\omega \tag{12c}$$

and $\mathbf{h}_L$ is fixed; $F_0(e^{j\omega})$ is given by

$$F_0(e^{j\omega}) = \mathbf{f}_L^T\mathbf{c}_L(\omega) \tag{12d}$$

where

$$\mathbf{f}_L = [f_0\ f_1\ \cdots\ f_{N-1}]^T \tag{12e}$$

After some fairly extensive manipulation, (11) and (12) give

$$E_L' = \mathbf{f}_L^T\mathbf{Q}_L\mathbf{f}_L - 2\mathbf{h}_L^T\mathbf{S}\mathbf{f}_L + \pi \tag{13}$$

where

$$\mathbf{Q}_L = \mathbf{R} + \alpha\mathbf{R}_s \tag{14a}$$

with entry (i,j) of $\mathbf{R}$ given by

$$r_{ij} = 2\pi\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} h_n h_m[1+(-1)^{j+m}]\delta(-i+j-n+m) \tag{14b}$$

where $\mathbf{R}_s$ is given by (14c) at the bottom of the next page and

$$\mathbf{S} = 2\pi\begin{bmatrix} 0 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ 0 & & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & & & \vdots \\ 1 & \cdot^{\cdot^{\cdot}} & & & & & \\ 0 & & & & & & \\ \vdots & & & & & & \vdots \\ 0 & \cdots & & & & \cdots & 0 \end{bmatrix} \tag{14d}$$

Evidently, $\mathbf{S}$ has unity entries on the (k_d+1)th northeast-to-southwest subdiagonal.

Since $\mathbf{Q}_L$ is a positive-definite symmetric matrix, for a given $\mathbf{h}_L$ the global minimum of E_L' is achieved if

$$\mathbf{f}_L = \mathbf{Q}_L^{-1}\mathbf{S}\mathbf{h}_L \tag{15}$$

$$u_{ij}^{(s)} = \begin{cases} (\pi - \omega_s)/2 - \sin[(2i-N-1)\omega_s]/(4i-2N-2), & i = j \\ \sin[(i-j)\omega_s]/(2j-2i) - \sin[(i+j-N-1)\omega_s]/(2i+2j-2N-2), & i \neq j \end{cases} \tag{7}$$

Note that matrices $\mathbf{R}_s$ and $\mathbf{S}$ are *independent* of $\mathbf{h}_L$ and can be pre-calculated as long as filter length N, stopband edge ω_s, and reconstruction delay k_d are specified.

An algorithm for the design of a QMF bank with low reconstruction delay is as follows:

Algorithm 2

Step 1 Use a least-squares approach to design a low-pass FIR filter of length N with stopband edge ω_s and group delay $k_d/2 < (N-1)/2$, and use the coefficient vector of the filter obtained to initialize $\mathbf{h}_L$.

Step 2 Use (14b) to compute matrix $\mathbf{R}$.

Step 3 Use (14a), (14c), and $\mathbf{R}$ obtained from Step 2 to form matrix $\mathbf{Q}_L$; then use (15) to compute $\mathbf{f}_L$.

Step 4 If $||\mathbf{h}_L - \mathbf{f}_L|| < \epsilon$, where ϵ is a prescribed tolerance, output $\mathbf{f}_L$ as the design result and stop. Otherwise, update $\mathbf{h}_L$ as

$$\mathbf{h}_L := (1-\tau)\mathbf{h}_L + \tau\mathbf{f}_L \tag{16}$$

with a τ close to 0.5, and repeat from Step 2.

Since $\mathbf{R}$ is symmetric and each entry is a linear combination of unit-impulse functions, which are very sparse and easy to determine, the algorithm can be easily implemented as will be seen in Section III.

We have observed from experiments that undesirable artifacts can occur in the transition region of filter H_0 when a low reconstruction delay k_d is required, which was also noticed in [16]. An effective approach for the elimination of this problem is to modify the objective function in (12a) by including an additional term $\alpha_1 E'_{L3}$, i.e.,

$$E'_L = E'_{L1} + \alpha E'_{L2} + \alpha_1 E'_{L3} \tag{17}$$

where

$$E'_{L3} = \int_{\omega_{t1}}^{\omega_{t2}} |F_0(e^{j\omega}) - e^{-j\omega k_d/2}|^2 d\omega \tag{18}$$

and $[\omega_{t1}, \omega_{t2}]$ is an interval in the transition region where the artifacts occur. It can be readily shown that with this modification the design algorithm, i.e., Algorithm 2, can still be used with the modifications in (13)–(15) as follows:

$$E'_L = \mathbf{f}_L \mathbf{Q}_L \mathbf{f}_L - 2(\mathbf{h}_L^T \mathbf{S} + \alpha_1 \mathbf{b}^T)\mathbf{f}_L + \pi + \alpha_1(\omega_{t2} - \omega_{t1}) \tag{19}$$

where

$$\mathbf{Q}_L = \mathbf{R} + \alpha \mathbf{R}_s + \alpha_1 \mathbf{R}_t. \tag{20}$$

$\mathbf{R}$ and $\mathbf{R}_s$ are given by (14b), (14c) and $\mathbf{R}_s$ is given by (20a) at the bottom of the next page where

$$\phi(\omega_{t2}, \omega_{t1}, k) = \frac{1}{k}[\sin k\omega_{t2} - \sin k\omega_{t1}]$$

and $\mathbf{b}$ is given by (20b) at the bottom of the next page. The global minimum of E'_L is achieved if

$$\mathbf{f}_L = \mathbf{Q}_L^{-1}(\mathbf{S}\mathbf{h}_L + \alpha_1 \mathbf{b}) \tag{21}$$

We conclude this section with a qualitative comparison of the class of filter banks that can be designed with Algorithm 2 with the class of FIR power symmetric filter banks [21], [22]. A common feature of these two classes of filter banks is that their impulse responses are not symmetrical and, as a consequence, their phase responses are not perfectly linear. In most other respects, they are quite different. In an FIR power symmetric filter bank, the signal reconstruction is always perfect whereas in a low-delay QMF bank proposed it is approximate. In an FIR power symmetric bank, the reconstruction delay should be at least $N-1$ while in a low-delay QMF bank it can be less than $(N-1)/2$. Finally, a low-delay QMF bank, unlike a corresponding FIR power symmetric bank, is amenable to a polyphase implementation which is both highly efficient and economical.

D. Design of QMF Banks with Equiripple Complex Reconstruction Error

If a QMF bank with equiripple complex reconstruction error is desired, the objective function E'_L in (12a) should be modified to include a nontrivial weighting function $W(\omega)$ in the first integral, i.e.,

$$\begin{aligned} E_L'^{(w)} &= E_{L1}'^{(w)} + \alpha E'_{L2} \\ &= \int_0^\pi W(\omega) \big| H_0(e^{j\omega})F_0(e^{j\omega}) - H_0\big(e^{j(\omega+\pi)}\big) \\ &\qquad \times F_0\big(e^{j(\omega+\pi)}\big) - e^{-jk_d\omega}\big|^2 d\omega \\ &\quad + \alpha \int_{\omega_s}^\pi |F_0(e^{j\omega})|^2 d\omega \end{aligned} \tag{22}$$

where $W(\omega)$ is adjusted during each iteration so as to assure a minimax design [17], [18]. Note that term $E_{L1}'^{(w)}$ can be written as

$$E_{L1}'^{(w)} = \mathbf{f}_L^T \mathbf{R}^{(w)} \mathbf{f}_L - 2\mathbf{h}_L^T \mathbf{S}^{(w)} \mathbf{f}_L + K \tag{23}$$

where K is a constant

$$\mathbf{R}^{(w)} = \mathrm{Re} \int_0^\pi W(\omega)\mathbf{u}(\omega)\mathbf{u}^H(\omega) d\omega$$

$$\mathbf{u}(\omega) = \big[\mathbf{c}_L(\omega)\mathbf{c}_L^T(\omega) - \mathbf{c}_L(\omega+\pi)\mathbf{c}_L^T(\omega+\pi)\big]\mathbf{h}_L$$

and

$$\begin{aligned} \mathbf{S}^{(w)} = \mathrm{Re} \int_0^\pi W(\omega)\big\{\big[\mathbf{c}_L(\omega)\mathbf{c}_L^T(\omega) - \mathbf{c}_L(\omega+\pi) \\ \times \mathbf{c}_L^T(\omega+\pi)\big] e^{jk_d\omega}\big\} d\omega \end{aligned}$$

$$\mathbf{R}_s = \begin{bmatrix} \pi - \omega_s & -\sin\omega_s & -\frac{1}{2}\sin 2\omega_s & \cdots & -\frac{1}{N-1}\sin(N-1)\omega_s \\ -\sin\omega_s & \pi - \omega_s & -\sin\omega_s & \cdots & -\frac{1}{N-2}\sin(N-2)\omega_s \\ \vdots & & \ddots & & \vdots \\ -\frac{1}{N-1}\sin(N-1)\omega_s & \cdots & \cdots & \cdots & \pi - \omega_s \end{bmatrix} \tag{14c}$$

It follows that both $\mathbf{R}^{(w)}$ and $\mathbf{S}^{(w)}$ are symmetric, and that entries (k,l) of $\mathbf{R}^{(w)}$ and $\mathbf{S}^{(w)}$ are given by

$$r_{kl}^{(w)} = \mathbf{h}_L^T \mathbf{B}_{kl}^{(w)} \mathbf{h}_L \tag{24a}$$

and

$$s_{kl}^{(w)} = \int_0^\pi W(\omega)\left[1-(-1)^{(k+l-2)}\right] \times [\cos(k+l-2-k_d)\omega]d\omega \tag{24b}$$

respectively, where $\mathbf{B}_{kl}^{(w)}$ is a matrix, which is *independent* of $\mathbf{h}_L$, given by

$$\mathbf{B}_{kl}^{(w)} = \mathrm{Re}\int_0^\pi W(\omega)\left[e^{-j\omega(k-1)}\mathbf{c}_L(\omega) - e^{-j(\omega+\pi)(k-1)}\mathbf{c}_L(\omega+\pi)\right] \cdot \left[e^{j\omega(l-1)}\mathbf{c}_L^H(\omega) - e^{j(\omega+\pi)(l-1)}\mathbf{c}_L^H(\omega+\pi)\right]d\omega$$

From (22), (23), and (14c), we obtain

$$E_L'^{(w)} = \mathbf{f}_L^T \mathbf{Q}_L^{(w)} \mathbf{f}_L - 2\mathbf{h}_L^T \mathbf{S}^{(w)} \mathbf{f}_L + K \tag{25}$$

where

$$\mathbf{Q}_L^{(w)} = \mathbf{R}^{(w)} + \alpha \mathbf{R}_s \tag{26}$$

is positive definite. Therefore, for a fixed $\mathbf{h}_L$ the global minimum of $E_L'^{(w)}$ is achieved if

$$\mathbf{f}_L = \mathbf{Q}_L^{(w)-1}\mathbf{S}^{(w)}\mathbf{h}_L \tag{27}$$

With the weighting function $W(\omega)$ included, closed-form formulas for $s_{kl}^{(w)}$ and $\mathbf{B}_{kl}^{(w)}$ do not exist in general. However, accurate values of the integrals that are involved in evaluating $s_{kl}^{(w)}$ and $\mathbf{B}_{kl}^{(w)}$ can be obtained by numerical integration using an adaptive recursive version of Simpson's rule or Newton-Cotes rule. These reliable numerical techniques are available in several software packages including MATLAB (version 3.5 or higher).

The steps that need to be taken to accomplish a minimax design are summarized as follows:

Algorithm 3

Step 1 Design a low-pass FIR filter of length N with stopband edge ω_s and group delay $k_d/2$, and use the coefficient vector of the filter obtained as an initial $\mathbf{h}_L$. The initial weighting is set $W(\omega)=1$.

Step 2 Use (24a) to compute matrix $\mathbf{R}^{(w)}$.

Step 3 Form matrix $\mathbf{Q}_L^{(w)}$ using (26) and compute $\mathbf{f}_L$ using (27).

Step 4 If $||\mathbf{h}_L - \mathbf{f}_L|| < \epsilon$, go to Step 5. Otherwise, update $\mathbf{h}_L$ using (16) with a τ close to 0.5 and repeat from Step 2.

Step 5 If the termination condition (37) of [17] is satisfied, output $\mathbf{f}_L$ as the design result and stop. Otherwise, update $W(\omega)$ using (33)–(36) of [17] where the reconstruction error is defined as

$$e_r(\omega) = H_0^2(e^{j\omega}) - H_0^2\left(e^{j(\omega+\pi)}\right) - e^{-jk_d\omega} \tag{28}$$

and repeat from Step 2.

Since the first integrand in (22) is proportional to the difference between the frequency response of the filter bank and that of an ideal allpass channel, Algorithm 3 yields a design in which the magnitude of the complex reconstruction error is equiripple. This does not correspond to an equiripple error in the amplitude or phase response of the filter bank in general when $k_d < N-1$. However, when $k_d = N-1$ and H_0 has a symmetrical impulse response, Algorithm 3 leads to a QMF bank with equiripple amplitude response.

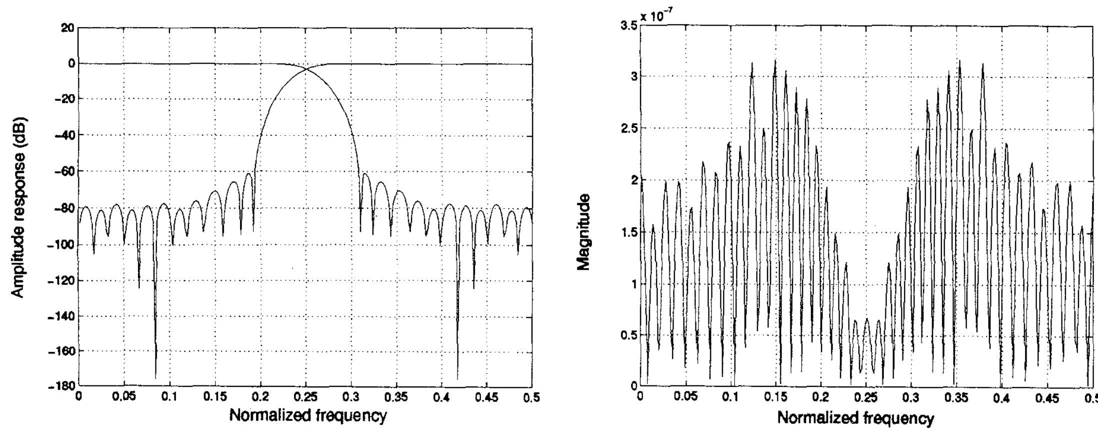
III. Design Examples

In this section, the algorithms proposed in Section II are applied to three design examples and comparisons are carried out in terms of design efficiency and performance of the filter banks obtained. The performance evaluations were made in terms of

- the number of floating-point operations in millions (MFLOPS) used
- the minimum stopband attenuation
$$A_a = \min_{\omega_s \le \omega \le \pi}\left[-20\log_{10}|H_0(e^{j\omega})|\right]$$
- the peak-to-peak passband ripple
$$A_p = \max_{0\le\omega\le\omega_p}\left[20\log_{10}|H_0(e^{j\omega})|\right] - \min_{0\le\omega\le\omega_p}\left[20\log_{10}|H_0(e^{j\omega})|\right]$$
where ω_p is the passband edge,
- the peak reconstruction error
$$\mathrm{PRE} = \max_{0\le\omega\le\pi}|e_r(\omega)|$$
where $e_r(\omega)$ is defined in (28),

$$\mathbf{R}_t = \begin{bmatrix} \omega_{t2}-\omega_{t1} & \phi(\omega_{t2},\omega_{t1},1) & \cdots & \phi(\omega_{t2},\omega_{t1},N-1) \\ \phi(\omega_{t2},\omega_{t1},1) & \omega_{t2}-\omega_{t1} & & \\ \vdots & & \ddots & \\ \phi(\omega_{t2},\omega_{t1},N-1) & \cdots & \cdots & \omega_{t2}-\omega_{t1} \end{bmatrix} \tag{20a}$$

$$\mathbf{b} = \begin{bmatrix} \frac{2}{k_d}\left[\sin\left(\frac{k_d\omega_{t2}}{2}\right) - \sin\left(\frac{k_d\omega_{t1}}{2}\right)\right] \\ \frac{2}{k_d-2}\left\{\sin\left[\left(\frac{k_d}{2}-1\right)\omega_{t2}\right] - \sin\left[\left(\frac{k_d}{2}-1\right)\omega_{t1}\right]\right\} \\ \vdots \\ \frac{2}{k_d-2N+2}\left\{\sin\left[\left(\frac{k_d}{2}-N+1\right)\omega_{t2}\right] - \sin\left[\left(\frac{k_d}{2}-N+1\right)\omega_{t1}\right]\right\} \end{bmatrix} \tag{20b}$$



(a) (b)

Fig. 2. Example 1. (a) Amplitude responses of the analysis filters. (b) Magnitude of the reconstruction error.

- the signal-to-noise ratio

$$\begin{aligned}\mathrm{SNR} &= 10\log_{10}\left(\frac{\text{energy of the signal}}{\text{energy of the reconstruction noise}}\right)\\ &= 10\log_{10}\left\{\frac{\sum x^2(n)}{\sum[x(n)-\hat{x}(n+k_d)]^2}\right\}\end{aligned}$$

A. Conventional QMF Bank

As Example 1, Algorithm 1 was used to design a conventional QMF bank with design parameters

$$N=64, \alpha=9.5\times10^{-7}, \omega_s=0.62\pi, \tau=0.5, \epsilon=10^{-5}$$

The results are summarized in Table I, where SNR_{r1} and SNR_{r2} denote the SNR with a ramp input and a random input with an amplitude which is uniformly distributed between 0 and 1, respectively. The amplitude responses of the analysis filters and the magnitude of the reconstruction error are depicted in Figs. 2(a) and (b), respectively. It is observed that the minimum stopband attenuation of the filter obtained is larger than 60 dB and the peak reconstruction error is of the order of 10^{-7}. In theory, a smaller reconstruction error may be obtained by using a smaller weight α in (2a). However, matrix **Q** defined by (9) can become ill-conditioned if α is too small. It is this potential numerical difficulty that prevents us from obtaining a design with the perfect reconstruction property. Nevertheless, this example demonstrates that our method can be used to design QMF banks with a reconstruction error which is small enough for many applications.

TABLE I
RESULTS FOR EXAMPLE 1

MFLOPS	11.9
A_a (dB)	60.68
A_p (dB)	5.82×10^{-6}
PRE	3.16×10^{-7}
SNR_{r1} (dB)	133.5
SNR_{r2} (dB)	134.2

B. QMF Bank With Low Reconstruction Delay

As Example 2, Algorithm 2 was used to design an FIR QMF bank with low reconstruction delay. The design parameters were

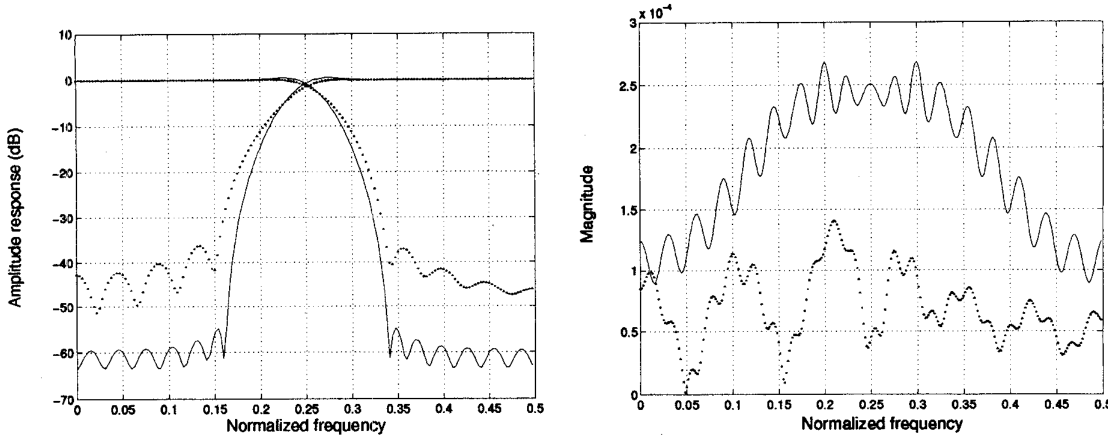
$$\begin{aligned}&N=32, k_d=15, \alpha=0.05, \alpha_1=1.38\times10^{-4},\\ &\omega_s=0.67\pi, \omega_{t1}=0.35\pi, \omega_{t2}=0.46\pi, \tau=0.5, \epsilon=10^{-3}\end{aligned}$$

For comparison purposes, we refer to Example 6.6.1 presented in [16] which was designed with a time-domain approach, and the results obtained are summarized in Table II. The amplitude responses of the analysis and synthesis low-pass and high-pass filters designed by the proposed method and the method of [16] are depicted in Fig. 3(a). The magnitude of the complex reconstruction error $e_r(\omega)$ of the proposed method and the method of [16] are plotted in Fig. 3(b). Fig. 4(a) and (b) shows the passband error of the low-pass analysis and synthesis filters and high-pass analysis and synthesis filters, respectively. Fig. 5 shows the magnitude of aliasing given by [22]

$$e_a(\omega)=H_0(e^{j(\omega+\pi)})G_0(e^{j\omega})+H_1(e^{j(\omega+\pi)})G_1(e^{j\omega})$$

Its peak value $E_a=\max_{0\le\omega\le\pi}|e_a(\omega)|$ is listed in Table II.

As can be observed from Table II, the proposed method increased the minimum stopband attenuation by 18 dB over the design in [16]. The passband ripple is slightly better in the design of [16] with respect to a low-pass passband edge greater than 0.43π or a high-pass passband edge less than 0.57π but is slightly inferior for lower low-pass passband edges or higher high-pass passband edges (see Fig. 4). The aliasing is completely cancelled in our design due to the fact that the quadrature mirror-image relationship is satisfied. The design in [16] does not satisfy this relationship so only the aliasing is minimized. Therefore, although our design has a larger PRE value, the reconstruction performance is still comparable with



(a) (b)

Fig. 3. Example 2, (a) amplitude responses of the analysis and synthesis filters, (b) magnitude of the reconstruction error. Solid line: Our design. Dotted line: Design of [16].

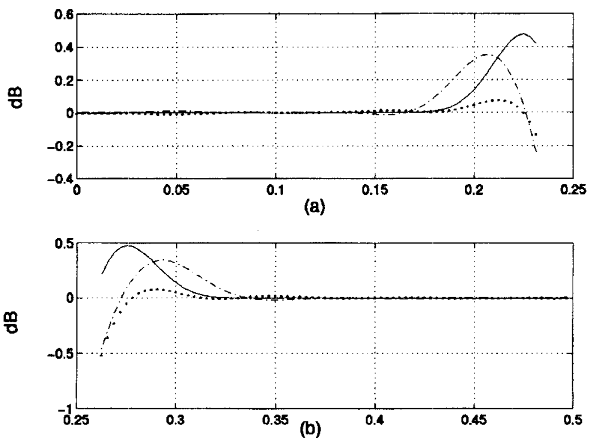


Fig. 4. Example 2. (a) Amplitude responses of the low-pass filters over the passband. (b) Amplitude responses of the high-pass filters over the passband. Solid line: Our design.Dotted line: Analysis filters of [16]. Dash and dotted line: Synthesis filters of [16].

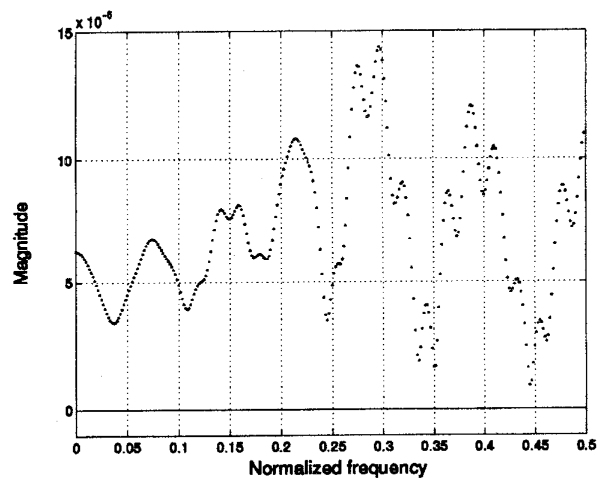


Fig. 5. Example 2. Magnitude of aliasing $e_a(\omega)$. Solid line: Our design. Dotted line: Design of [16].

that in [16] since the SNR values are almost the same. In addition, the quadrature mirror structure of our design is amenable to the efficient polyphase implementation [13], [20] which requires only $N/2$ additions and $N/2$ multiplications per input sample, as compared to N additions and N multiplications for the low-delay filter banks of [16]. In effect, we have been able to obtain improved two-channel low-delay QMF banks while using a much smaller set of independent filter coefficients, namely, 1/4 of the coefficients used in [16].

C. QMF Bank with Equiripple Complex Reconstruction Error

As Example 3, Algorithm 3 was used to design an FIR QMF bank with equiripple complex reconstruction error. The design parameters were

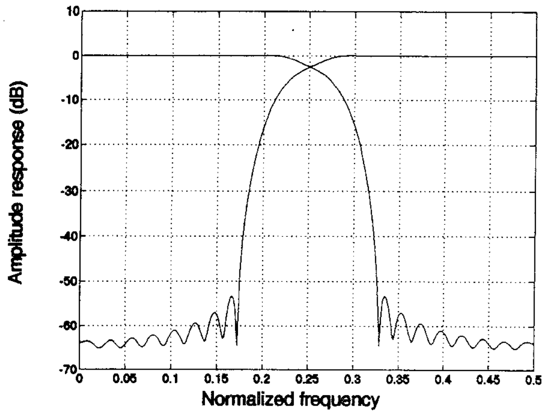
$$N = 40, k_d = 19, \alpha = 0.01, \omega_s = 0.66\pi,$$
$$\tau = 0.5, \kappa = 0.02, \epsilon = 10^{-4}$$

where κ is defined in [17]. The results are listed in Table III. The amplitude responses of the analysis filters are depicted in Fig. 6(a). The magnitude of the complex reconstruction error is plotted in Fig. 6(b) and, as can be seen, it is equiripple.

IV. Conclusion

An improved implementation of an algorithm due to Chen and Lee [17] has been proposed, namely, Algorithm 1, in which the objective function is expressed in terms of closed-form integrals that can be precisely evaluated with minimal computational effort. The avoidance of discretization enables the design of filter banks with improved performance and increases the efficiency of the algorithm quite significantly.

Algorithm 1 has been extended to the case where a QMF filter bank with low reconstruction delay is required. While the algorithm obtained, namely, Algorithm 2, yields filter banks in which the signal reconstruction is near-perfect, reconstruction delays lower than $(N-1)$ sampling periods can be achieved. These designs are amenable to economical and efficient polyphase implementation.



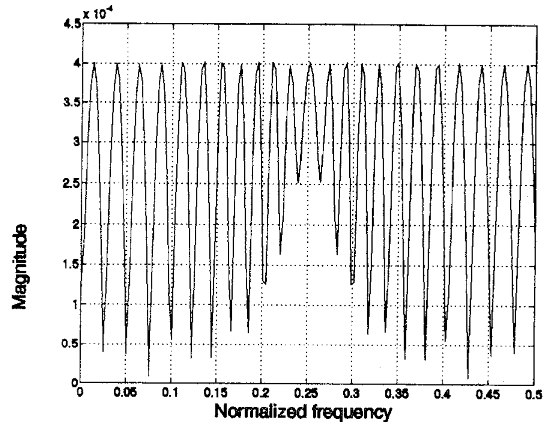


Fig. 6. Example 3. (a) Amplitude responses of analysis filters. (b) Magnitude of the reconstruction error.

TABLE II
COMPARISON OF THE PROPOSED METHOD WITH THE METHOD OF [16]

Parameter	Proposed	Method of [16]
k_d	15	15
MFLOPS	0.96	-
A_a (dB)	54.81	36.97
A_p (dB)	0.48	0.37
E_a	0	1.44×10^{-4}
PRE	2.69×10^{-4}	1.41×10^{-4}
SNR_{r1} (dB)	77.5	75.8
SNR_{r2} (dB)	76.7	77.5

TABLE III
RESULTS FOR EXAMPLE 3

A_a (dB)	53.29
A_p (dB)	0.095
PRE	4.1×10^{-4}
SNR_{r1} (dB)	82.5
SNR_{r2} (dB)	75.03

The approach was extended to the case where a minimax design is required and a suitable algorithm, namely, Algorithm 3, has been constructed. In this case, the reconstruction error is defined in terms of the difference between the frequency response of the filter bank and that of an ideal allpass channel, and the magnitude of the complex reconstruction error so obtained is forced to become equiripple.

The algorithms have been used to design many filter banks and it has been established that the design efficiency can be increased by as much as 75% while the minimum stopband attenuation of the individual filters can be increased by as much as 18 dB. The algorithms and the quality of the filter banks that can be achieved have been illustrated by three typical design examples in Section III.

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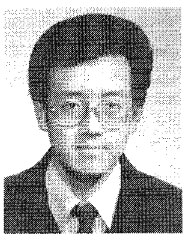

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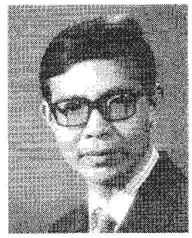

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Andreas Antoniou (M'69–SM'79–F'82) for a photograph and biography, see p. 95 of the February 1996 issue of this TRANSACTIONS.