Stability of 2-D Digital Filters under Parameter Variations

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Abstract —**The relationship between the stability margins of a 2-D digital filter (or discrete system) and the norm of the transition matrix of its minimum-norm realization is considered. Upper bounds on parameter variations, which guarantee the stability of a perturbed 2-D digital filter, are then derived in terms of the minimum norm. The results obtained are illustrated by two examples.**

I. Introduction

In the design of two-dimensional (2-D) digital filters and discrete systems in general, the designer is interested not only whether the system is stable or not but also whether the system will remain stable in the presence of system parameter variations. This type of stability analysis can be carried out in terms of stability margins which are measures of the degree to which a system will tolerate system parameter variations without becoming unstable. Such parameter variations may be due to parameter quantization in digital filters or to system uncertainties in control systems.

In [1], a sensitivity analysis of 2-D systems, including analytical expressions for the variations of the characteristic roots of a 2-D filter, has been presented. The definition and calculation of stability margins as well as the interrelation of stability margins with the settling time of the impulse response have been considered in [2], [3]. In [4], lower bounds for the stability margins have been obtained by using the positive-definite solutions of the 2-D Lyapunov equation.

In this contribution, the stability margins defined in [2] are related to the norm of the transition matrix of the state-space minimum-norm realization. Upper bounds on parameter variations, which guarantee the stability of a perturbed 2-D digital filter (or discrete system), are then derived in terms of the minimum norm of the filter. The results obtained are illustrated by two examples.

Minimum-norm realizations are of considerable interest in practice since they lead to 2-D digital-filter implementations which are free of overflow oscillations when implemented using finite-wordlength arithmetic [5].

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II. Notation

The 2-D digital filters considered in this paper are quarter-plane causal filters with support on the first quadrant of the (m, n) plane. The open unit disk, the open unit bidisk and the distinguished boundary of the latter are denoted by U, U^2, and T^2, respectively. The closure of a set S is denoted by $\bar{S}$. The singular values of a matrix M are defined as the eigenvalues of matrix M^tM where M^t denotes the transpose of M. The spectral norm of M, i.e., the largest singular value of M, is denoted by $\|M\|$. The condition number of a nonsingular matrix M is given by $K(M) = \|M\| \, \|M^{-1}\|$.

III. Stability Margins for 2-D Digital Filters in State-Space Representation

A single-input, single-output 2-D digital filter (or discrete system) can be represented by the Roesser model [6] as

$$\begin{bmatrix} h(m+1,n) \\ v(m,n+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} h(m,n) \\ v(m,n) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(m,n)$$

$$\equiv A \begin{bmatrix} h(m,n) \\ v(m,n) \end{bmatrix} + bu(m,n)$$

$$y(m,n) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} h(m,n) \\ v(m,n) \end{bmatrix} \equiv c \begin{bmatrix} h(m,n) \\ v(m,n) \end{bmatrix} \quad (3.1)$$

where h and v are real vectors of dimensions M and N, respectively. The 2-D z-transform of (3.1) gives the transfer function of the digital filter as

$$H(z_1, z_2) = \frac{g(z_1, z_2)}{a(z_1, z_2)} \quad (3.2)$$

where

$$a(z_1, z_2) = \det \begin{bmatrix} I_M - z_1 A_1 & -z_1 A_2 \\ -z_2 A_3 & I_N - z_2 A_4 \end{bmatrix}$$

$$= \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} z_1^i z_2^j, \qquad a_{00} = 1. \quad (3.3)$$

In the rest of the paper, it is assumed that $a(z_1, z_2)$ and $g(z_1, z_2)$ in (3.2) are factor coprime and that there exists a block-diagonal positive-definite matrix $G = G_1 \oplus G_2$ such that

$$W \equiv G - A^tGA \quad (3.4)$$

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is positive definite, where $G_1 \in R^{M\times M}$, $G_2 \in R^{N\times N}$, and $\oplus$ represents direct sum. It can readily be shown that this assumption implies the bounded-input bounded-output (BIBO) stability of the filter [5], [7], [8].

The stability margins, denoted by σ_1, σ_2, and σ, are defined to be the largest values of σ_1, σ_2, and σ for which

$$a(z_1,z_2)\neq 0 \quad \text{in } U^2_{\sigma_1}=\{(z_1,z_2)\,|\,|z_1|<1+\sigma_1,\ |z_2|<1\} \tag{3.5}$$

$$a(z_1,z_2)\neq 0 \quad \text{in } U^2_{\sigma_2}=\{(z_1,z_2)\,|\,|z_1|<1,\ |z_2|<1+\sigma_2\} \tag{3.6}$$

$$a(z_1,z_2)\neq 0 \quad \text{in } U^2_{\sigma}=\{(z_1,z_2)\,|\,|z_1|<1+\sigma,\ |z_2|<1+\sigma\} \tag{3.7}$$

respectively, [2], [3].

We now consider the norm of the transition matrix in (3.1) under all nonsingular state-variable transformations $T=T_1\oplus T_2$ with $T_1\in R^{M\times M}$, $T_2\in R^{N\times N}$. The minimum norm of the state-space realization is defined as

$$\mu=\min_T\|TAT^{-1}\|. \tag{3.8}$$

Although a nonsingular minimizer $T=\tilde{T}$ such that

$$\mu=\|\tilde{T}A\tilde{T}^{-1}\|$$

may not exist (see Example 1), an approximate solution of the minimization problem is always possible [5]. This is similar to the 1-D case where the minimum-norm realization for a given transfer function may not exist, but can be approximated [9].

Consider now a 2-D digital filter characterized by (3.1) and assume that 2-D Lyapunov equation (3.4) is satisfied. By applying 2-D similarity transformation $T=G^{1/2}$, where G is the positive-definite matrix in (3.4), we obtain

$$\|TAT^{-1}\|<1 \tag{3.9}$$

and, therefore, $\mu<1$. Conversely, since the singular values of a given matrix are continuous functions of its entries, $\mu<1$ implies that there is a nonsingular $T=T_1\oplus T_2$ such that $\|TAT^{-1}\|<1$. In other words, an approximate solution of the optimization problem (3.8) will lead to a state-space realization $(\bar{A},\bar{b},\bar{c})=(TAT^{-1},Tb,cT^{-1})$ in which $\|\bar{A}\|<1$. Such a realization is free of overflow oscillations [5], [8].

With the minimum norm of a state-space realization defined as in (3.8), stability margin σ can be related to μ. On the other hand, σ_1 and σ_2 can be related to parameter $\bar{\mu}$ given by

$$\bar{\mu}(\alpha,\beta)=\min_T\|\mathrm{diag}(\alpha I_M,\beta I_N)TAT^{-1}\|$$

where α and β are real parameters and $\mathrm{diag}(\alpha I_M,\beta I_N)=\alpha I_M\oplus\beta I_N$.

Parameters μ and $\bar{\mu}$ are closely related. For instance, one may easily show that

$$\bar{\mu}(1,1)=\mu$$
$$\bar{\mu}(\alpha,\alpha)=|\alpha|\mu$$
$$\min(|\alpha|,|\beta|)\mu\leqslant\bar{\mu}(\alpha,\beta)\leqslant\max(|\alpha|,|\beta|)\mu$$

and for

$$\alpha_1\geqslant\alpha_2\geqslant 0,\quad \text{and}\quad \beta_1\geqslant\beta_2\geqslant 0$$
$$\bar{\mu}(1+\alpha_1,1)\geqslant\bar{\mu}(1+\alpha_2,1)$$
$$\bar{\mu}(1,1+\beta_1)\geqslant\bar{\mu}(1,1+\beta_2). \tag{3.10}$$

The stability margins can now be related to μ and $\bar{\mu}$ as follows.

Theorem 1

If the digital filter (or discrete system) represented by (3.1) satisfies (3.4), then

(i) $$\sigma\geqslant\frac{1}{\mu}-1. \tag{3.11a}$$

(ii) $$\sigma_1\geqslant\alpha \tag{3.11b}$$

where α is the smallest positive number such that

$$\bar{\mu}(1+\alpha,1)=1.$$

(iii) $$\sigma_2\geqslant\beta \tag{3.11c}$$

where β is the smallest positive number such that

$$\bar{\mu}(1+\beta,1)=1.$$

Proof: For any ν such that $\nu>\mu$, if we pick ν_1 in the range $\nu>\nu_1>\mu$, we have

$$\min_T\left\|T\begin{bmatrix}\dfrac{A_1}{\nu_1} & \dfrac{A_2}{\nu_1}\\ \dfrac{A_3}{\nu_1} & \dfrac{A_4}{\nu_1}\end{bmatrix}T^{-1}\right\|<1.$$

In other words

$$a\left(\frac{z_1}{\nu_1},\frac{z_2}{\nu_1}\right)\neq 0 \qquad \text{for } (z_1,z_2)\in\bar{U}^2$$

and, thus

$$a\left(\frac{z_1}{\nu},\frac{z_2}{\nu}\right)\neq 0 \quad \text{in } \left\{(z_1,z_2)\,|\,|z_1|\leqslant\frac{\nu}{\nu_1},|z_2|\leqslant\frac{\nu}{\nu_1}\right\}.$$

But for fixed ν,

$$\{(z_1,z_2)\,|\,|z_1|<\nu(1+\sigma),\ |z_2|<\nu(1+\sigma)\}$$

is the largest bidisk in which

$$a\left(\frac{z_1}{\nu},\frac{z_2}{\nu}\right)\neq 0.$$

Hence $\nu(1+\sigma)\geqslant\nu/\nu_1$, which gives

$$\nu>\frac{1}{1+\sigma}.$$

We have shown that

$$\nu>\mu\Rightarrow\nu>\frac{1}{1+\sigma}$$

and, therefore, (3.11a) is established.

To prove (ii), assume that $\sigma_1<\alpha$. In such a case, there exists an $\bar{\alpha}$ such that

$$\sigma_1<\bar{\alpha}<\alpha.$$

By the definition of α and (3.10), we have

$$\bar{\mu}(1+\bar{\alpha},1)<1$$

which implies that

$$a(z_1,z_2)\neq 0 \quad \text{in } \{(z_1,z_2) \mid |z_1|\leqslant 1+\bar{\alpha},\ |z_2|\leqslant 1\}.$$

This contradicts the fact that $\{(z_1,z_2) \mid |z_1|\leqslant 1+\sigma_1,\ |z_2|\leqslant 1\}$ is the largest bidisk in which $a(z_1,z_2)\neq 0$. Therefore, $\sigma_1\geqslant\alpha$. A similar argument can be applied for the proof of (3.11c). □

If $M=N=1$, then stability margins σ, σ_1, and σ_2 can be determined exactly in terms of μ and $\bar{\mu}$, as stated in the following corollary.

Corollary 1.1 If $M=N=1$, then inequalities (3.11a)–(3.11c) become equalities.

Proof: For any fixed ν such that $\nu>1/(1+\sigma)$, (3.7) implies that

$$a\left(\frac{z_1}{\nu},\frac{z_2}{\nu}\right)\neq 0 \qquad \text{for}(z_1,z_2)\in\bar{U}^2. \tag{3.12}$$

It was shown in [10] that

$$a(z_1,z_2)\neq 0 \qquad \text{for } (z_1,z_2)\in\bar{U}^2$$

if, and only if, (3.4) holds. Hence (3.12) implies that

$$\min_T\left\| T\begin{bmatrix}\dfrac{A_1}{\nu} & \dfrac{A_2}{\nu}\\ \dfrac{A_3}{\nu} & \dfrac{A_4}{\nu}\end{bmatrix}T^{-1}\right\| = \frac{1}{\nu}\min_T\|TAT^{-1}\|<1$$

i.e., $\nu>\mu$. Therefore,

$$\frac{1}{1+\sigma}\geqslant\mu.$$

By combining this inequality with (3.11a), we have

$$\sigma=\frac{1}{\mu}-1.$$

Furthermore, σ_1 must be the smallest number for which $\bar{\mu}(1+\alpha,1)=1$ since if this were not the case, one would be able to find a point $(z_1,z_2)\in\bar{U}\sigma_1^2$ such that $a(z_1,z_2)=0$, which contradicts (3.5). Therefore, (3.11b) becomes an equality. A similar argument can be applied to show that $\sigma_2=\beta$. □

We conclude this section with a brief discussion of the above results. In the 1-D case, a digital filter with eigenvalues $\{\lambda_i,\ 1\leqslant i\leqslant k\}$ is called a minimum-norm filter if the norm of its transition matrix $\bar{F}$ satisfies

$$\|\bar{F}\|=|\lambda|_{\max}$$

where $|\lambda|_{\max}$ is the largest value of $|\lambda_i|$ for $1\leqslant i\leqslant k$ [9]. The name "minimum norm" originates from the fact that for any fixed square matrix F with eigenvalues $\{\lambda_i,\ 1\leqslant i\leqslant k\}$, we have

$$\min_T\|TFT^{-1}\|=|\lambda|_{\max}. \tag{3.13}$$

In the 2-D case, relation (3.13) no longer holds since the nonsingular transformations are restricted to be block diagonal. Therefore, minimum-norm realizations of system (3.1) are those in which transition matrix A satisfies $\|A\|=\mu\geqslant|\lambda|_{\max}$.

If a 1-D digital filter is stable, then its stability margin [11] is

$$\sigma=\frac{1}{|\lambda|_{\max}}-1=\frac{1}{\min\limits_T\|TFT^{-1}\|}-1.$$

On the other hand, if a 2-D digital filter is stable and satisfies (3.4), then (3.11a) gives

$$\sigma\geqslant\frac{1}{\min\limits_T\|TAT^{-1}\|}-1.$$

It should be pointed out here that the results presented in Theorem 1 can easily be extended to the N-dimensional case, where $N\geqslant 3$, provided that the general state-space model proposed in [12] is used.

IV. Bounds on Parameter Variations

In this section we use the concept of the minimum-norm of a 2-D state-space digital filter to obtain an upper bound on parameter variations, which guarantees the stability of a perturbed system. In addition, various special cases are considered in which tighter bounds are possible.

Assume that a stable 2-D digital filter or (discrete-time system) is represented by (3.1) and let

$$\Delta A=\begin{bmatrix}\Delta A_1 & \Delta A_2\\ \Delta A_3 & \Delta A_4\end{bmatrix} \tag{4.1}$$

be variations of the transition matrix A in (3.1), where $\Delta A_1\in R^{M\times M}$, $\Delta A_4\in R^{N\times N}$. Such variations occur when a digital filter is implemented by finite-wordlength arithmetic. If, on the other hand, model (3.1) represents a control system, ΔA may be caused by inevitable errors in estimating the parameters of the system.

As can be seen in Section III, if state-space characterization (3.1) satisfies (3.4), then $\mu<1$ and one can find numerically a nonsingular $\tilde{T}=\tilde{T}_1\oplus\tilde{T}_2$ such that $\|\tilde{T}A\tilde{T}^{-1}\|=\mu$. Denoting the transition matrix of a perturbed 2-D digital filter by $\tilde{A}=A+\Delta A$, one may observe that

$$\|\tilde{T}\tilde{A}\tilde{T}^{-1}\|\leqslant\|\tilde{T}A\tilde{T}^{-1}\|+\|\tilde{T}\Delta A\tilde{T}^{-1}\|=\mu+\|\tilde{T}\Delta A\tilde{T}^{-1}\|. \tag{4.2}$$

Thus

$$\|\tilde{T}\tilde{A}\tilde{T}^{-1}\|<1 \qquad \text{when } \|\tilde{T}\Delta A\tilde{T}^{-1}\|<1-\mu. \tag{4.3}$$

In particular, if

$$\|\Delta A\|<\frac{1-\mu}{K(\tilde{T})} \tag{4.4}$$

where $K(\tilde{T})$ is the condition number of $\tilde{T}$, then (4.2) and (4.4) lead to

$$\min_T\|T\tilde{A}T^{-1}\|\leqslant\|\tilde{T}\tilde{A}\tilde{T}^{-1}\|<1$$

and, therefore, the perturbed digital filter is stable. We thus have the following theorem.

Theorem 2

Assume that the 2-D digital filter (or discrete system) represented by (3.1) satisfies (3.4). Any perturbed version of digital filter with $\tilde{A}=A+\Delta A$ will remain stable provided that parameter variations ΔA satisfy (4.4). Moreover, the state-space realization with transition matrix $\tilde{T}\tilde{A}\tilde{T}^{-1}$ is free of overflow oscillations.

By applying Corollary 1.1, we can prove the following corollary.

Corollary 2.1 If $M=N=1$, then any perturbed 2-D digital filter described in Theorem 2 will remain stable provided that parameter variations ΔA satisfy

$$\|\Delta A\| < \frac{\sigma}{(1+\sigma)K(\tilde{T})}. \tag{4.5}$$

A 2-D polynomial

$$a(z_1, z_2) = \sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij} z_1^i z_2^j, \qquad a_{00}=1$$

is said to be stable if $a(z_1, z_2) \neq 0$ in $\bar{U}^2$. One may seek a bound for the coefficient variations of a stable polynomial such that the perturbed polynomial will remain stable. For the transfer function $1/a(z_1, z_2)$ one can find a controller-form realization with minimal order in which the transition matrix has the form [13]

$$A=\left[\begin{array}{cccc|cccc} -a_{10} & \cdots & & -a_{M0} & -1 & & & \\ 1 & & & 0 & & & & \\ & \ddots & & & & 0 & & \\ 0 & & 1 & 0 & & & & \\ \hline \bar{a}_{11} & \cdots & & \bar{a}_{M1} & -a_{01} & 1 & & 0 \\ \vdots & & & \vdots & \vdots & 0 & \ddots & 1 \\ \bar{a}_{1N} & \cdots & & \bar{a}_{MN} & -a_{0N} & \cdots & & 0 \end{array}\right] \tag{4.6}$$

where

$$\bar{a}_{ij} = a_{ij} - a_{i0}a_{0j}, \qquad 1 \leqslant i \leqslant M;\ 1 \leqslant j \leqslant N.$$

Thus a perturbed 2-D polynomial

$$\begin{aligned}\tilde{a}(z_1, z_2) &= \sum_{i=0}^{M}\sum_{j=0}^{N} \tilde{a}_{ij} z_1^i z_2^j \\ &\equiv \sum_{i=0}^{M}\sum_{j=0}^{N} (a_{ij}+\delta_{ij}) z_1^i z_2^j \end{aligned} \tag{4.7}$$

has a controller-form realization with transition matrix $\tilde{A}=A+\Delta A$, where

$$\Delta A=\left[\begin{array}{ccc|ccc} -\delta_{10} & \cdots & -\delta_{M0} & & & \\ & 0 & & & 0 & \\ \hline \bar{\delta}_{11} & \cdots & \bar{\delta}_{M1} & -\delta_{01} & & \\ \vdots & & \vdots & \vdots & 0 & \\ \bar{\delta}_{1N} & \cdots & \bar{\delta}_{MN} & -\delta_{0N} & & \end{array}\right] \tag{4.8}$$

and

$$\bar{\delta}_{ij} = \delta_{ij} - \delta_{i0}\delta_{0j} - (a_{i0}\delta_{0j} + a_{0j}\delta_{i0}). \tag{4.9}$$

It is now evident that the following corollary is an immediate consequence of Theorem 2.

Corollary 2.2: If matrix A in (4.6) satisfies (3.4), then the perturbed polynomial $\tilde{a}(z_1, z_2)$ in (4.7) remains stable if variations ΔA in (4.8) satisfy inequality (4.4), where $\tilde{T}$ is a nonsingular block diagonal matrix such that

$$\|\tilde{T}A\tilde{T}^{-1}\| = \min_{T}\|TAT^{-1}\|.$$

Several special cases can be considered through the above corollary, as follows.

Corollary 2.3: If $\{\tilde{a}_{i0}, 1 \leqslant i \leqslant M\}$ are the only perturbed coefficients, then $\tilde{a}(z_1, z_2)$ remains stable if

$$\left[\sum_{i=1}^{M} \delta_{i0}^2\right]^{1/2} < \frac{1-\mu}{(1+\|a_1\|)K(\tilde{T})} \tag{4.10}$$

where $\|a_1\|$ is the Euclidean norm of vector $a_1 \equiv [a_{01} \cdots a_{0N}]$.

Proof: In this case

$$\Delta A=\left[\begin{array}{ccc|c} -\delta_{10} & \cdots & -\delta_{M0} & \\ & 0 & & 0 \\ \hline & (\bar{\delta}_{ij}) & & 0 \end{array}\right]$$

where $\bar{\delta}_{ij} = -a_{0j}\delta_{i0}$. Thus

$$(\bar{\delta}_{ij}) = -a_1^T[\delta_{10} \cdots \delta_{M0}]$$

which leads to

$$\|\Delta A\| \leqslant \left[\sum_{i=1}^{M} \delta_{i0}^2\right]^{1/2}(1+\|a_1\|).$$

Therefore, the inequality (4.4) will be satisfied if (4.10) holds. □

Corollary 2.4: If $\{\tilde{a}_{0j}, 1 \leqslant j \leqslant N\}$ are the only perturbed coefficients, then $\tilde{a}(z_1, z_2)$ remains stable if

$$\left[\sum_{i=1}^{N} \delta_{0j}^2\right]^{1/2} < \frac{1-\mu}{(1+\|a_2\|)K(\tilde{T})} \tag{4.11}$$

where $a_2 \equiv [a_{10} \cdots a_{M0}]$.

Corollary 2.5: If coefficients $\{a_{i0}, 1 \leqslant i \leqslant M\}$ and $\{a_{0j}, 1 \leqslant j \leqslant N\}$ are not perturbed, then $\tilde{a}(z_1, z_2)$ remains stable if

$$\|\Delta A_3\| < \frac{1-\mu}{K(\tilde{T})} \tag{4.12}$$

where $\Delta A_3 = (\delta_{ij})$, $1 \leqslant i \leqslant M$, $1 \leqslant j \leqslant N$.

Proof: In this case $\delta_{i0} = \delta_{0j} = 0$ for $1 \leqslant i \leqslant M$, $1 \leqslant j \leqslant N$ and $\bar{\delta}_{ij} = \delta_{ij}$ which gives $\|\Delta A\| = \|\Delta A_3\|$. Therefore, this corollary follows from Corollary 2.2. □

Example 1: Consider a first-order 2-D digital filter characterized by

$$H(z_1, z_2) = \frac{1}{a(z_1, z_2)} = \frac{1}{1+az_1+bz_2+cz_1z_2}. \tag{4.13}$$

The transition matrix of its controller-form realization is

$$A = \begin{bmatrix} -a & -1 \\ c-ab & -b \end{bmatrix}.$$

We can write

$$\min_{\substack{t_1\neq 0\\ t_2\neq 0}}\left\|\begin{bmatrix} t_1 & 0\\ 0 & t_2\end{bmatrix}A\begin{bmatrix} t_1^{-1} & 0\\ 0 & t_2^{-1}\end{bmatrix}\right\|$$

$$=\min_{t>0}\left\|\begin{bmatrix} 1 & 0\\ 0 & t\end{bmatrix}A\begin{bmatrix} 1 & 0\\ 0 & t^{-1}\end{bmatrix}\right\|=\min_{t>0}\left\|\begin{bmatrix} a & t^{-1}\\ (ab-c)t & b\end{bmatrix}\right\|.$$

Simple algebra indicates that the above minimum is achieved when

$$t=|ab-c|^{-1/2}$$

$$\mu=\min_T\|TAT^{-1}\|$$

$$=\left\{\frac{a^2+b^2+2|ab-c|+\left[\left(a^2+b^2+2|ab-c|\right)^2-4c^2\right]^{1/2}}{2}\right\}^{1/2} \tag{4.14}$$

and

$$\tilde{T}=\begin{bmatrix} 1 & 0\\ 0 & |ab-c|^{-1/2}\end{bmatrix}. \tag{4.15}$$

We now conclude that the first-order filter (4.13) is stable if, and only if, μ given in (4.14) is less than one; the state-space realization with transition matrix

$$\tilde{T}A\tilde{T}^{-1}=\begin{bmatrix} -a & |ab-c|^{1/2}\\ \dfrac{c-ab}{|ab-c|^{1/2}} & -b\end{bmatrix}$$

has minimum norm and is, therefore, free of overflow oscillations if, and only if, $\mu<1$.

To obtain σ_1, we note that for any fixed $\alpha>0$

$$\bar{\mu}(\alpha,1)=\min_{\substack{t_1\neq 0\\ t_2\neq 0}}\left\|\begin{bmatrix} \alpha & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix} t_1 & 0\\ 0 & t_2\end{bmatrix}A\begin{bmatrix} t_1^{-1} & 0\\ 0 & t_2^{-1}\end{bmatrix}\right\|$$

$$=\min_{t>0}\left\|\begin{bmatrix} \alpha & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix} t & 0\\ 0 & 1\end{bmatrix}A\begin{bmatrix} t^{-1} & 0\\ 0 & 1\end{bmatrix}\right\|$$

$$=\min_{t>0}\left\|\begin{bmatrix} \alpha a & \alpha t\\ (ab-c)t^{-1} & b\end{bmatrix}\right\|.$$

The minimum is achieved if

$$t=\left[\frac{|ab-c|}{\alpha}\right]^{1/2}$$

and

$$\bar{\mu}(\alpha,1)=\left\{\frac{\alpha^2a^2+b^2+2\alpha|ab-c|+\left[\left(\alpha^2a^2+b^2+2\alpha|ab-c|\right)^2-4\alpha^2c^2\right]^{1/2}}{2}\right\}^{1/2}. \tag{4.16}$$

Similarly, one can compute

$$\bar{\mu}(1,\beta)=\left\{\frac{a^2+\beta^2b^2+2\beta|ab-c|+\left[\left(a^2+\beta^2b^2+2\beta|ab-c|\right)^2-4\beta^2c^2\right]^{1/2}}{2}\right\}^{1/2}. \tag{4.17}$$

By expression (4.16), it is easy to see that the smallest positive number σ_1 that makes $\bar{\mu}(1+\sigma_1,1)=1$ is the minimum positive value of

$$\left\{\frac{|ab-c|\pm|a-bc|}{c^2-a^2}\right\}$$

and, therefore,

$$\sigma_1=\min\left\{\left|\frac{1+b}{a+c}\right|,\left|\frac{1-b}{a-c}\right|\right\}-1. \tag{4.18}$$

Similarly by (4.17), the smallest positive number that makes $\bar{\mu}(1,1+\sigma_2)=1$ is the minimum positive value of

$$\left\{\frac{|ab-c|\pm|b-ac|}{c^2-b^2}\right\}-1$$

and, therefore,

$$\sigma_2=\min\left\{\left|\frac{1+a}{b+c}\right|,\left|\frac{1-a}{b-c}\right|\right\}-1. \tag{4.19}$$

Evidently, these results coincide with the results reported in [2].

It should be mentioned that if $ab=c$, matrix $\tilde{T}$ in (4.15) is no longer nonsingular and so a minimum-norm realization does not exist. However, one may pick

$$T=\begin{bmatrix} 1 & 0\\ 0 & t\end{bmatrix}$$

and if t is chosen to be large enough, the norm of

$$TAT^{-1}=\begin{bmatrix} -a & -t^{-1}\\ 0 & -b\end{bmatrix}$$

will be close to the minimum norm given by $\mu=\max\{|a|,|b|\}$.

We now assume that $ab-c\neq 0$ so that $\tilde{T}A\tilde{T}^{-1}$, where $\tilde{T}$ is given by (4.15), represents a minimum-norm realization of $H(z_1,z_2)$.

Since

$$K(\tilde{T})=\max\{|ab-c|^{1/2},|ab-c|^{-1/2}\} \tag{4.20}$$

Corollaries 2.2–2.5 can be used to obtain variation bounds on coefficients a, b, and c. The method is illustrated by considering the following transfer function [2]:

$$H(z_1, z_2) = \frac{1}{1+0.5z_1+0.01z_2+0.4z_1z_2}. \quad (4.21)$$

Equations (4.14), (4.18), and (4.19) give the stability margins of $H(z_1. z_2)$ as $\sigma = 0.083$, $\sigma_1 = 0.122$, and $\sigma_2 = 0.282$. The digital filter has a controller-form realization with transition matrix

$$A = \begin{bmatrix} -0.5 & -1 \\ 0.395 & -0.01 \end{bmatrix}.$$

By (4.15), one may compute the transition matrix of a minimum-norm realization as

$$\begin{aligned} \tilde{A} &= \tilde{T}A\tilde{T}^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1.591 \end{bmatrix}\begin{bmatrix} -0.5 & -1 \\ 0.395 & -0.01 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.628 \end{bmatrix} \\ &= \begin{bmatrix} -0.5 & -0.628 \\ 0.628 & -0.01 \end{bmatrix} \end{aligned}$$

whose norm is $\mu = 0.923$. Finally, (4.20) gives $K(\tilde{T}) = 1.591$. Thus the allowable single variations of the coefficients in (4.21) denoted by δa, δb, and δc are

$$|\delta a| < 0.048, \quad |\delta b| < 0.032, \quad \text{and} \quad |\delta c| < 0.048$$

respectively.

Example 2: We now consider a second-order filter designed in [14] and characterized by

$$H(z_1, z_2) = \frac{1}{\begin{bmatrix} 1 & z_1 & z_1^2 \end{bmatrix}\begin{bmatrix} 1 & -1.88899 & 0.91219 \\ -1.88899 & 3.59599 & -1.74892 \\ 0.91219 & -1.74892 & 0.85640 \end{bmatrix}\begin{bmatrix} 1 \\ z_2 \\ z_2^2 \end{bmatrix}}. \quad (4.22)$$

The transition matrix A of its controller-form realization is given by

$$A = \left[\begin{array}{cc|cc} 1.88899 & -0.91219 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0.02771 & -0.02580 & 1.88899 & 1 \\ -0.02580 & 0.02431 & -0.91219 & 0 \end{array}\right]. \quad (4.23)$$

By the use of their computation approach, Lodge and Fahmy [5] were able to find a similarity transformation $T = T_1 \oplus T_2$ such that

$$\begin{aligned} A &\equiv TAT^{-1} \\ &= \left[\begin{array}{cc|cc} 0.95776 & -0.14277 & -0.02571 & 0.16849 \\ 0.14215 & 0.93123 & 0 & 0 \\ \hline 0.02466 & 0.08282 & 0.93168 & 0.13884 \\ -0.16070 & 0.01245 & -0.14609 & 0.95731 \end{array}\right] \end{aligned} \quad (4.24)$$

with norm $\mu = \|TAT^{-1}\| = 0.982$ in contrast with $\|A\| = 7.609$. Having done this, one can verify that the above transformation matrix has the form

$$T = \left[\begin{array}{cc|cc} 1 & -0.93123 & & \\ 0 & 0.14215 & \multicolumn{2}{c}{0} \\ \hline & & 38.88864 & 40.81211 \\ \multicolumn{2}{c|}{0} & 0 & 6.22868 \end{array}\right]^{T} \quad (4.25)$$

where τ is a nonzero real parameter. Note that the condition number of T is fairly large, i.e., $K(T) = 545.016$ so that the bound given by (4.10) will be quite conservative. However, one can obtain tighter bounds for many special cases by directly using inequality (4.3). For example, if we suppose that no variations occur except in the coefficient of $z_1^2 z_2^2$ in (4.22), then (4.3) becomes

$$\left\| T \left[\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0 \\ \hline 0 & 0 & \\ & & 0 \\ 0 & \delta_{22} & \end{array}\right] T^{-1} \right\| < 1 - \mu = 0.018$$

which leads to

$$|\delta_{22}| < 0.0031.$$

V. Conclusions

The stability margins defined in [2], [3] have been related to the norm of the state-space minimum-norm realization. The results obtained are given in Theorem 1.

Then by using the concept of the minimum norm of a 2-D state-space digital filter (or discrete system), upper bounds on parameter variations have been derived which guarantee the stability of a perturbed digital filter.

The results obtained were illustrated by considering two low-order 2-D digital filters. They are thought to be of practical significance since minimum-norm realizations are known to be free of overflow oscillations.

The main results presented in this paper can be extended to the N-dimensional case where $N \geqslant 3$ in a straightforward manner, provided that the general state-space model proposed in [12] is used.

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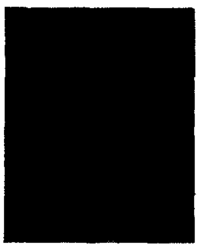

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