III. Output Noise and the 2-D Noise Matrix

Under zero initial conditions, the formula to compute the output noise $e_y(i,j)$ of (2) is given [1, eq. (11)] by

$$e_y(k,l)=C\left(\sum_{i=0}^{k}\sum_{j=0}^{l}\bar{A}(i,j)\,w_{AB}(k-i,l-j)\right)+w_{CD}(k,l) \tag{3}$$

where

$$\bar{A}(i,j)=A^{i-1,j}\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix}+A^{i,j-1}\begin{bmatrix} 0 & 0\\ 0 & I_{n_2}\end{bmatrix} \tag{4}$$

and the matrix $A^{i,j}$ (denoted by A_{ij} in [2]) is recursively computed as follows [3]:

$$A^{1,0}=\begin{bmatrix} A_1 & A_2\\ 0 & 0\end{bmatrix},\quad A^{0,1}=\begin{bmatrix} 0 & 0\\ A_3 & A_4\end{bmatrix} \tag{5a}$$

$$A^{i,j}=A^{1,0}A^{i-1,j}+A^{0,1}A^{i,j-1},\quad (i,j)>(0,0) \tag{5b}$$

$$A^{0,0}=I \tag{5c}$$

$$A^{-i,j}=A^{i,-j}=0,\quad \text{for } i>0,\ j>0. \tag{5d}$$

Since $\bar{A}(0,0)=0$, (3) can be written as

$$e_y(k,l)=C\left(\sum\sum_{(0,0)<(i,j)\leqslant(k,l)}\left(A^{i-1,j}\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix}+A^{i,j-1}\begin{bmatrix} 0 & 0\\ 0 & I_{n_2}\end{bmatrix}\right)w_{AB}(k-i,l-j)\right)+w_{CD}(k,l). \tag{6}$$

To derive (3) or (6), we only need to follow the same procedure as given in [3] to derive the *general response formula* for $y(k,l)$ of (1) [3, p. 3].

On the other hand, the formula given in [2, eq. (13)] to compute the output noise is

$$y(k,l)=C\left(\sum\sum_{(0,0)<(i,j)\leqslant(k,l)}\left(A^{i-1,j}\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix}w_{AB}\cdot(k-i-1,l-j)+A^{i,j-1}\begin{bmatrix} 0 & 0\\ 0 & I_{n_2}\end{bmatrix}w_{AB}\cdot((k-i,l-j-1))\right)\right)+w_{CD}(k,l). \tag{7}$$

Obviously, (7) is incorrect and both $w_{AB}(k-i-1,l-j)$ and $w_{AB}(k-i,l-j-1)$ should be $w_{AB}(k-i,l-j)$.

From (3), we can obtain the 2-D noise matrix [1, eq. (17)]:

$$W=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{A}^t(i,j)C^tC\bar{A}(i,j). \tag{8}$$

On the other hand, (7) results in the following incorrect formula for the 2-D noise matrix [2, eq. (18)]:

$$W=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(A^{i,j})^tC^tCA^{i,j}. \tag{9}$$

It can be easily seen that the 2-D noise matrix W given by (8) has the dual form to the 2-D covariance matrix [1, eq. (22)], [2, eq. (24)]:

$$K=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{A}(i,j)BB^t\bar{A}^t(i,j). \tag{10}$$

while the matrix W given by (9) does not.

IV. Concluding Remarks

It has been shown that the derivation of the formulas to compute the output noise and the 2-D noise matrix given in [2] is incorrect. Moreover, the formula for the 2-D noise matrix obtained in [2] does not have the dual form to the formula for the 2-D covariance matrix.

It should be noted that from the viewpoint of 2-D system theory, the 2-D noise matrix (2-D observability Gramian) W and the 2-D covariance matrix (2-D controllability Gramian) K are related to 2-D local observability and 2-D local controllability which are dual notions to each other. Therefore, the duality between W and K is reasonable. Moreover, the duality makes any algorithms for computing K (such as the algorithms bsed on the contour integral [1] and the frequency dependent Lyapunov equation [4]) be able to be applied directly to computing W.

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Author's Reply on the 2-D Unit Noise Matrix

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We thank Lin and Kawamata (pp. 610–611, this issue) and Hinamoto, Hamanaka, and Maekawa (pp. 609–610, this issue) for their interest in the results reported in [1].

In the one-dimensional (1-D) case, matrix W_1 is defined as

$$W_1=\sum_{i=0}^{\infty}g_i^tg_i \tag{1}$$

where $\{g_i=cA^i,\ i=0,1,2,\cdots\}$ is the impulse response from state to output [2]. A general analysis of roundoff noise in 1-D state-space digital filters was carried out by Kwang [3], [4], yielding the same matrix W_1 given by (1) (see (13) of [4]). With a few minor modifications, the derivation of the 2-D noise model

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and the computation of the output-noise power given by [1] are essentially the same as those reported in [3], [4].

A natural 2-D extension of matrix W_1 was proposed by Metzios [5] as

$$W=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g'(i,j)g(i,j) \tag{2a}$$

with

$$g(i,j)=cA^{i,j}, \qquad i,j \geqslant 0 \tag{2b}$$

representing the impulse response of the 2-D filter from state to output.

After re-examination of our noise model, the random error $\alpha(i,j)+\beta(i,j)$ generated by product quantization can also be expressed as

$$\begin{bmatrix} \tau_1(i+1,j) \\ \tau_2(i,j+1) \end{bmatrix} \tag{3}$$

where $\tau_1(i+1,j)$ and $\tau_2(i,j+1)$ are the random error generated in the computations of $v(i+1,j)$ and $h(i,j+1)$, respectively. Evidently, the error vector given by (3) reflects the nature of error propagation in a 2-D digital filter. With this modification, the error model in (12a) of [1] should be written as

$$\begin{bmatrix} \Delta v(i,j) \\ \Delta h(i,j) \end{bmatrix} = A_{10}\begin{bmatrix} \Delta v(i-1,j) \\ \Delta h(i-1,j) \end{bmatrix} + A_{01}\begin{bmatrix} \Delta v(i,j-1) \\ \Delta h(i,j-1) \end{bmatrix} + \begin{bmatrix} \tau_1(i,j) \\ \tau_2(i,j) \end{bmatrix}$$

or

$$\delta(i,j)=A_{10}\delta(i-1,j)+A_{01}\delta(i,j-1)+\tau(i,j) \tag{4}$$

where

$$\delta(i,j)=\begin{bmatrix} \Delta v(i,j) \\ \Delta h(i,j) \end{bmatrix} \quad \text{and} \quad \tau(i,j)=\begin{bmatrix} \tau_1(i,j) \\ \tau_2(i,j) \end{bmatrix}.$$

It now follows from (4) that equations (16)–(18) of [1] can readily be obtained where, (18) is the same as equation (18) of [5].

In conclusion, our formula for the 2-D unit noise matrix is not only correct but it is also consistent with other formulas used in [2]–[5]. While the formulas in [1] and [6] are different, they apply to different noise models. The correct relationship between the two forms of the 2-D unit noise matrix is discussed by Hinamoto, Hamanaka, and Maekawa (pp. 609–610, this issue).

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