A Weighted Balanced Approximation for 2-D Discrete Systems and Its Application to Model Reduction

Hong Luo, *Student Member, IEEE,* Wu–Sheng Lu, *Senior Member, IEEE,* and Andreas Antoniou, *Fellow, IEEE*

***Abstract*— Two auxiliary transfer-function matrices called *weighted-input-to-state* and *state-to-weighted-output* transfer-function matrices are first introduced. Based on these matrices, the controllability and observability Gramians of the weighted system are defined and the existence of the Gramians is justified. Then a method is developed for the computation of the Gramians in terms of *unconstrained* minimizations in which good initial points are provided for the minimizations. The objective function in the required minimizations is modified by including a term which depends on the Hankel singular values to be discarded. The method is then implemented as a step-by-step algorithm. As will be shown, the method generates a weighted structurally balanced approximation which yields a quadratically stable reduced-order system regardless of the system order. The method is illustrated by designing a lowpass 2-D filter of order (4, 8).**

I. Introduction

The BALANCED approximation is an effective and numerically economical method for the reduction of the system order in one-dimensional (1-D) and two-dimensional (2-D) systems. It has, in addition, several desirable properties such as a bounded error and preservation of stability [1]–[10] of the original system. If the requirements imposed on the approximation error are different for different frequency ranges, improved reduced-order systems can be obtained by applying a weighted balanced approximation method. The continuous-time 1-D case is considered in [2] and the discrete-time 1-D case is considered in [3] and [4]. In this paper, a weighted structurally balanced approximation for discrete 2-D systems is proposed.

The goal of the paper is to develop a model reduction algorithm that yields a weighted structurally balanced approximation which leads to a reduced-order stable transfer-function matrix that approximates the original transfer-function matrix to within a prescribed error. To this end, two auxiliary transfer-function matrices called the *weighted-input-to-state* and *state-to-weighted-output* transfer-function matrices are first introduced. It is shown that these matrices are quadratically stable (Q-stable) provided that the input and output weights and the system are Q-stable. Here quadratic stability (Q-stability) is the stability associated with the constant 2-D Lyapunov inequalities [7], which is known to be stronger than the bounded-input-bounded-output (BIBO) stability [11]. These auxiliary transfer-function matrices play a crucial role in the development of our method. On the basis of these matrices, the structured controllability and observability Gramians of the weighted system are defined and the existence of the Gramians is justified. The Gramians are the solutions of two 2-D Lyapunov inequalities, and solving them amounts to solving two *constrained* optimization problems.

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The authors are with the Department of Electrical and Computer Engineering, University of Victoria, Victoria, B.C. V8W 3P6 Canada.

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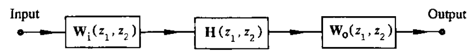

Fig. 1. A weighted 2-D discrete system.

In the proposed method these problems are reformulated as *unconstrained* minimizations and good initial points are deduced which improve the efficiency of the minimizations. It is important to emphasize that the controllability and observability Gramians obtained by solving 2-D Lyapunov inequalities are not unique and, consequently, the Hankel singular values are not state-transformation independent. Although no closed-form solution exists for choosing these Gramians in such a way as to achieve a good approximation, a solution is possible by modifying the objective function by including a term which is dependent on the Hankel singular values to be discarded.

As will be shown, our minimization procedure generates a weighted structurally balanced approximation which yields a reduced-order system that is always Q-stable regardless of the value of the reduced order. The balanced model reduction method proposed is illustrated by designing a lowpass 2-D filter of order (4, 8) used in [6].

II. Auxiliary Transfer-Function Matrices of a Weighted 2-D System

The system configuration to be considered here is shown in Fig. 1, where $\mathbf{H}(z_1, z_2) \in R^{p\times q}$ is the transfer-function matrix of the system, and $\mathbf{W}_i(z_1, z_2) \in R^{q\times t}$ and $\mathbf{W}_o(z_1, z_2) \in R^{s\times p}$ are the input and output weights, respectively.

If we use the Roesser state-space model [11] to describe $\mathbf{H}(z_1, z_2)$, $\mathbf{W}_i(z_1, z_2)$, and $\mathbf{W}_o(z_1, z_2)$, then

$$\mathbf{H}(z_1, z_2) = \mathbf{C}\,[\,\mathbf{I}(z_1, z_2) - \mathbf{A}\,]^{-1}\,\mathbf{B} + \mathbf{D} \tag{1a}$$

$$\mathbf{W}_i(z_1, z_2) = \mathbf{C}_i\,[\,\mathbf{I}(z_1, z_2) - \mathbf{A}_i\,]^{-1}\,\mathbf{B}_i + \mathbf{D}_i \tag{1b}$$

$$\mathbf{W}_o(z_1, z_2) = \mathbf{C}_o\,[\,\mathbf{I}(z_1, z_2) - \mathbf{A}_o\,]^{-1}\,\mathbf{B}_o + \mathbf{D}_o. \tag{1c}$$

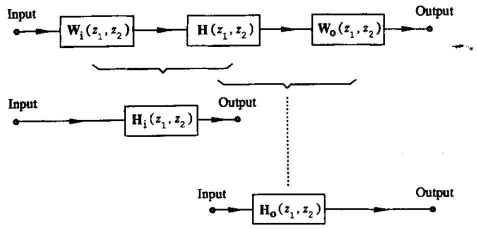

Fig. 2. Transfer functions $\mathbf{H}_i(z_1, z_2)$ and $\mathbf{H}_o(z_1, z_2)$.

The matrices involved can be partitioned as shown at the bottom of this page and $\mathbf{I}(z_1, z_2) = z_1\mathbf{I} \oplus z_2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix. In the rest of the paper, we assume that $\mathbf{H}(z_1, z_2)$, $\mathbf{W}_i(z_1, z_2)$, and $\mathbf{W}_o(z_1, z_2)$ are 2-D Q-stable, namely, there exist positive definite matrices $\mathbf{S}$, $\mathbf{S}_i$, and $\mathbf{S}_o$, and positive definite, block-diagonal matrices

$$\mathbf{G} = \mathbf{G}_1 \oplus \mathbf{G}_2, \quad \mathbf{G}_i = \mathbf{G}_{i1} \oplus \mathbf{G}_{i2}, \quad \mathbf{G}_o = \mathbf{G}_{o1} \oplus \mathbf{G}_{o2}$$

such that

$$\mathbf{A}\mathbf{G}\mathbf{A}^T - \mathbf{G} = -\mathbf{S} \tag{2a}$$

$$\mathbf{A}_i\mathbf{G}_i\mathbf{A}_i^T - \mathbf{G}_i = -\mathbf{S}_i \tag{2b}$$

$$\mathbf{A}_o\mathbf{G}_o\mathbf{A}_o^T - \mathbf{G}_o = -\mathbf{S}_o. \tag{2c}$$

Definition 1: Based on the configuration in Fig. 1, the *weighted-input-to-state* transfer-function matrix $\mathbf{H}_i(z_1, z_2)$ and *state-to-weighted-output* transfer-function matrix $\mathbf{H}_o(z_1, z_2)$ are defined as

$$\mathbf{H}_i(z_1, z_2) = [\, \mathbf{I}(z_1, z_2) - \mathbf{A}\,]^{-1}\, \mathbf{B}\, \mathbf{W}_i(z_1, z_2) \tag{3a}$$

and

$$\mathbf{H}_o(z_1, z_2) = \mathbf{W}_o(z_1, z_2)\, \mathbf{C}\, [\, \mathbf{I}(z_1, z_2) - \mathbf{A}\,]^{-1} \tag{3b}$$

respectively. These definitions are illustrated in Fig. 2.

By performing permutations for certain blocks of the matrices in (3a) and (3b), $\mathbf{H}_i(z_1, z_2)$ and $\mathbf{H}_o(z_1, z_2)$ can be written as

$$\mathbf{H}_i(z_1, z_2) = \hat{\mathbf{C}}_i[\mathbf{I}(z_1, z_2) - \hat{\mathbf{A}}_i]^{-1}\hat{\mathbf{B}}_i \tag{4a}$$

$$\mathbf{H}_o(z_1, z_2) = \hat{\mathbf{C}}_o[\mathbf{I}(z_1, z_2) - \hat{\mathbf{A}}_o]^{-1}\hat{\mathbf{B}}_o \tag{4b}$$

respectively, where

$$\hat{\mathbf{A}}_i = \begin{bmatrix} \hat{\mathbf{A}}_{i1} & \hat{\mathbf{A}}_{i2} \\ \hat{\mathbf{A}}_{i3} & \hat{\mathbf{A}}_{i4} \end{bmatrix} = \left[\begin{array}{cc|cc} \mathbf{A}_1 & \mathbf{B}_1\mathbf{C}_{i1} & \mathbf{A}_2 & \mathbf{B}_1\mathbf{C}_{i2} \\ \mathbf{0} & \mathbf{A}_{i1} & \mathbf{0} & \mathbf{A}_{i2} \\ \hline \mathbf{A}_3 & \mathbf{B}_2\mathbf{C}_{i1} & \mathbf{A}_4 & \mathbf{B}_2\mathbf{C}_{i2} \\ \mathbf{0} & \mathbf{A}_{i3} & \mathbf{0} & \mathbf{A}_{i4} \end{array}\right]$$

$$\hat{\mathbf{B}}_i = [\hat{\mathbf{B}}_{i1}^T \quad \hat{\mathbf{B}}_{i2}^T]^T = [\mathbf{D}_i^T\mathbf{B}_1^T \quad \mathbf{B}_{i1}^T \mid \mathbf{D}_i^T\mathbf{B}_2^T \quad \mathbf{B}_{i2}^T]^T$$

$$\hat{\mathbf{C}}_i = [\hat{\mathbf{C}}_{i1} \quad \hat{\mathbf{C}}_{i2}] = \left[\begin{array}{cc|cc} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{array}\right]$$

$$\hat{\mathbf{A}}_o = \begin{bmatrix} \hat{\mathbf{A}}_{o1} & \hat{\mathbf{A}}_{o2} \\ \hat{\mathbf{A}}_{o3} & \hat{\mathbf{A}}_{o4} \end{bmatrix} = \left[\begin{array}{cc|cc} \mathbf{A}_1 & \mathbf{0} & \mathbf{A}_2 & \mathbf{0} \\ \mathbf{B}_{o1}\mathbf{C}_1 & \mathbf{A}_{o1} & \mathbf{B}_{o1}\mathbf{C}_2 & \mathbf{A}_{o2} \\ \hline \mathbf{A}_3 & \mathbf{0} & \mathbf{A}_4 & \mathbf{0} \\ \mathbf{B}_{o2}\mathbf{C}_1 & \mathbf{A}_{o3} & \mathbf{B}_{o2}\mathbf{C}_2 & \mathbf{A}_{o4} \end{array}\right]$$

$$\hat{\mathbf{B}}_o = [\hat{\mathbf{B}}_{o1}^T \quad \hat{\mathbf{B}}_{o2}^T]^T = \left[\begin{array}{cc|cc} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{array}\right]^T$$

$$\hat{\mathbf{C}}_o = [\hat{\mathbf{C}}_{o1} \quad \hat{\mathbf{C}}_{o2}] = [\mathbf{D}_o\mathbf{C}_1 \quad \mathbf{C}_{o1} \mid \mathbf{D}_o\mathbf{C}_2 \quad \mathbf{C}_{o2}].$$

As indicated in Fig. 1, transfer-function matrix $\mathbf{H}_i(z_1, z_2)$ in (4a) relates a weighted input signal to a state of the system, which takes the input weight $\mathbf{W}_i(z_1, z_2)$ into account. Similarly, transfer-function matrix $\mathbf{H}_o(z_1, z_2)$ in (4b) relates a state to the weighted output, which takes the output weight $\mathbf{W}_o(z_1, z_2)$ into account. As will be seen in Section III, these two transfer-function matrices serve as the starting point for balancing the weighted 2-D system.

III. Weighted Structurally Balanced Approximation of 2-D Discrete Systems

We now propose a weighted structurally balanced approximation (WBA) of 2-D systems which is essentially an extension of the unweighted structurally balanced approximation proposed in [7].

A. Definition of WBA Using 2-D Lyapunov Inequalities

It can be shown that if transfer-function matrix $\mathbf{H}(z_1, z_2)$ and weights $\mathbf{W}_i(z_1, z_2)$ and $\mathbf{W}_o(z_1, z_2)$ are all 2-D Q-stable as was assumed in Section II, then $\mathbf{H}_i(z_1, z_2)$ and $\mathbf{H}_o(z_1, z_2)$ are also 2-D Q-stable. The proof of this fact is given in Appendix A. Consequently, there exist block-diagonal, positive definite matrices

$$\hat{\mathbf{P}} = \hat{\mathbf{P}}_1 \oplus \hat{\mathbf{P}}_2 = \left[\begin{array}{cc|cc} \hat{\mathbf{P}}_{11} & \hat{\mathbf{P}}_{21} & \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{P}}_{21}^T & \hat{\mathbf{P}}_{31} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \hat{\mathbf{P}}_{12} & \hat{\mathbf{P}}_{22} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{P}}_{22}^T & \hat{\mathbf{P}}_{32} \end{array}\right] \tag{5a}$$

$$\hat{\mathbf{Q}} = \hat{\mathbf{Q}}_1 \oplus \hat{\mathbf{Q}}_2 = \left[\begin{array}{cc|cc} \hat{\mathbf{Q}}_{11} & \hat{\mathbf{Q}}_{21} & \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{Q}}_{21}^T & \hat{\mathbf{Q}}_{31} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \hat{\mathbf{Q}}_{12} & \hat{\mathbf{Q}}_{22} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{Q}}_{22}^T & \hat{\mathbf{Q}}_{32} \end{array}\right] \tag{5b}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}_{(n+m)\times(n+m)} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}_{(n+m)\times q} \qquad \mathbf{C} = [\mathbf{C}_1 \quad \mathbf{C}_2]_{p\times(n+m)}$$

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{A}_{i1} & \mathbf{A}_{i2} \\ \mathbf{A}_{i3} & \mathbf{A}_{i4} \end{bmatrix}_{(n_i+m_i)\times(n_i+m_i)} \qquad \mathbf{B}_i = \begin{bmatrix} \mathbf{B}_{i1} \\ \mathbf{B}_{i2} \end{bmatrix}_{(n_i+m_i)\times t} \qquad \mathbf{C}_i = [\mathbf{C}_{i1} \quad \mathbf{C}_{i2}]_{q\times(n_i+m_i)}$$

$$\mathbf{A}_o = \begin{bmatrix} \mathbf{A}_{o1} & \mathbf{A}_{o2} \\ \mathbf{A}_{o3} & \mathbf{A}_{o4} \end{bmatrix}_{(n_o+m_o)\times(n_o+m_o)} \qquad \mathbf{B}_o = \begin{bmatrix} \mathbf{B}_{o1} \\ \mathbf{B}_{o2} \end{bmatrix}_{(n_o+m_o)\times p} \qquad \mathbf{C}_o = [\mathbf{C}_{o1} \quad \mathbf{C}_{o2}]_{s\times(n_o+m_o)}$$

that satisfy the 2-D Lyapunov inequalities

$$\hat{\mathbf{A}}_i \hat{\mathbf{P}} \hat{\mathbf{A}}_i^T - \hat{\mathbf{P}} + \hat{\mathbf{B}}_i \hat{\mathbf{B}}_i^T < 0 \tag{6a}$$

$$\hat{\mathbf{A}}_o^T \hat{\mathbf{Q}} \hat{\mathbf{A}}_o - \hat{\mathbf{Q}} + \hat{\mathbf{C}}_o^T \hat{\mathbf{C}}_o < 0. \tag{6b}$$

From (4a) and (4b), it is observed that the four blocks of the system matrix of $\mathbf{H}(z_1, z_2)$, namely, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, and $\mathbf{A}_4$ of matrix $\mathbf{A}$ defined in (1a), occur in both $\hat{\mathbf{A}}_i$ and $\hat{\mathbf{A}}_o$ as the (1, 1), (1, 3), (3, 1), and (3, 3) blocks. Therefore, the *structured controllability and observability Gramians of the weighted system*, denoted as $\mathbf{P}$ and $\mathbf{Q}$, respectively, can be obtained from $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ as

$$\mathbf{P} = \hat{\mathbf{I}} \hat{\mathbf{P}} \hat{\mathbf{I}}^T, \; \mathbf{Q} = \hat{\mathbf{I}} \hat{\mathbf{Q}} \hat{\mathbf{I}}^T \tag{7}$$

where

$$\hat{\mathbf{I}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}$$

that is,

$$\mathbf{P} = \begin{bmatrix} \hat{\mathbf{P}}_{11} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{P}}_{12} \end{bmatrix}_{(n+m)\times(n+m)} \equiv \mathbf{P}_1 \oplus \mathbf{P}_2 =$$

$$\mathbf{Q} = \begin{bmatrix} \hat{\mathbf{Q}}_{11} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Q}}_{12} \end{bmatrix}_{(n+m)\times(n+m)} \equiv \mathbf{Q}_1 \oplus \mathbf{Q}_2$$

which are positive definite, block-diagonal matrices. Hence a nonsingular, block-diagonal matrix $\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2$ can be found [12], [5] such that

$$\mathbf{T}_1^{-1}\mathbf{P}_1\mathbf{T}_1^{-T} = \mathbf{T}_1^T\mathbf{Q}_1\mathbf{T}_1 = \mathbf{\Sigma}_1 = \text{diag}\,(\sigma_1, \sigma_2, \cdots, \sigma_n) \tag{8a}$$

$$\mathbf{T}_2^{-1}\mathbf{P}_2\mathbf{T}_2^{-T} = \mathbf{T}_2^T\mathbf{Q}_2\mathbf{T}_2 = \mathbf{\Sigma}_2 = \text{diag}\,(\mu_1, \mu_2, \cdots, \mu_m) \tag{8b}$$

where

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \text{ and } \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0.$$

Having found balancing transformation matrix $\mathbf{T}$, a WBA of the system $\mathbf{H}(z_1, z_2)$ in the Roesser state-space model can be characterized by the set $\{\mathbf{A}_b, \mathbf{B}_b, \mathbf{C}_b, \mathbf{D}\}$ with

$$\mathbf{A}_b = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \quad \mathbf{B}_b = \mathbf{T}^{-1}\mathbf{B}, \quad \mathbf{C}_b = \mathbf{C}\mathbf{T} \tag{9}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are given in (1a).

B. Computation of $\hat{\mathbf{P}}$ *and* $\hat{\mathbf{Q}}$

In principle, matrices $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ that satisfy (6a) and (6b) can be obtained by solving the *constrained* convex minimization problems

$$\min_{\hat{\mathbf{P}}>0} \rho_{\max}(\hat{\mathbf{A}}_i\hat{\mathbf{P}}\hat{\mathbf{A}}_i^T - \hat{\mathbf{P}} + \hat{\mathbf{B}}_i\hat{\mathbf{B}}_i^T) < 0 \tag{10a}$$

$$\min_{\hat{\mathbf{Q}}>0} \rho_{\max}(\hat{\mathbf{A}}_o^T\hat{\mathbf{Q}}\hat{\mathbf{A}}_o - \hat{\mathbf{Q}} + \hat{\mathbf{C}}_o^T\hat{\mathbf{C}}_o) < 0 \tag{10b}$$

where $\rho_{\max}$ denotes the largest eigenvalue of a matrix. Using the Cholesky factorization, we obtain

$$\hat{\mathbf{P}} = \hat{\mathbf{L}}_p\hat{\mathbf{L}}_p^T \text{ and } \hat{\mathbf{Q}} = \hat{\mathbf{L}}_q\hat{\mathbf{L}}_q^T \tag{11}$$

where $\hat{\mathbf{L}}_p$ and $\hat{\mathbf{L}}_q$ are block-diagonal lower-triangular matrices. Thus, (6a) and (6b) can be reformulated as

$$\| \hat{\mathbf{L}}_p^{-1}[\hat{\mathbf{A}}_i\hat{\mathbf{L}}_p \quad \hat{\mathbf{B}}_i] \| < 1 \text{ and } \| \hat{\mathbf{L}}_q^{-1}[\hat{\mathbf{A}}_o^T\hat{\mathbf{L}}_q \quad \hat{\mathbf{C}}_o^T] \| < 1.$$

Therefore, instead of solving the constrained minimization problems in (10a) and (10b), we can work on the following *unconstrained* optimization problems

$$\min_{\hat{\mathbf{L}}_p} \| \hat{\mathbf{L}}_p^{-1}[\hat{\mathbf{A}}_i\hat{\mathbf{L}}_p \quad \hat{\mathbf{B}}_i] \| \tag{12a}$$

and

$$\min_{\hat{\mathbf{L}}_q} \| \hat{\mathbf{L}}_q^{-1}[\hat{\mathbf{A}}_o^T\hat{\mathbf{L}}_q \quad \hat{\mathbf{C}}_o^T] \|. \tag{12b}$$

Obviously, local minimum points of (12a) and (12b), $\hat{\mathbf{L}}_p^*$ and $\hat{\mathbf{L}}_q^*$, are acceptable if and only if the norms defined in (12a) and (12b) at $\hat{\mathbf{L}}_p^*$ and $\hat{\mathbf{L}}_q^*$ are strictly less than one. This is because only then matrices $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$, formed by (11), will satisfy the Lyapunov inequalities (6a) and (6b).

Note that even for a 2-D system of modest order, the number of parameters involved in each of the above minimization problems is quite large. For example, if $n = m = 15$, $n_i = m_i = 10$, and $n_o = m_o = 10$, we have

$$\frac{(n+n_i)(n+n_i+1)+(m+m_i)(m+m_i+1)}{2} = 650.$$

It would, therefore, be useful if a good initial point could be deduced before one starts minimizing the norms in (12a) and (12b). Let us consider the two Lyapunov-like equations

$$\hat{\mathbf{A}}_{i1}\hat{\mathbf{P}}_1\hat{\mathbf{A}}_{i1}^T - \hat{\mathbf{P}}_1 + \hat{\mathbf{A}}_{i2}\hat{\mathbf{P}}_2\hat{\mathbf{A}}_{i2}^T = -\hat{\mathbf{B}}_{i1}\hat{\mathbf{B}}_{i1}^T \tag{13a}$$

$$\hat{\mathbf{A}}_{i4}\hat{\mathbf{P}}_2\hat{\mathbf{A}}_{i4}^T - \hat{\mathbf{P}}_2 + \hat{\mathbf{A}}_{i3}\hat{\mathbf{P}}_1\hat{\mathbf{A}}_{i3}^T = -\hat{\mathbf{B}}_{i2}\hat{\mathbf{B}}_{i2}^T. \tag{13b}$$

These equations were first utilized in [6] in order to introduce the concept of *quasibalanced* 2-D systems. Note that (13a) and (13b) are linear in $\hat{\mathbf{P}}_1$ and $\hat{\mathbf{P}}_2$. It can be shown that if system (4a) is Q-stable and both $(\hat{\mathbf{A}}_{i1}, \hat{\mathbf{B}}_{i1})$ and $(\hat{\mathbf{A}}_{i4}, \hat{\mathbf{B}}_{i2})$ are controllable, then there exist unique positive definite $\hat{\mathbf{P}}_1$ and $\hat{\mathbf{P}}_2$ that satisfy (13a) and (13b), respectively. The proof of this fact is given in Appendix B. Having obtained $\hat{\mathbf{P}}_1$ and $\hat{\mathbf{P}}_2$, the Cholesky decomposition can be used to write

$$\hat{\mathbf{P}}_1 = \hat{\mathbf{L}}_{p1}\hat{\mathbf{L}}_{p1}^T, \; \hat{\mathbf{P}}_2 = \hat{\mathbf{L}}_{p2}\hat{\mathbf{L}}_{p2}^T$$

and (13a) and 13(b) imply that

$$\| \hat{\mathbf{L}}_{p1}^{-1}[\hat{\mathbf{A}}_{i1}\hat{\mathbf{L}}_{p1} \quad \hat{\mathbf{A}}_{i2}\hat{\mathbf{L}}_{p2} \quad \hat{\mathbf{B}}_{i1}] \| = 1 \tag{14a}$$

$$\| \hat{\mathbf{L}}_{p2}^{-1}[\hat{\mathbf{A}}_{i3}\hat{\mathbf{L}}_{p1} \quad \hat{\mathbf{A}}_{i4}\hat{\mathbf{L}}_{p2} \quad \hat{\mathbf{B}}_{i2}] \| = 1. \tag{14b}$$

Hence

$$\| \hat{\mathbf{L}}_p^{-1}[\hat{\mathbf{A}}_i\hat{\mathbf{L}}_p \quad \hat{\mathbf{B}}_i] \| \leq 2 \tag{15}$$

where

$$\hat{\mathbf{L}}_p = \hat{\mathbf{L}}_{p1} \oplus \hat{\mathbf{L}}_{p2}.$$

In the light of (15), the $\hat{\mathbf{L}}_p$ so obtained can now be used as a good initial point for the minimization problem (12a). A good initial point for problem 12(b) can be found in a similar manner.

IV. Weighted Structurally Balanced Model Reduction of 2-D Discrete Systems

The weighted structurally balanced model reduction (WBMR) problem entails finding a reduced-order (r_1, r_2), Q-stable transfer-function matrix $\mathbf{H}_r(z_1, z_2)$ such that for given Q-stable $\mathbf{H}(z_1, z_2)$, $\mathbf{W}_i(z_1, z_2)$, and $\mathbf{W}_o(z_1, z_2)$, the weighted approximation error

$$\| \mathbf{W}_o(z_1, z_2) \left[\mathbf{H}(z_1, z_2) - \mathbf{H}_r(z_1, z_2) \right] \mathbf{W}_i(z_1, z_2) \|_\infty \tag{16}$$

is minimized with respect to all Q-stable transfer-function matrices of order (r_1, r_2).

Having obtained a WBA $\{\mathbf{A}_b, \mathbf{B}_b, \mathbf{C}_b, \mathbf{D}\}$ as given in (9), we can partition it as

$$\mathbf{A}_b = \left[\begin{array}{cc|cc} \mathbf{A}_{1r} & \mathbf{A}_{12} & \mathbf{A}_{2r} & \mathbf{A}_{22} \\ \mathbf{A}_{13} & \mathbf{A}_{14} & \mathbf{A}_{23} & \mathbf{A}_{24} \\ \hline \mathbf{A}_{3r} & \mathbf{A}_{32} & \mathbf{A}_{4r} & \mathbf{A}_{42} \\ \mathbf{A}_{33} & \mathbf{A}_{34} & \mathbf{A}_{43} & \mathbf{A}_{44} \end{array}\right], \quad \mathbf{B}_b = \left[\begin{array}{c} \mathbf{B}_{1r} \\ \mathbf{B}_{12} \\ \hline \mathbf{B}_{2r} \\ \mathbf{B}_{22} \end{array}\right],$$
$$\mathbf{C}_b = \left[\begin{array}{cc|cc} \mathbf{C}_{1r} & \mathbf{C}_{12} & \mathbf{C}_{2r} & \mathbf{C}_{22} \end{array}\right] \tag{17}$$

where $\mathbf{A}_{1r} \in R^{r_1 \times r_1}, \mathbf{B}_{1r} \in R^{r_1 \times q}, \mathbf{C}_{1r} \in R^{p \times r_1}, \mathbf{A}_{4r} \in R^{r_2 \times r_2}, \mathbf{B}_{2r} \in R^{r_2 \times q}, \mathbf{C}_{2r} \in R^{p \times r_2}$. The reduced-order 2-D system of order (r_1, r_2) can now be formed as $\{\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}\}$ with

$$\mathbf{A}_r = \begin{bmatrix} \mathbf{A}_{1r} & \mathbf{A}_{2r} \\ \mathbf{A}_{3r} & \mathbf{A}_{4r} \end{bmatrix}, \quad \mathbf{B}_r = \begin{bmatrix} \mathbf{B}_{1r} \\ \mathbf{B}_{2r} \end{bmatrix}, \quad \mathbf{C}_r = [\mathbf{C}_{1r} \quad \mathbf{C}_{2r}]. \tag{18}$$

Unlike the 1-D case, the reduced-order system of order (r_1, r_2) in (18) obtained by truncating a WBA may not be satisfactory in terms of weighted approximation error. This is mainly because there are infinitely many solutions to the Lyapunov inequalities (6a) and (6b), and the analytic relation of these solutions to the approximation error introduced by a simple truncation remains unclear. In what follows, the Q-stability of the reduced-order system obtained from a WBA is first demonstrated. Then, we propose a modified weighted structurally balanced approximation that takes both the stability issue and the approximation error into account.

A. Q-Stability of the Reduced-Order System

In this section, we show that for the single-input-single-output (SISO) case, the reduced-order systems are Q-stable for any Q-stable input and output weights $\mathbf{W}_i(z_1, z_2)$ and $\mathbf{W}_o(z_1, z_2)$; and that for the multi-input-multi-output (MIMO) case, the Q-stability of the reduced-order system holds under some additional conditions.

Theorem 1: If $\mathbf{H}(z_1, z_2)$ is Q-stable with either unit input weight, $\mathbf{W}_i(z_1, z_2) = \mathbf{I}$, or unit output weight, $\mathbf{W}_o(z_1, z_2) = \mathbf{I}$, then the reduced-order system obtained from the weighted structurally balanced realization in (18) is also Q-stable.

Proof: If $\mathbf{W}_i(z_1, z_2) = \mathbf{I}$, then (3a) becomes

$$\mathbf{H}_i(z_1, z_2) = \left[\mathbf{I}(z_1, z_2) - \mathbf{A} \right]^{-1} \mathbf{B}. \tag{19}$$

Comparing (19) with (4a), we have

$$\mathbf{A}_i = \mathbf{A} \text{ and } \mathbf{B}_i = \mathbf{B}$$

and (6a) becomes

$$\mathbf{A}\mathbf{P}\mathbf{A}^T - \mathbf{P} + \mathbf{B}\mathbf{B}^T < 0. \tag{20}$$

Pre-multiplying (20) by $\mathbf{T}^{-1}$ and post-multiplying by $\mathbf{T}^{-T}$, it follows that

$$\mathbf{A}_b \mathbf{\Sigma} \mathbf{A}_b^T - \mathbf{\Sigma} + \mathbf{B}_b \mathbf{B}_b^T < 0. \tag{21}$$

By performing permutations, (21) can be rewritten as

$$\tilde{\mathbf{A}}_b \tilde{\mathbf{\Sigma}} \tilde{\mathbf{A}}_b^T - \tilde{\mathbf{\Sigma}} + \tilde{\mathbf{B}}_b \tilde{\mathbf{B}}_b^T < 0 \tag{22}$$

where

$$\tilde{\mathbf{A}}_b = \begin{bmatrix} \mathbf{A}_r & \tilde{\mathbf{A}}_2 \\ \tilde{\mathbf{A}}_3 & \tilde{\mathbf{A}}_4 \end{bmatrix}, \quad \tilde{\mathbf{B}}_b = \begin{bmatrix} \mathbf{B}_r \\ \tilde{\mathbf{B}}_2 \end{bmatrix}$$

and

$$\tilde{\mathbf{\Sigma}} = \mathbf{\Sigma}_r \oplus \tilde{\mathbf{\Sigma}}_2 = \operatorname{diag}(\sigma_1, \cdots, \sigma_{r_1}, \mu_1, \cdots, \mu_{r_2}, \sigma_{r_1+1}, \cdots, \sigma_n, \mu_{r_2+1}, \cdots, \mu_m). \tag{23}$$

By (22) we obtain

$$\mathbf{A}_r \mathbf{\Sigma}_r \mathbf{A}_r^T - \mathbf{\Sigma}_r + \mathbf{B}_r \mathbf{B}_r^T < 0 \tag{24}$$

which implies the Q-stability of the reduced-order system of order (r_1, r_2). □

An immediate consequence of Theorem 1 is the Q-stability of the reduced-order systems for the SISO case.

Lemma 1: For the SISO case, if $\mathbf{H}(z_1, z_2)$, $\mathbf{W}_i(z_1, z_2)$, and $\mathbf{W}_o(z_1, z_2)$ are Q-stable, then the reduced-order system obtained in (18) is also Q-stable.

Theorem 2: If $\mathbf{H}(z_1, z_2)$, $\mathbf{W}_i(z_1, z_2)$, and $\mathbf{W}_o(z_1, z_2)$ are Q-stable, and if $\hat{\mathbf{P}}_1$ and $\hat{\mathbf{P}}_2$ in (5a) are block-diagonal (or $\hat{\mathbf{Q}}_1$ and $\hat{\mathbf{Q}}_2$ in (5b) are block-diagonal), then the reduced-order system obtained in (18) is also Q-stable.

Proof: Write $\hat{\mathbf{P}}_1$ and $\hat{\mathbf{P}}_2$ in (5a) as follows:

$$\hat{\mathbf{P}}_1 = \begin{bmatrix} \hat{\mathbf{P}}_{11} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{P}}_{31} \end{bmatrix} \text{ and } \hat{\mathbf{P}}_2 = \begin{bmatrix} \hat{\mathbf{P}}_{12} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{P}}_{32} \end{bmatrix}.$$

The proof for the existence of block-diagonal $\hat{\mathbf{P}}_1$ and $\hat{\mathbf{P}}_2$ has been justified in Appendix A. By performing permutations, (6a) can be written as

$$\begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 \\ \mathbf{\Phi}_2^T & \mathbf{\Phi}_3 \end{bmatrix} < 0 \tag{25}$$

where

$$\mathbf{\Phi}_1 = \mathbf{A}\mathbf{P}\mathbf{A}^T - \mathbf{P} + \mathbf{B}\mathbf{C}_i\tilde{\mathbf{P}}_3\mathbf{C}_i^T\mathbf{B}^T + \mathbf{B}\mathbf{D}_i\mathbf{D}_i^T\mathbf{B}^T$$
$$\mathbf{\Phi}_2 = \mathbf{B}\mathbf{C}_i\tilde{\mathbf{P}}_3\mathbf{A}_i^T + \mathbf{B}\mathbf{D}_i\mathbf{B}_i^T$$
$$\mathbf{\Phi}_3 = \mathbf{A}_i\tilde{\mathbf{P}}_3\mathbf{A}_i^T - \tilde{\mathbf{P}}_3 + \mathbf{B}_i\mathbf{B}_i^T$$

and $\tilde{\mathbf{P}}_3 = \hat{\mathbf{P}}_{31} \oplus \hat{\mathbf{P}}_{32}$. Pre-multiplying (25) with $\mathbf{X}$

$$\mathbf{X} = \begin{bmatrix} \mathbf{I} & -\mathbf{\Phi}_2\mathbf{\Phi}_3^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

and post-multiplying with $\mathbf{X}^T$, gives

$$\begin{bmatrix} \mathbf{\Phi}_1 - \mathbf{\Phi}_2\mathbf{\Phi}_3^{-1}\mathbf{\Phi}_2^T & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}_3 \end{bmatrix} < 0. \tag{26}$$

Since $\hat{\mathbf{P}}_3 > 0$ and $\mathbf{\Phi}_3 < 0$, we have

$$\mathbf{APA}^T - \mathbf{P} \le \mathbf{\Phi}_1 \le \mathbf{\Phi}_1 - \mathbf{\Phi}_2\mathbf{\Phi}_3^{-1}\mathbf{\Phi}_2^T < 0.$$

It follows that

$$\mathbf{A}_b\mathbf{\Sigma}\mathbf{A}_b^T - \mathbf{\Sigma} < 0.$$

Hence

$$\mathbf{A}_r\mathbf{\Sigma}_r\mathbf{A}_r^T - \mathbf{\Sigma}_r < 0$$

which implies the Q-stability of the reduced-order system of order (r_1, r_2). □

B. WBMR Method Based on WBA

For the 1-D case, the Hankel singular values, defined as the square roots of the eigenvalues of the product of the controllability and observability Gramians, are a set of system invariants. Namely, they are invariant under state-variable transformations. More importantly, it is well known that the approximation error introduced by a balanced approximation is controlled by twice the sum of the discard Hankel singular values.

Let $\mathbf{P}$ and $\mathbf{Q}$ be positive definite solutions of (6a), (6b), and (7). If we define the square roots of the eigenvalues of $\mathbf{PQ}$, i.e.,

$$\sigma_1, \sigma_2, \cdots, \sigma_n, \mu_1, \mu_2, \cdots, \mu_m$$

as the Hankel singular values (h.s.v.'s) of the weighted system, then these parameters are in general not invariant under state-variable transformations since the solutions of inequalities (6a) and (6b) are not unique. In other words, the h.s.v.'s so defined are $\mathbf{P}$ and $\mathbf{Q}$ dependent. However, this dependence provides an approach to specify a suitable solution of (6a) and (6b) so as to achieve a small approximation error. We consider an objective function defined by

$$f(\hat{\mathbf{L}}_p, \hat{\mathbf{L}}_q) = ||\hat{\mathbf{L}}_p^{-1}[\hat{\mathbf{A}}_i\hat{\mathbf{L}}_p \quad \hat{\mathbf{B}}_i]|| + k_1||\hat{\mathbf{L}}_q^{-1}[\hat{\mathbf{A}}_o^T\hat{\mathbf{L}}_q \quad \hat{\mathbf{C}}_o^T]|| + k_2\left(\sum_{k=r_1+1}^{n}\sigma_k + \sum_{l=r_2+1}^{m}\mu_l\right) \tag{27}$$

where $k_1 > 0$ and $k_2 > 0$ are weighted scalars. By minimizing the objective function $f(\hat{\mathbf{L}}_p, \hat{\mathbf{L}}_q)$ with an appropriate selection of k_1 and k_2, block-diagonal matrices $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ can be found such that

$$||\,\hat{\mathbf{L}}_p^{-1}[\hat{\mathbf{A}}_i\hat{\mathbf{L}}_p \quad \hat{\mathbf{B}}_i]\,|| < 1 \quad \text{and} \quad ||\,\hat{\mathbf{L}}_q^{-1}[\hat{\mathbf{A}}_o^T\hat{\mathbf{L}}_q \quad \hat{\mathbf{C}}_o^T]\,|| < 1 \tag{28}$$

so that the Q-stability of the reduced-order system is guaranteed and the least h.s.v.'s $\sigma_{r_1+1}, \cdots, \sigma_n, \mu_{r_2+1}, \cdots, \mu_m$ are suppressed so that the approximation error is expected to be small as well. The following is a step-by-step summary of the proposed WBMR algorithm.

Algorithm:

Step 1: Find a local minimum of $f(\hat{\mathbf{L}}_p, \hat{\mathbf{L}}_q)$ in (27) such that (28) holds. The optimization is unconstrained and can be carried out using established numerical optimization techniques [13]. Weights k_1 and k_2 can be adjusted to satisfy (28).

Step 2: Compute $\hat{\mathbf{P}} = \hat{\mathbf{L}}_p\hat{\mathbf{L}}_p^T$ and $\hat{\mathbf{Q}} = \hat{\mathbf{L}}_q\hat{\mathbf{L}}_q^T$.

Step 3: Obtain the structured Gramians of the weighted system, $\mathbf{P}$ and $\mathbf{Q}$, from the relations

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{P}}_{11} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{P}}_{12} \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{Q}}_{11} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Q}}_{12} \end{bmatrix}.$$

Step 4: Find nonsingular matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ such that (8a) and (8b) are satisfied and construct balancing transformation matrix $\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2$.

Step 5: Obtain the weighted structurally balanced approximation $\{\mathbf{A}_b, \mathbf{B}_b, \mathbf{C}_b, \mathbf{D}\}$ with

$$\mathbf{A}_b = \mathbf{T}^{-1}\mathbf{AT}, \quad \mathbf{B}_b = \mathbf{T}^{-1}\mathbf{B}, \quad \mathbf{C}_b = \mathbf{CT}$$

and partition it as in (17).

Step 6: The reduced-order 2-D system of order (r_1, r_2) can now be formed as $\{\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}\}$.

The balancing transformations $\mathbf{T}_1$ and $\mathbf{T}_2$ in step 4 can be found by using known algorithms [12].

Some remarks are in order at this point. It is noted that the objective function in (27) is not convex. However, the optimization problem formulated is unconstrained and there are several efficient optimization methods such as the class of quasi-Newton methods [13] that are applicable to noncovex but unconstrained objective functions. In addition, as was shown in Section III-B, we can always find a good initial point for the problem at hand, which makes the optimization more efficient.

An alternative approach to solving optimization problems (6a) and (6b) is to use the linear matrix inequality (LMI) techniques that have recently been developed in [14]–[16]. For example, (6a) can be reformulated as

$$\min_{\nu\mathbf{I}-\mathbf{F}(\mathbf{x})>0,\, \mathbf{G}(\mathbf{x})>0} \nu \tag{29}$$

where $\mathbf{x}$ is the k-dimensional vector that collects all unknown entries of $\hat{\mathbf{P}}$, matrix $\mathbf{F}(\mathbf{x})$ is an equivalent representation of $\hat{\mathbf{A}}_i\,\hat{\mathbf{P}}\hat{\mathbf{A}}_i^T - \hat{\mathbf{P}} + \hat{\mathbf{B}}_i\hat{\mathbf{B}}_i^T$ which depends affinely on $\mathbf{x}$, i.e.,

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + \sum_{i=1}^{k}\mathbf{x}_i\mathbf{F}_i$$

and matrix $\mathbf{G}(\mathbf{x})$ is an equivalent representation of $\hat{\mathbf{P}}$, which also depends affinely on $\mathbf{x}$, i.e.,

$$\mathbf{G}(\mathbf{x}) = \mathbf{G}_0 + \sum_{i=1}^{k}\mathbf{x}_i\mathbf{G}_i.$$

Note that (29) is a standard formulation for the minimization problem that can be solved using the interior-point algorithms developed in [15] and [16].

V. EXAMPLE

In this section, a lowpass filter discussed in [6] is used to demonstrate the proposed weighted model reduction algorithm. The state-space model of the filter is given by (1a) with

$$\mathbf{A}_1 = \begin{bmatrix} 0.537000 & -0.068810 & 0.985510 & 0.503880 \\ 1.000000 & 0 & 0 & 0 \\ 0 & 0 & 0.538819 & -0.066576 \\ 0 & 0 & 1.000000 & 0 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} -0.390721 & 0.244967 & -0.483603 & -0.247260 \\ 0.251223 & -0.145125 & 0.026974 & 0.013792 \\ 1.270478 & 1.106765 & 0.198140 & 0.101307 \\ 1.796448 & 0.421991 & 0.592117 & 0.302742 \\ 0 & 0 & -0.393352 & 0.242461 \\ 0 & 0 & 0.252014 & -0.145254 \\ 0 & 0 & 1.270845 & 1.107215 \\ 0 & 0 & 1.797547 & 0.423333 \end{bmatrix}$$

$$\mathbf{A}_4 = \begin{bmatrix} 0.490714 & 1 & 0 & 0 & 0.490714 & 0 & -0.490714 & 0 \\ -0.027371 & 0 & 0 & 0 & -0.027371 & 0 & 0.027371 & 0 \\ -0.201054 & 0 & 0 & 1 & -0.201054 & 0 & 0.201054 & 0 \\ -0.600823 & 0 & 0 & 0 & -0.600823 & 0 & 0.600823 & 0 \\ 0 & 0 & 0 & 0 & 0.491231 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.028179 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.201054 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -0.600823 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{B}_1 &= 10^{-3}[1.34044 \quad 0 \quad 1.34044 \quad 0]^T \\ \mathbf{B}_{21} &= [-0.657773 \quad 0.036689 \quad 0.269501 \quad 0.805367] \\ \mathbf{B}_{22} &= [-0.658465 \quad 0.037772 \quad 0.269501 \quad 0.805367] \\ \mathbf{B}_2 &= 10^{-3}[\mathbf{B}_{21} \quad \mathbf{B}_{22}]^T \\ \mathbf{C}_1 &= [0.983681 \quad 0.501644 \quad 0.985510 \quad 0.503879] \\ \mathbf{C}_2 &= [-1 \quad 0 \quad 1 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0] \\ \mathbf{D} &= 10^{-3} \times 1.34044. \end{aligned}$$

The amplitude response of the original filter with order of (4, 8) is depicted in Fig. 3. First, the unweighted model reduction (UWMR) with

$$\mathbf{W}_i(z_1, z_2) = \mathbf{I} \text{ and } \mathbf{W}_o(z_1, z_2) = \mathbf{I}$$

is carried out to obtain a (4, 4) reduced-order filter, characterized by $\mathbf{H}_{uw}(z_1, z_2)$, whose state-space model is given by (1a) with

$$\mathbf{A}_{uw1} = \begin{bmatrix} 0.398395 & 0.067647 & -0.041205 & -0.047699 \\ -0.165376 & 0.097911 & 0.056528 & 0.057307 \\ 0.054464 & -0.386181 & 0.439015 & 0.198796 \\ -0.300009 & 0.004979 & -0.079289 & 0.140498 \end{bmatrix}$$

$$\mathbf{A}_{uw2} = \begin{bmatrix} -0.493470 & 0.111113 & 0.433265 & -0.097438 \\ 0.164658 & -0.074752 & 0.539074 & 0.478618 \\ -0.106099 & 0.031564 & -0.046091 & -0.111813 \\ -0.328834 & 0.085950 & 0.072648 & -0.205922 \end{bmatrix}$$

$$\mathbf{A}_{uw3} = \begin{bmatrix} -0.503071 & 0.027229 & 0.054018 & -0.078483 \\ 0.034454 & -0.185564 & 0.310070 & 0.350054 \\ 0.042530 & 0.446818 & -0.018492 & 0.407874 \\ 0.027084 & 0.270648 & 0.381488 & -0.371722 \end{bmatrix}$$

$$\mathbf{A}_{uw4} = \begin{bmatrix} 0.340191 & 0.384383 & 0.158443 & -0.101164 \\ 0.000524 & 0.366455 & 0.024461 & -0.015399 \\ 0.113994 & -0.139845 & 0.034981 & -0.044875 \\ -0.045414 & 0.299108 & -0.065030 & 0.087856 \end{bmatrix}$$

$$\begin{aligned} \mathbf{B}_{uw1} &= [0.304264 \quad 0.359372 \quad -0.028458 \quad 0.057085]^T \\ \mathbf{B}_{uw2} &= [-0.150370 \quad -0.189039 \quad 0.230790 \quad 0.157800]^T \\ \mathbf{C}_{uw1} &= [0.100857 \quad -0.057272 \quad 0.132552 \quad -0.064706] \\ \mathbf{C}_{uw2} &= [-0.398981 \quad 0.357827 \quad -0.336220 \quad 0.456789] \\ \mathbf{D}_{uw} &= 10^{-3} \times 1.34044. \end{aligned}$$

It is easy to verify that the (4, 4) reduced-order transfer function $\mathbf{H}_{uw}(z_1, z_2)$ is Q-stable. The amplitude response is shown in Fig. 4. Next, a (4, 4) Q-stable lowpass filter is used as the input weight $\mathbf{W}_i(z_1, z_2)$ to emphasize the reduced-order system over the low frequency region. This input weight is modeled by (1a) with

$$\mathbf{A}_{i1} = \begin{bmatrix} 0.865266 & -0.320202 & -0.038859 & -0.029287 \\ 0.320207 & 0.669571 & -0.345662 & -0.047159 \\ -0.038856 & 0.345681 & 0.488474 & -0.405720 \\ 0.029288 & -0.047162 & 0.405698 & 0.359184 \end{bmatrix}$$

$$\mathbf{A}_{i2} = \begin{bmatrix} 0.376026 & 0.327476 & 0.168483 & 0.057372 \\ -0.329114 & -0.286621 & -0.147464 & -0.050214 \\ 0.168784 & 0.146991 & 0.075626 & 0.025752 \\ -0.057005 & -0.049645 & -0.025542 & -0.008697 \end{bmatrix}$$

$$\mathbf{A}_{i3} = \begin{bmatrix} 0.002027 & 0.002826 & 0.003092 & 0.002087 \\ -0.002002 & -0.003350 & -0.004149 & -0.002547 \\ 0.000960 & 0.002303 & 0.003304 & 0.001774 \\ -0.001168 & -0.002485 & -0.002790 & -0.000828 \end{bmatrix}$$

$$\mathbf{A}_{i4} = \begin{bmatrix} 0.865687 & -0.320519 & -0.039609 & -0.029425 \\ 0.320210 & 0.669343 & -0.346448 & -0.046520 \\ -0.038457 & 0.345556 & 0.488348 & -0.403728 \\ 0.029377 & -0.047308 & 0.407151 & 0.359117 \end{bmatrix}$$

$$\begin{aligned} \mathbf{B}_{i1} &= [-0.014868 \quad 0.013013 \quad -0.006674 \quad 0.002254]^T \\ \mathbf{B}_{i2} &= [-0.263117 \quad 0.228922 \quad -0.118278 \quad 0.039798]^T \\ \mathbf{C}_{i1} &= [-0.263607 \quad -0.228362 \quad -0.118098 \quad -0.039582] \\ \mathbf{C}_{i2} &= [-0.014877 \quad -0.012956 \quad -0.006666 \quad -0.002270] \\ \mathbf{D}_i &= 10^{-4} \times 5.8826. \end{aligned}$$

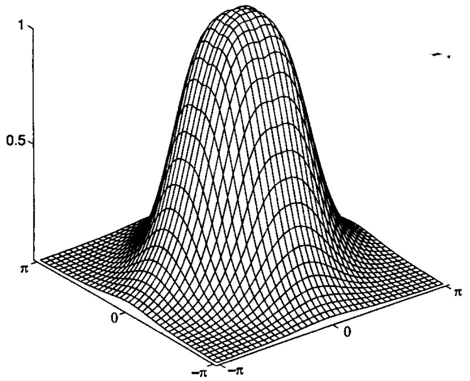

Fig. 3. Amplitude response of the original (4, 8) filter.

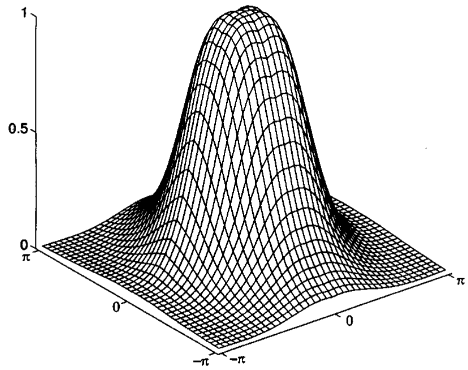

Fig. 4. Amplitude response of the (4, 4) filter obtained from UWMR.

The amplitude response of the input weight is depicted in Fig. 5. The WBMR with input weight $\mathbf{W}_i(z_1, z_2)$ and unit output weight

$$\mathbf{W}_o(z_1, z_2) = \mathbf{I}$$

leads to a weighted (4, 4) transfer function $\mathbf{H}_w(z_1, z_2)$. The state-space model of the filter is given by (1a) with

$$\mathbf{A}_{w1} = \begin{bmatrix} 0.377205 & -0.029050 & -0.031851 & -0.020335 \\ -0.263009 & 0.233722 & 0.303009 & 0.169255 \\ 0.172187 & 0.021959 & 0.430882 & 0.129212 \\ -0.180399 & -0.077882 & -0.287881 & 0.034010 \end{bmatrix}$$

$$\mathbf{A}_{w2} = \begin{bmatrix} -0.479988 & 0.458503 & 0.293809 & -0.119313 \\ -0.100826 & 0.195997 & 0.250964 & 0.148250 \\ -0.093182 & -0.143636 & -0.384634 & -0.427647 \\ -0.229037 & 0.021276 & -0.234765 & -0.400324 \end{bmatrix}$$

$$\mathbf{A}_{w3} = \begin{bmatrix} -0.486739 & 0.068631 & 0.073475 & -0.035764 \\ 0.174899 & 0.439104 & 0.094213 & 0.276255 \\ -0.054904 & 0.241124 & -0.424258 & 0.004268 \\ 0.031853 & 0.135682 & 0.104322 & -0.530429 \end{bmatrix}$$

$$\mathbf{A}_{w4} = \begin{bmatrix} 0.400167 & 0.430839 & -0.045702 & -0.098613 \\ 0.051712 & 0.162037 & -0.060519 & -0.023039 \\ 0.095767 & -0.273725 & 0.226859 & -0.057916 \\ 0.040648 & 0.232954 & -0.170824 & 0.078143 \end{bmatrix}$$

$$\mathbf{B}_{w1} = [0.277210 \quad 0.201779 \quad -0.281204 \quad -0.152142]^T$$
$$\mathbf{B}_{w2} = [-0.157826 \quad -0.010694 \quad 0.263263 \quad 0.142137]^T$$
$$\mathbf{C}_{w1} = [0.129210 \quad -0.026848 \quad 0.146844 \quad -0.088941]^T$$
$$\mathbf{C}_{w2} = [-0.419206 \quad 0.103558 \quad -0.591489 \quad 0.448008]^T$$
$$\mathbf{D}_w = 10^{-3} \times 1.34044.$$

This weighted (4, 4) transfer function $\mathbf{H}_w(z_1, z_2)$ is also Q-stable. The amplitude response is shown in Fig. 6. The contour plots for the original filter, the reduced-order (4, 4) filter by the UWMR, and the reduced-order (4, 4) filter using WBMR are given in Figs. 7–10.

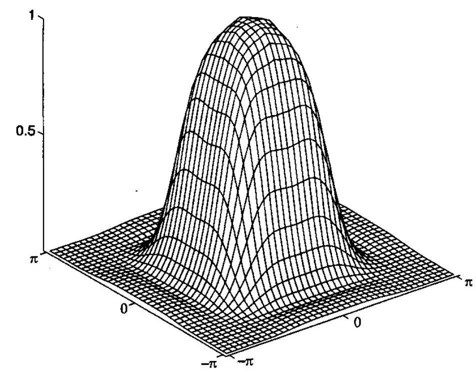

Fig. 5. Amplitude response of the input weight.

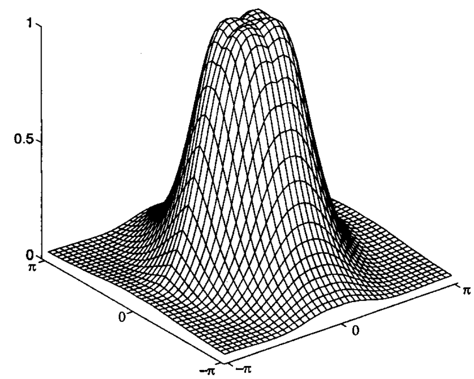

Fig. 6. Amplitude response of the (4, 4) filter obtained from WBMR.

Defining the error matrices of the UWMR and WBMR by

$$\mathbf{E}_{uw} = (E_{uwkl}) \text{ and } \mathbf{E}_w = (E_{wkl})$$

as

$$E_{uw_{kl}} = |\, \mathbf{H}(e^{j2\pi k/K}, e^{j2\pi l/L}) - \mathbf{H}_{uw}(e^{j2\pi k/K}, e^{j2\pi l/L}) \,| \tag{30a}$$

$$E_{w_{kl}} = |\, \mathbf{H}(e^{j2\pi k/K}, e^{j2\pi l/L}) - \mathbf{H}_w(e^{j2\pi k/K}, e^{j2\pi l/L}) \,| \tag{30b}$$

with $k = 0, 1, \cdots, K$ and $l = 0, 1, \cdots, L$ where K and L are the numbers of frequency response samples in the normalized

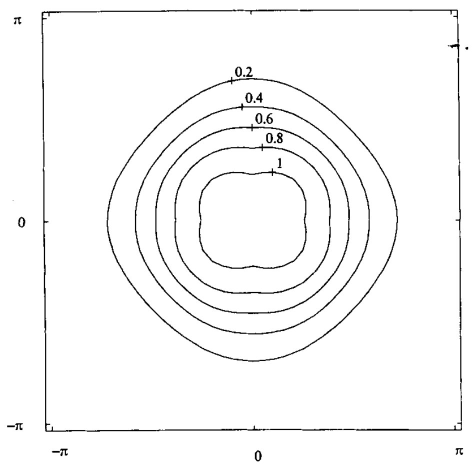

Fig. 7. Contour plot of the original (4, 8) filter.

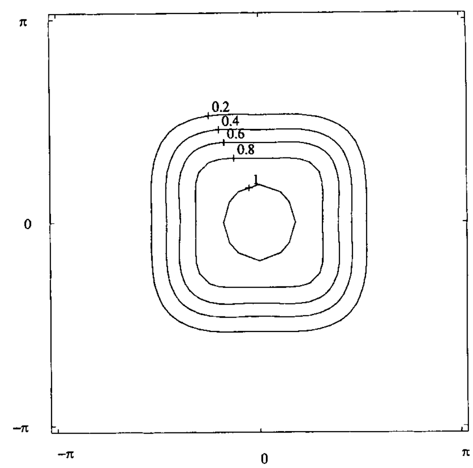

Fig. 9. Contour plot of the input weight.

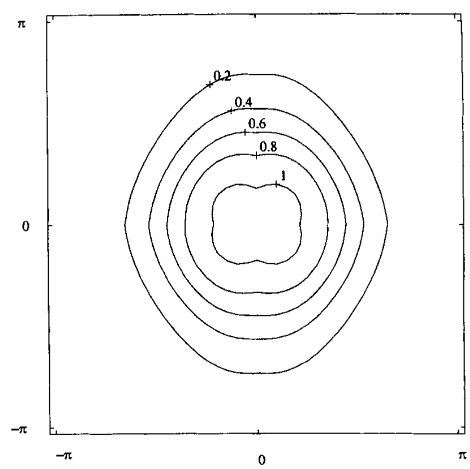

Fig. 8. Contour plot of the (4, 4) filter obtained from UWMR.

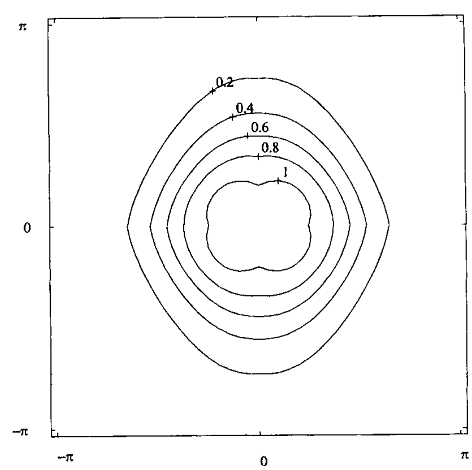

Fig. 10. Contour plot of the (4, 4) filter obtained from WBMR.

frequency ranges $\omega_1, \omega_2 \in [0, 1.0]$ ($\omega_1 = \frac{k}{K}$, $\omega_2 = \frac{l}{L}$). The Frobenius and l_∞ norms of $\mathbf{E}_{uw}$ and $\mathbf{E}_w$ are given by

$$e_f = ||\mathbf{E}_{uw}||_F = \left(\sum_{k=1}^{K}\sum_{l=1}^{L} E_{uw_{kl}}^2\right)^{1/2},$$

$$e_{wf} = ||\mathbf{E}_w||_F = \left(\sum_{k=1}^{K}\sum_{k=1}^{L} E_{w_{kl}}^2\right)^{1/2};$$

$$e_\infty = ||\mathbf{E}_{uw}||_\infty = \max_{1\le k\le K, 1\le l\le L} E_{uw_{kl}},$$

$$e_{w\infty} = ||\mathbf{E}_w||_\infty = \max_{1\le k\le K, 1\le l\le L} E_{w_{kl}}.$$

The errors of the UWMR and WBMR have been computed in three frequency ranges and are summarized in Table I, where ω denotes the normalized frequency $\omega_1 = \omega_2 = \omega$.

As is expected, Table I shows that the error e_{wf} from WBMR is smaller than the error e_f from UWMR in the low and intermediate frequency ranges but it is larger than the error from UWMR in the high frequency range due to the use of the lowpass input weight.

TABLE I
ERRORS OF THE UNWEIGHTED AND WEIGHTED MODEL REDUCTION

$r_1 = r_2 = 4$	$\omega \in [0, 0.30]$	$\omega \in [0.30, 0.55]$	$\omega \in [0.55, 1.0]$	$\omega \in [0, 1.0]$
e_f	0.4214	1.2775	3.0288	3.3141
e_{wf}	0.2684	1.2470	3.6109	3.8295
e_∞	0.0562	0.1440	0.1595	0.1595
$e_{\omega\infty}$	0.0551	0.1662	0.1891	0.1891

If the weighted error matrices of the UWMR and WBMR are defined by

$$\mathbf{E}_{uw}^w = (E_{uwkl}^w) \text{ and } \mathbf{E}_w^w = (E_{wkl}^w)$$

where

$$\begin{aligned} E_{uw_{kl}}^w &= |\,\mathbf{W}_o(e^{j2\pi k/K}, e^{j2\pi l/L})\,[\,\mathbf{H}(e^{j2\pi k/K}, e^{j2\pi l/L}) \\ &\quad - \mathbf{H}_{uw}(e^{j2\pi k/K}, e^{j2\pi l/L})\,]\,\mathbf{W}_i(e^{j2\pi k/K}, e^{j2\pi l/L})\,| \\ E_{w_{kl}}^w &= |\,\mathbf{W}_o(e^{j2\pi k/K}, e^{j2\pi l/L})\,[\,\mathbf{H}(e^{j2\pi k/K}, e^{j2\pi l/L}) \\ &\quad - \mathbf{H}_w(e^{j2\pi k/K}, e^{j2\pi l/L})\,]\,\mathbf{W}_i(e^{j2\pi k/K}, e^{j2\pi l/L})\,| \end{aligned}$$

TABLE II
WEIGHTED ERRORS OF THE UNWEIGHTED AND WEIGHTED MODEL REDUCTIONS

$r_1 = r_2 = 4$	$\omega \in [0, 0.30]$	$\omega \in [0.30, 0.55]$	$\omega \in [0.55, 1.0]$	$\omega \in [0, 1.0]$
e_f^w	0.4248	0.8793	0.5751	1.1333
e_{wf}^w	0.2744	0.8221	0.6713	1.0963
e_∞^w	0.0582	0.0911	0.0839	0.0911
$e_{w\infty}^w$	0.0572	0.1057	0.0990	0.1057

then, the Frobenius and l_∞ norms of $\mathbf{E}_{uw}^w$ and $\mathbf{E}_w^w$ are given by

$$e_f^w = \|\mathbf{E}_{uw}^w\|_F = \left[\sum_{k=1}^{K}\sum_{l=1}^{L}(E_{uw_{kl}}^w)^2\right]^{1/2},$$
$$e_\infty^w = \|\mathbf{E}_{uw}^w\|_\infty = \max_{1\le k\le K, 1\le l\le L} E_{uw_{kl}}^w;$$
$$e_{wf}^w = \|\mathbf{E}_w^w\|_F = \left[\sum_{k=1}^{K}\sum_{l=1}^{L}(E_{w_{kl}}^w)^2\right]^{1/2},$$
$$e_{w\infty}^w = \|\mathbf{E}_w^w\|_\infty = \max_{1\le k\le K, 1\le l\le L} E_{w_{kl}}^w.$$

The weighted errors of the UWMR and WBMR in three frequency ranges in this case are given in Table II.

From Table II, it is noted that the weighted error e_{wf}^w from WBMR is smaller than the weighted error e_f^w from UWMR in the low and intermediate frequency ranges but it is larger than that from UWMR in the high frequency range due to the input weighting chosen.

VI. CONCLUSIONS

A weighted structurally balanced approximation for 2-D discrete systems has been proposed which is based on two 2-D Lyapunov inequalities, and an algorithm for computing the inherent weighted controllability and observability Gramians of the 2-D system has been developed. Then a corresponding weighted structurally balanced model reduction has been proposed, which takes into account both the stability issue and the approximation error.

It has been shown that if one starts with a Q-stable 2-D transfer-function matrix with either unit input weight or unit output weight, the reduced-order system is also Q-stable. It has also been shown that if there exist block-diagonal matrices $\hat{\mathbf{P}}_1$ and $\hat{\mathbf{P}}_2$ in (5a) that satisfy the 2-D Lyapunov inequality (6a) (or block-diagonal matrices $\hat{\mathbf{Q}}_1$ and $\hat{\mathbf{Q}}_2$ in (5b) that satisfy the 2-D Lyapunov inequality (6b), then, the reduced-order system is also Q-stable. A (4, 8)-order lowpass filter has been used to demonstrate that the error of the weighted reduced-order filter can be reduced relative to the corresponding unweighted reduced-order filter in a desirable frequency range.

APPENDIX A

This appendix presents a proof of the Q-stability of the weighted-input-to-state-output and state-input-to-weighted-output transfer-function matrices $\mathbf{H}_i(z_1, z_2)$ and $\mathbf{H}_o(z_1, z_2)$ defined by (3a) and (3b) in order to justify the existence of the balanced approximation of the 2-D system described in Section III.

Let $\mathbf{H}(z_1, z_2)$ be a 2-D Q-stable transfer-function matrix and assume that output weight $\mathbf{W}_o(z_1, z_2)$ satisfies the 2-D Lyapunov equations (2a) and (2b). By Lemma 4 of [7], (2a) and (2b) imply that there exist block-diagonal matrices

$$\mathbf{Y} = \mathbf{Y}_1 \oplus \mathbf{Y}_2 > \mathbf{0} \text{ and } \mathbf{Y}_i = \mathbf{Y}_{i1} \oplus \mathbf{Y}_{i2} > \mathbf{0}$$

such that

$$\mathbf{\Psi} = \mathbf{AYA}^T - \mathbf{Y} + \mathbf{BB}^T < \mathbf{0} \tag{A.1}$$

and

$$\mathbf{\Psi}_i = \mathbf{A}_i\mathbf{Y}_i\mathbf{A}_i^T - \mathbf{Y}_i + \mathbf{B}_i\mathbf{B}_i^T < \mathbf{0}. \tag{A.2}$$

Define

$$\hat{\mathbf{Y}} = \left[\begin{array}{cc|cc} \mathbf{Y}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_{i1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Y}_{i2} \end{array}\right] = \hat{\mathbf{Y}}_1 \oplus \hat{\mathbf{Y}}_2 \tag{A.3}$$

and

$$\hat{\mathbf{\Psi}} = \hat{\mathbf{A}}_i\hat{\mathbf{Y}}\hat{\mathbf{A}}_i^T - \hat{\mathbf{Y}} + \hat{\mathbf{B}}_i\hat{\mathbf{B}}_i^T \tag{A.4}$$

It is found that

$$\mathbf{X}_1\hat{\mathbf{\Psi}}\mathbf{X}_1^T = \begin{bmatrix} \mathbf{AYA}^T - \mathbf{Y} + \mathbf{B}(\mathbf{C}_i\mathbf{Y}_i\mathbf{C}_i^T + \mathbf{D}_i\mathbf{D}_i^T)\mathbf{B}^T & \mathbf{BK} \\ \mathbf{K}^T\mathbf{B}^T & \mathbf{\Psi}_i \end{bmatrix} \tag{A.5}$$

where

$$\mathbf{X}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathbf{K} = \mathbf{C}_i\mathbf{Y}_i\mathbf{A}_i^T + \mathbf{D}_i\mathbf{B}_i^T$$

and $\mathbf{\Psi}_i$ is defined by (A.2). As $\mathbf{\Psi}_i$ is nonsingular, $\mathbf{X}_1\hat{\mathbf{\Psi}}\mathbf{X}_1^T$ in (A.5) can be block-diagonalized by pre-multiplying it with

$$\mathbf{X}_2 = \begin{bmatrix} \mathbf{I} & -\mathbf{BK}\mathbf{\Psi}_i^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

and then post-multiplying with $\mathbf{X}_2^T$, leading to

$$\mathbf{X}_2(\mathbf{X}_1\hat{\mathbf{\Psi}}\mathbf{X}_1^T)\mathbf{X}_2^T = \begin{bmatrix} \mathbf{AYA}^T - \mathbf{Y} + \mathbf{B}\hat{\mathbf{K}}\mathbf{B}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{\Psi}_i \end{bmatrix}$$

where

$$\hat{\mathbf{K}} = \mathbf{C}_i\mathbf{Y}_i\mathbf{C}_i^T + \mathbf{D}_i\mathbf{D}_i^T - \mathbf{K}\mathbf{\Psi}_i^{-1}\mathbf{K}^T. \tag{A.6}$$

As $\mathbf{\Psi}_i < \mathbf{0}$, matrix $\mathbf{X}_2(\mathbf{X}_1\hat{\mathbf{\Psi}}\mathbf{X}_1^T)\mathbf{X}_2^T$ is negative definite (and hence $\hat{\mathbf{\Psi}} < \mathbf{0}$) if and only if

$$\mathbf{AYA}^T - \mathbf{Y} + \mathbf{B}\hat{\mathbf{K}}\mathbf{B}^T < \mathbf{0}. \tag{A.7}$$

With a $\mathbf{Y}_i$ that satisfies (A.2) fixed, matrix $\hat{\mathbf{K}}$ in (A.6) is a known matrix and $\|\hat{\mathbf{K}}\| \le \beta$ for some $\beta > \mathbf{0}$. From (A.1) it follows that for any scalar $\alpha > 1$

$$\mathbf{A}(\alpha\mathbf{Y})\mathbf{A}^T - \alpha\mathbf{Y} + \mathbf{B}\hat{\mathbf{K}}\mathbf{B}^T = \alpha\mathbf{\Psi} + \mathbf{B}(\hat{\mathbf{K}} - \alpha\mathbf{I})\mathbf{B}^T \le \alpha\mathbf{\Psi} + \mathbf{B}\hat{\mathbf{K}}\mathbf{B}^T.$$

Since $\mathbf{\Psi} < \mathbf{0}$, if α is chosen such that

$$\alpha > \frac{\beta\,\|\mathbf{B}\|^2}{|\rho_{\max}(\mathbf{\Psi})|} \tag{A.8}$$

then we have

$$\mathbf{A}(\alpha\mathbf{Y})\mathbf{A}^T - \alpha\mathbf{Y} + \mathbf{B}\hat{\mathbf{K}}\mathbf{B}^T < \mathbf{0}. \quad \text{(A.9)}$$

In other words, (A.7) holds if $\mathbf{Y}$ is scaled to $(\alpha\mathbf{Y})$ as is seen in (A.9). We conclude that

$$\hat{\mathbf{A}}_i\hat{\mathbf{P}}\hat{\mathbf{A}}_i^T - \hat{\mathbf{P}} + \hat{\mathbf{B}}_i\hat{\mathbf{B}}_i^T < \mathbf{0}$$

where

$$\hat{\mathbf{P}} = \left[\begin{array}{cc|cc} \alpha\mathbf{Y}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_{i1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \alpha\mathbf{Y}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Y}_{i2} \end{array}\right]$$

with α satisfying (A.8) and, therefore, $\mathbf{H}_i(z_1, z_2)$ is 2-D Q-stable. Likewise, by using (2a) and (2c) in conjunction with an argument similar to the above, one can show that $\mathbf{H}_o(z_1, z_2)$ is also a Q-stable 2-D transfer-function matrix.

Appendix B

In this appendix, we show that if system (3a) is Q-stable, then there exist unique positive definite matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ that satisfy (13a) and (13b). For the sake of simplicity, we assume that

$$\|\hat{\mathbf{A}}_i\| = \delta < 1, \quad \hat{\mathbf{B}}_{i1}\hat{\mathbf{B}}_{i1}^T = \mathbf{I}, \quad \hat{\mathbf{B}}_{i2}\hat{\mathbf{B}}_{i2}^T = \mathbf{I}.$$

Equations (13a) and (13b) can then be written as

$$[\hat{\mathbf{A}}_{i1} \quad \hat{\mathbf{A}}_{i2}]\begin{bmatrix}\mathbf{P}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_2\end{bmatrix}\begin{bmatrix}\hat{\mathbf{A}}_{i1}^T\\ \hat{\mathbf{A}}_{i2}^T\end{bmatrix} - \mathbf{P}_1 = -\mathbf{I} \quad \text{(B.1a)}$$

$$[\hat{\mathbf{A}}_{i3} \quad \hat{\mathbf{A}}_{i4}]\begin{bmatrix}\mathbf{P}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_2\end{bmatrix}\begin{bmatrix}\hat{\mathbf{A}}_{i3}^T\\ \hat{\mathbf{A}}_{i4}^T\end{bmatrix} - \mathbf{P}_2 = -\mathbf{I}. \quad \text{(B.1b)}$$

From (B.1a) and (B.1b), a recursive formula for solving (B.1a) and (B.1b) is suggested as follows:

$$\mathbf{P}_1^{(k)} = \mathbf{I} + [\hat{\mathbf{A}}_{i1} \quad \hat{\mathbf{A}}_{i2}]\begin{bmatrix}\mathbf{P}_1^{(k-1)} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_2^{(k-1)}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{A}}_{i1}^T\\ \hat{\mathbf{A}}_{i2}^T\end{bmatrix} \quad \text{(B.2a)}$$

$$\mathbf{P}_2^{(k)} = \mathbf{I} + [\hat{\mathbf{A}}_{i3} \quad \hat{\mathbf{A}}_{i4}]\begin{bmatrix}\mathbf{P}_1^{(k-1)} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_2^{(k-1)}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{A}}_{i3}^T\\ \hat{\mathbf{A}}_{i4}^T\end{bmatrix} \quad \text{(B.2b)}$$

where

$$k = 1, 2, \cdots, \text{ with } \mathbf{P}_1^{(0)} = \mathbf{I} \text{ and } \mathbf{P}_2^{(0)} = \mathbf{I}.$$

Obviously, $\mathbf{P}_1^{(k)}$ and $\mathbf{P}_2^{(k)}$ can be expressed as

$$\mathbf{P}_1^{(k)} = \sum_{l=0}^{k}\mathbf{F}_1^{(l)} \text{ and } \mathbf{P}_2^{(k)} = \sum_{l=0}^{k}\mathbf{F}_2^{(l)} \quad \text{(B.3)}$$

where

$$\mathbf{F}_1^{(l)} = [\hat{\mathbf{A}}_{i1} \quad \hat{\mathbf{A}}_{i2}]\begin{bmatrix}\mathbf{F}_1^{(l-1)} & \mathbf{0}\\ \mathbf{0} & \mathbf{F}_2^{(l-1)}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{A}}_{i1}^T\\ \hat{\mathbf{A}}_{i2}^T\end{bmatrix} \quad \text{(B.4a)}$$

$$\mathbf{F}_2^{(l)} = [\hat{\mathbf{A}}_{i3} \quad \hat{\mathbf{A}}_{i4}]\begin{bmatrix}\mathbf{F}_1^{(l-1)} & \mathbf{0}\\ \mathbf{0} & \mathbf{F}_2^{(l-1)}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{A}}_{i3}^T\\ \hat{\mathbf{A}}_{i4}^T\end{bmatrix} \quad \text{(B.4b)}$$

with $\mathbf{F}_1^{(0)} = \mathbf{I}$ and $\mathbf{F}_2^{(0)} = \mathbf{I}$. Since

$$\|[\hat{\mathbf{A}}_{i1} \quad \hat{\mathbf{A}}_{i2}]\| \le \|\hat{\mathbf{A}}_i\| = \delta < 1$$
$$\|[\hat{\mathbf{A}}_{i3} \quad \hat{\mathbf{A}}_{i4}]\| \le \|\hat{\mathbf{A}}_i\| = \delta < 1$$

it follows that

$$\|\mathbf{F}_1^{(l)}\| \le \delta^l, \ \|\mathbf{F}_2^{(l)}\| \le \delta^l$$

and hence

$$\|\mathbf{P}_1^{(k)}\| \le \frac{1}{1-\delta}, \ \|\mathbf{P}_2^{(k)}\| \le \frac{1}{1-\delta}. \quad \text{(B.5)}$$

By (B.4a) and (B.4b), each $\mathbf{F}_1^{(l)}$ and $\mathbf{F}_2^{(l)}$ is positive semi-definite and, therefore, $\mathbf{P}_1^{(k)}$ and $\mathbf{P}_2^{(k)}$ are positive definite and the two sequences

$$\mathbf{P}_1^{(0)} \le \mathbf{P}_1^{(1)} \le \cdots \le \mathbf{P}_1^{(k-1)} \le \mathbf{P}_1^{(k)} \le \cdots$$
$$\mathbf{P}_2^{(0)} \le \mathbf{P}_2^{(1)} \le \cdots \le \mathbf{P}_2^{(k-1)} \le \mathbf{P}_2^{(k)} \le \cdots$$

are monotonically increasing and bounded because of (B.5). From (B.2a) and (B.2b), it is observed that the limits of these two sequences, which are given by

$$\mathbf{P}_1 = \sum_{l=0}\mathbf{F}_1^{(l)} \text{ and } \mathbf{P}_2 = \sum_{l=0}\mathbf{F}_2^{(l)}$$

satisfy (B.1a) and (B.1b).

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Hong Luo (S'90) received the B.Sc. and M.Sc. degrees from Shanghai Jiao Tong University, P.R. China, in mechanical engineering, in 1983 and 1986, respectively.

She is now a Ph.D. candidate with the Department of Electrical and Computer Engineering, University of Victoria, Canada. Her current research interests include digital signal and image processing, model reduction methods, and communication applications.

Wu-Sheng Lu (S'81–M'85–SM'90) received the B.S. degree in mathematics from Fudan University, China in 1964, the M.S. degree in electrical engineering, and the Ph.D. degree in control science from the University of Minnesota, Minneapolis, in 1983 and 1984, respectively.

He was a post-Doctoral Fellow at the University of Victoria, Victoria, B.C., Canada, in 1985 and held a visiting assistant professorship at the University of Minnesota in 1986. Since 1987, he has been with the University of Victoria where he is currently Professor. His teaching and research interests are in the areas of digital signal processing and robotics. He is the co-author with A. Antoniou of *Two-Dimensional Digital Filters* (Marcel Dekker). He served as Associate Editor of the *Canadian Journal of Electrical and Computer Engineering* from 1989 to 1990 and was Editor of the same journal from 1990 to 1992. He is presently an Associate Editor of the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS—PART II.

Andreas Antoniou (M'69–SM'79–F'82) received the B.Sc. (Eng.) and Ph.D. degrees in electrical engineering from London University in 1963 and 1966, respectively.

From 1966 to 1969 he was Senior Scientific Officer at the Post Office Research Department, London, and from 1969 to 1970, he was a member of the Scientific Staff at the R&D Laboratories of Northern Electric Company Ltd., Ottawa, Ontario, Canada. From 1970 to 1983 he was with the Department of Electrical and Computer Engineering, Concordia University, Montreal, Quebec, Canada, as Professor from June 1973 and as Chairman from December 1977. He served as founding Chairman of the Department of Electrical and Computer Engineering, University of Victoria, Victoria, British Columbia, Canada, from July 1, 1983 to June 30, 1990 and is now Professor in the same department. His teaching and research interests are in the areas of electronics, network synthesis, digital system design, active and digital filters, and digital signal processing. He published extensively in these areas. One of his papers on gyrator circuits was awarded the Ambrose Fleming Premium by the Institution of Electrical Engineers, UK. He is the author of *Digital Filters: Analysis, Design, and Applications* (McGraw-Hill) and the co-author with W.-S. Lu of *Two-Dimensional Digital Filters* (Marcel Dekker).

Dr. Antoniou is a Member of the Association of Professional Engineers and Geoscientists of B.C., and a Fellow of the Institution of Electrical Engineers. He was elected Fellow of the Institute of Electrical and Electronics Engineers for contributions to active and digital filters, and to electrical engineering education. He served as Associate Editor, IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS from June 1983 to May 1985 and as Editor from June 1985 to May 1987, and is currently serving as a member of the Board of Governors of the Circuits and Systems Society.