TABLE II
CONSTRAINED CONTROLLERS

CONTROLLER	GAIN MATRIX				EIGENVALUES	R.M.S. MODAL SHIFT
A	-0.0803	-1.3497	0.6512	0.	-1.0139	0.011
	0.5598	0.2763	-2.2346	0.5099	-3.0428 -1.2506±j1.7499	
B	-0.0803	-1.3497	0.6512	0.	-1.0108	0.029
	0.5633	0.	-1.9031	0.5219	-3.0832 -1.2995±j1.7222	
C	0.	-1.3478	0.6486	0.	-0.9830	0.075
	0.5633	0.	-1.9031	0.5219	-3.2958 -1.2436±j1.7108	
D	0.	-1.3478	0.6486	0.	-1.1673	0.106
	0.5586	0.	0.	0.5172	-2.7177 -1.4252±j1.6797	
E	0.	-1.1783	0.	0.	-1.2192	0.150
	0.5586	0.	0.	0.5172	-2.6979 -1.2570±j1.4184	

TABLE III
CHANGES IN EIGENVECTORS

MODE	A	B	C	D	E
Spiral	0.9998	0.9999	0.9994	0.9966	0.9944
Roll Subsid.	0.9997	0.9996	0.9954	0.9958	0.9953
Dutch roll	1.0000	0.8236	0.8347	0.3331	0.2805

VI. CONCLUSIONS

We have established a systematic method to design a constrained output feedback system by approaching an *a priori* prescribed eigenstructure. The main contribution of this note is the use of residue analysis (based on right and left eigenvectors) to estimate the effect on the eigenvalues of constraints in the feedback gains. Numeric results show that some eigenvectors can be also approximately preserved, although eigenvectors sensitivities have not been taken into account.

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Measuring How Far a Controllable System is from an Uncontrollable One

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Abstract—**The main goal of this note is to explore some of the properties of a controllable system which, in a precise sense to be defined, is near an uncontrollable pair. To do this, we first discuss the notion of the distance between a controllable system and the nearest uncontrollable one in both real and complex cases [2]–[5]. When this distance is relatively small, we show to what extent this distance corresponds to certain properties of the uncontrollability matrix and the reachability Grammian.**

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I. INTRODUCTION

Consider a linear time-invariant system with mathematical model

$$\dot{x} = Ax + Bu \tag{1.1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$. Define the controllability matrix of (1.1) by $G \equiv [B \; AB \cdots A^{n-1}B]$. Also define $P(s) \equiv [sI - A, B]$. It is well known that the system (1.1) is controllable if and only if rank $G = n$, or equivalently, rank $P(s) = n$ for all complex s. Also, one may recall that for a given n-dimensional pair (A, B), if rank $G = r < n$, then there exists a nonsingular Q such that

$$\bar{A} = QAQ^{-1} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ 0 & \bar{A}_4 \end{bmatrix}, \quad \bar{B} = QB = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$$

where $\bar{A}_3 = 0$, $\bar{B}_2 = 0$, and $(\bar{A}_1, \bar{B}_1)$ is an r-dimensional controllable system [6]. Throughout this note, we denote the set of all controllable systems by Y; when $m = 1$ the set will be labeled by Y_1.

The main goal of this note is to explore some of the properties of a controllable system which, in a precise sense to be defined, is near an uncontrollable pair. To do this, we first discuss the notion of the distance between a pair $(A, B) \in Y$ and the nearest uncontrollable pair in both real and complex cases [2]–[5]. When this distance is relatively small, we show to what extent this distance corresponds to certain properties of the controllability matrix and the reachability Grammian.

II. THE DISTANCE $\mu(A, B)$

Controllability is a generic property [1, p. 100], which means that if a pair (A, B) is considered as a point in a finite-dimensional parameter space, the set Y of controllable pairs is open and dense in the whole parameter space. In some neighborhood of a pair $(A, B) \in Y$, however, there may not exist any uncontrollable pair. Therefore, it makes sense to consider the distance to the "nearest uncontrollable pair."

In the remainder of this note, the vector and matrix norms we refer to are the 2-norms; the spectrum of a matrix A is denoted by $\Lambda(A)$; the set of all singular values of A is labeled by $\Sigma(A) = \{\sigma_i, 1 \le i \le n\}$ where $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n \ge 0$. We also follow the convention of numbering the definitions, lemmas, theorems, and examples together in one series.

Definition 1: For a given pair $(A, B) \in Y$, define the distance between (A, B) and a nearest uncontrollable pair by

$$\mu(A, B) = \min_{\delta A, \delta B} \|[\delta A, \delta B]\|$$

where $\delta A \in C^{n\times n}$, $\delta B \in C^{n\times m}$ such that $(A + \delta A, B + \delta B)$ is uncontrollable.

If we restrict ourselves to real perturbations, we use

Definition 2: For a given pair $(A, B) \in Y$, define

$$\mu_r(A, B) = \min_{\delta A, \delta B \text{ real}} \|[\delta A, \delta B]\|$$

where $\delta A \in R^{n\times n}$, $\delta B \in R^{n\times m}$ such that $(A + \delta A, B + \delta B)$ is uncontrollable.

These definitions give a characterization of when a system is "hard to control" in a sense that is different from the more classical characterizations such as energy or feedback gain. The definitions are significant in certain situations, especially when the data defining the coefficient matrices are not known to great accuracy, or when we are carrying out digital computer simulations, which involve the accumulation of roundoff error. In addition, if one computes the nearby uncontrollable system, the controllable part of that system will be a lower order approximation to the original system. Hence, this concept can be used for model reduction. This particular application will be explored in a separate paper.

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We are attempting to measure how perturbations to the coefficients might affect the structural properties of the original system model. In this note, we explore the relationships that exist between this characterization and the more classical points of view. We begin with a formulation equivalent to the above definition and then discuss some consequences of it.

Theorem 3 [2] [4]:

$$\mu(A,\ B)=\min_{s\in C}\ \sigma_n(sI-A,\ B)$$

$$\mu_r(A,\ B)=\min_{s\in R}\ \sigma_n(sI-A,\ B)$$

where $\sigma_n(sI - A, B)$ is the smallest singular value of $[sI - A, B]$.

Remark: The proof is a direct application of the theory of the singular value decomposition. In fact, algorithms to compute the quantities μ and μ_r have been proposed in [2] and [3].

In most physical systems, only real coefficients make physical sense, hence, the need to consider only real perturbations. However, it is easy to demonstrate that in restricting oneself to only real perturbations, one may substantially overestimate the distance to the nearest uncontrollable system. The following example shows the necessity of considering complex perturbation pairs when finding the nearest uncontrollable system.

Example 4: Consider a pair $(A, b) \in Y$, with

$$A=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ b=\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

A straightforward computation indicates that $\min_{s\in C}\sigma_2(sI - A, b) = 0.6614$ is achieved when $s = s^* = j\sqrt{15}/4$ (or, when $s = s^* = -j\sqrt{15}/4$). Since s^* is not real, the perturbation δA, δb to reduce the rank of $P(s^*)$ will also not be real. To obtain a real δA, δb, we must restrict s to be real. In this case, a straightforward computation involving $P(s)P(s)^T$ shows that the minimum $\min_{s\in R}\sigma_2(sI - A, b) = 1$ is achieved for $s = 0$. Hence, in this case, there is no real perturbation of norm 0.6614 that yields an uncontrollable system. In fact, one needs a real pertubation 50 percent larger. That is, the estimate μ_r is over than 50 percent larger than μ. This example also shows that the minimum which yields the measure μ is not achieved when s is an eigenvalue of A, so that the minimization must be carried out over the whole complex plane.

III. Relation to the Controllability Matrix

As mentioned above, complex perturbations to (1.1) yield a system with complex coefficients which may not correspond to any physical process. To be consistent with many physical situations, we must restrict ourselves to *real* perturbations. Unfortunately, as was discovered in [3], finding the value of μ_r can be a very involved process. However, we can give a simple bound for this quantity, in terms of classical quantities, namely the singular values of G. Contrary to what one may expect, however, this bound is not simply based on the smallest singular value of G. In other words, it is *not* true that one can obtain a nearby uncontrollable system by applying a perturbation δA, δB with norm bounded by the smallest nonzero singular value of G.

To see what bound one can obtain, we first take the S.V.D. of the controllability matrix G associated with a given pair $(A, B) \in Y$

$$G=P^T[\Sigma|0]Q \tag{3.1}$$

where $\Sigma = \text{diag}\ \{\sigma_1, \cdots, \sigma_n\}$, $P \in R^{n\times n}$ and $Q \in R^{nm\times nm}$ are orthogonal matrices. Let $\hat{A} = PAP^T$ and $\hat{B} = PB$. Since P is orthogonal, it is evident that $\mu_r(A, B) = \mu_r(\hat{A}, \hat{B})$, and (3.1) becomes

$$\hat{G}=[\hat{B}\ \hat{A}\hat{B}\ \cdots\ \hat{A}^{n-1}\hat{B}]=[\Sigma|0]Q. \tag{3.2}$$

We can now state the following lemma.

Lemma 5: For a given pair $(A, B) \in Y$, there exists an orthogonal matrix P such that

$$\hat{A}=PAP^T=\begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix},\ \hat{B}=PB=\begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}$$

where $\hat{A}_1 \in R^{r\times r}$, $\hat{B}_1 \in R^{r\times m}$, with

$$\|\hat{A}_3\| \le \|A_F\|\ \frac{\sigma_{r+1}}{\sigma_r},\ \text{and}\ \|\hat{B}_2\| \le \sigma_{r+1}. \tag{3.3}$$

Here A_F denotes the companion matrix for A.

Proof: By (3.2)

$$\hat{B}=[\Sigma|0]Q\begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

where Σ is as defined in (3.1). Partition

$$Q\begin{bmatrix} I_m \\ 0 \end{bmatrix},\ =\begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix},$$

where $S_1 \in R^{r\times r}$, $S_2 \in R^{(n-r)\times(n-r)}$, $S_3 \in R^{(m-1)n\times(m-1)n}$, and $I_m \in R^{m\times m}$. Partition Σ into two diagonal blocks: $\Sigma_r = \text{diag}\ \{\sigma_1 \cdots \sigma_r\}$, $\Sigma_{\bar{r}} = \text{diag}\ \{\sigma_{r+1} \cdots \sigma_n\}$. Then B can be written

$$\hat{B}=\begin{bmatrix} \Sigma_r S_1 \\ \Sigma_{\bar{r}} S_2 \end{bmatrix}=\begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}.$$

Then the second part of (3.3) follows from $\|S_2\| \le 1$. To obtain a bound for $\|\hat{A}_3\|$, notice that

$$\hat{A}[\hat{B}\ \hat{A}\hat{B}\ \cdots\ \hat{A}^{n-1}\hat{B}]=[\hat{B}\ \hat{A}\hat{B}\ \cdots\ \hat{A}^{n-1}\hat{B}](A_F\otimes I_m) \tag{3.4}$$

where $\otimes$ denotes the Kronecker product. Equations (3.4) and (3.2) yield

$$\hat{A}=\begin{bmatrix} \Sigma_r & |0 \\ & \Sigma_{\bar{r}}|0 \end{bmatrix} Q(A_F\otimes I_m)Q^T \begin{bmatrix} \Sigma_r & |0 \\ & \Sigma_{\bar{r}}|0 \end{bmatrix}^{\dagger}$$

where $\dagger$ denotes the Moore-Penrose inverse [7]. We now denote the upper left $n \times n$ submatrix of $Q(A_F \otimes I_m)Q^T$ by Z and partition it as

$$Z=\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix},\ \text{with}\ Z_1 \in R^{r\times r},\ Z_4 \in R^{(n-r)\times(n-r)}.$$

Then $\hat{A}$ can be written as

$$\hat{A}=PAP^T\equiv\begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}=\begin{bmatrix} \Sigma_r Z_1\Sigma_r^{-1} & \Sigma_r Z_2\Sigma_{\bar{r}}^{-1} \\ \Sigma_{\bar{r}} Z_3\Sigma_r^{-1} & \Sigma_{\bar{r}} Z_4\Sigma_{\bar{r}}^{-1} \end{bmatrix}.$$

It is easy to see that $\|\Sigma_r^{-1}\| = \sigma_r^{-1}$ and $\|A_F \otimes I_m\| = \|A_F\|$, hence, we have the first part of (3.3). Q.E.D.

We can now state our main result.

Theorem 6: For a given pair $(A, B) \in Y$,

$$\mu(A,\ B)\le\mu_r(A,\ B)\le\left(1+\frac{\|A_F\|}{\sigma_{n-1}}\right)\sigma_n. \tag{3.5}$$

Proof: In Lemma 5, by setting $\hat{A}_3$ and $\hat{B}_2$ to zero, we commit a real perturbation whose norm is bounded by the right-hand side of (3.5). By setting those two blocks to zero, we obtain a uncontrollable system, hence, this perturbation must be at least μ_r. Q.E.D.

We recall that σ_i are the singular values of the controllability matrix G, and the nonzero entries of A_F are just the coefficients $\alpha_0, \cdots, \alpha_n$ of the characteristic polynomial of A, so $\|A_F\| \le \max|\alpha_j|$. It should be pointed out that the computation of the singular values of G can be a difficult numerical problem, but the accuracy can be enhanced by applying some of the reduction techniques in, e.g., [2], [5], [9] such as the "staircase algorithm" to first reduce the matrices A, B to a simpler structure.

IV. Examples

For most systems (1.1), the coefficients of the characteristic polynomial of A, and hence the entries of the companion form A_F are larger than 1, so the formula (3.5) would be dominated by the second term $\|A_F\|\sigma_{r+1}/\sigma_r$. This formula shows that the size of the perturbation μ_r is not bounded by the smallest singular value of G; rather it is bounded by the ratio between two consecutive singular values. To find a small real perturbation to obtain an uncontrollable system, one must find a gap among the singular values. We illustrate this point with two examples.

Example 7: In [8], there was given an 11×11 example with $A = \text{diag}\{32, 16, 8, -64, -32, 4, 2, -2, -4, -8, 16\}$ and B equal to the vector $B = (10^{-14}, 2.3, 0.4, 10^{-14}, 1.2, 0.2, 0.5, 3.7, -1.1, 0.1, 10^{-14})^T$. The controllability matrix G for this example has singular values with orders of magnitude $\{10^0, 10^{-4}, 10^{-9}, 10^{-10}, 10^{-12}, 10^{-13}, 10^{-15}, 10^{-15}, 10^{-18}, 10^{-28}, 0\} \times 10^{15}$. It was shown in [8] that one can cannot use directly the singular values of the matrix G to decide how close this system is to an uncontrollable system. However, we see that the *greatest gap* between consecutive singular values is 10^{-10}, so we can safely say using Theorem 6 that there is a perturbation of the order of 10^{-10} to our original system that will yield an uncontrollable system.

Example 8: In [9], a 2×2 stable, controllable example was defined by

$$A = \begin{bmatrix} -1/2 & 0 \\ -\sqrt{\epsilon} & -1/2 \end{bmatrix}, \quad B = \begin{bmatrix} \sqrt{\epsilon} \\ 0 \end{bmatrix}$$

where ϵ is a small number. It is easy to verify that for this example the singular values of G are on the order of $\sqrt{\epsilon}$ and ϵ, yet it was shown in [9] that the norm μ of the smallest perturbation to an uncontrollable system is of the order $\sqrt{\epsilon}$, exactly the ratio between the two singular values of G.

V. Variations on Above Results

One may give an interpretation of Theorem 6 in terms of the energy necessary to control certain modes, as represented by the singular values of the reachability or controllability Grammian for the system. For the discrete-time system $x_{k-1} = Ax_k + Bu_k$, the controllability reachability Grammian is approximated by GG^T. Hence, the squares of the singular values of the matrix G correspond to the eigenvalues of the discrete-time controllability Grammian. Hence, Lemma 5 gives a relationship between the energy modes of (5.1) and the distance μ_r. We can also obtain a similar result for the continuous time case (1.1). In case A is a stable matrix, we may also define the continuous-time reachability Grammian

$$W = \int_0^\infty e^{At}BB^Te^{A^Tt}\, dt.$$

Using this Grammian, we may obtain a similar bound for the distance μ_r in terms of the energy modes of (1.1) as represented by the eigenvalues of W.

Theorem 9: Given a pair $(A, B) \in Y$ with A stable. Let Q be an orthogonal matrix such that $QWQ^T = \text{diag}\{\delta_1^2 \cdots \delta_n^2\} \equiv \Delta^2$ with $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n > 0$. Then the equivalent pair (F, H) determined by $F = QAQ^T = [\begin{smallmatrix} F_1 & F_2 \\ F_3 & F_4 \end{smallmatrix}]$, $H = QB = [\begin{smallmatrix} H_1 \\ H_2 \end{smallmatrix}]$, where $F_1 \in R^{r \times r}$, $H_1 \in R^{r \times m}$ has estimates

$$\|F_3\| \leq c_1 \frac{\delta_{r+1}}{\delta_r} \text{ and } \|H_2\| \leq \delta_{r+1}, \tag{5.1}$$

where $c_1 = \|H_1\|\sqrt{2\|F_4\|} + \|F_2\|$, $c_2 = \sqrt{2\|F_4\|}$. The subsystem (F_1, H_1) has reachability Grammian $\Delta_r^2 \equiv \text{diag}\{\delta_1^2 \cdots \delta_r^2\}$.

Proof: Define $\Delta_{\bar{r}}^2 \equiv \text{diag}\{\delta_{r-1}^2 \cdots \delta_n^2\}$. It is known that the reachability Grammian W satisfies the Lyapunov equation $AW + WA^T + BB^T = 0$, or, with the partitioning given above,

$$\begin{bmatrix} F_1\Delta_r^2 + \Delta_r^2F_1^T + H_1H_1^T & F_2\Delta_{\bar{r}}^2 + \Delta_r^2F_3^T + H_1H_2^T \\ F_3\Delta_r^2 + \Delta_{\bar{r}}^2F_2^T + H_2H_1^T & F_4\Delta_{\bar{r}}^2 + \Delta_{\bar{r}}^2F_4^T + H_2H_2^T \end{bmatrix} = 0.$$

Solving for F_3, one obtains

$$F_3 = -(\Delta_{\bar{r}}^2F_2^T + H_2H_1^T)(\Delta_r^2)^{-1} \tag{5.2}$$

and

$$H_2H_2^T = -F_4\Delta_{\bar{r}}^2 - \Delta_{\bar{r}}^2F_4^T,$$

which immediately gives the second part of (5.1). The estimate (5.1) then follows from (5.2). Q.E.D.

We can also obtain a simple relation between the distance μ and the feedback gain necessary to move a pole of the system. We will see that this relationship is limited in the sense that it holds only as the feedback gain applied (or else the distance the pole is moved) becomes sufficiently small.

Theorem 10: Assume the pair $(A, B) \in Y$ and that λ_n is a simple eigenvalue of A. Then for any sufficiently small $h > 0$, there exists a feedback matrix K with norm bounded by h such that all the eigenvalues $\nu_1, \cdots, \nu_n$ of the closed-loop matrix $A + BK$ differ from λ_n by at least $h\mu(A, B)$.

Proof: Let $\alpha = 1/2 \min_{1 \leq i \leq n-1} |\lambda_i - \lambda_n|$. Let the Schur decomposition of the pair (A, B) be

$$\bar{A} \equiv PAP^H = \begin{bmatrix} A_{11} & A_{12} \\ 0 & \lambda_n \end{bmatrix} \cdot \bar{B} \equiv PB = \begin{bmatrix} B_1 \\ b_n \end{bmatrix}$$

where P is a unitary matrix, and $b_n \in C^{1-m}$ is a row vector. It is trivial to note that $\mu \equiv \mu(\bar{A}, \bar{B}) = \mu(A, B)$, and also that $\|b_n\| \geq \mu$. Choose a positive $h < \alpha/\mu$. Define $K = [0, \cdots, 0, k_n]P$, where $k_n = b_n^T h/\|b_n\|$. then Schur decomposition of $A + BK$ is

$$A + BK = P^H \begin{bmatrix} A_{11} & A_{12} + B_1k_n \\ 0 & \lambda_n + b_nk_n \end{bmatrix} P$$

and hence the eigenvalues $\nu_1, \cdots, \nu_n$ of $A + BK$ are, respectively, $\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + b_nk_n$. It is easy to see that λ_n differs from $\nu_1, \cdots, \nu_{n-1}$ by at least $2\alpha > h\mu$. The value λ_n differs from ν_n by $\|b_nk_n\| = h\|b_n\| \geq h\mu$. Q.E.D.

This theorem relates the distance measure μ to the feedback gain necessary to move a pole. Specifically, it says that if we want to move a simple pole by ϵ, we can find a feedback matrix K whose norm is bounded by $h = \epsilon/\mu$. In the limit as the gains applied become sufficiently small, we can say that if we are far from an uncontrollable system, then μ is large, and the gain needed is of the same order of magnitude as the amount we move the pole. Conversely, it says that if we have a "hard-to-move" pole, i.e., any feedback of norm h moves the pole by no more than ϵh, then it follows that we have a bound on μ, namely $\mu \leq \epsilon$. In words, if the gain needed to move a pole is much larger than the amount the pole is moved (by a factor $1/\epsilon$), then the original system must be close (within ϵ) to an uncontrollable system.

VI. Concluding Remarks

We have given a relationship between the distance from a controllable system, as measured by perturbations to its coefficients, and schemes used classically to indicate when a system is "hard to control" in terms of energy. We have shown that one cannot blindly interpret the singular values of the controllability matrix in the most direct way if one is interested in perturbations to the coefficients; rather one must look for gaps among these singular values. We have concluded with a relationship between the distance to the nearest uncontrollable system and the state feedback gains needed to move a simple pole. These results, although not unexpected, provide a more solid foundation on which to interpret the distance measures μ and μ_r.

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