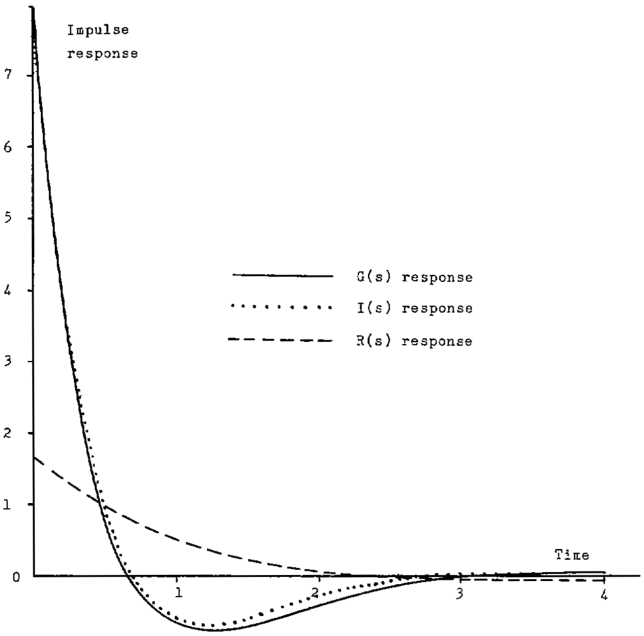

Fig. 1.

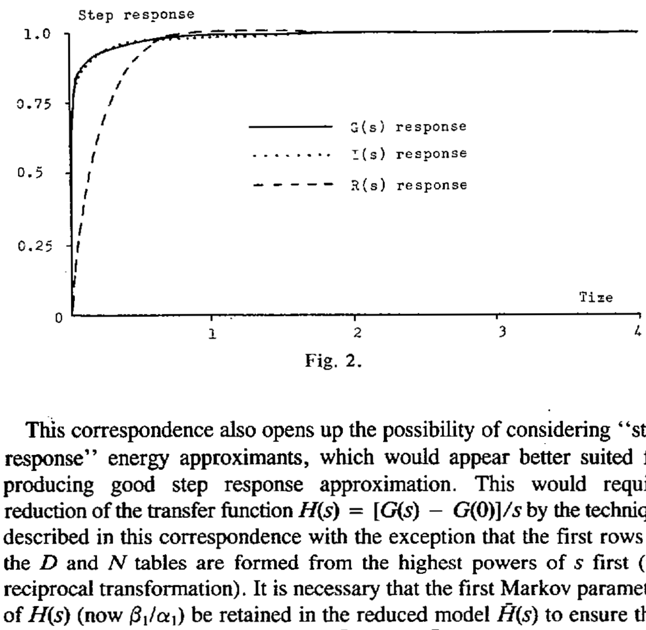

Fig. 2.

This correspondence also opens up the possibility of considering "step response" energy approximants, which would appear better suited for producing good step response approximation. This would require reduction of the transfer function $H(s) = [G(s) - G(0)]/s$ by the technique described in this correspondence with the exception that the first rows in the D and N tables are formed from the highest powers of s first (no reciprocal transformation). It is necessary that the first Markov parameter of $H(s)$ (now β_1/α_1) be retained in the reduced model $\bar{H}(s)$ to ensure that the final reduced model, given by $\bar{G}(s) = s\bar{H}(s) + G(0)$, is a proper rational transfer function. However, initial work in this direction seems to indicate that step energy approximants are not significantly better for step responses than impulse energy approximants (when the first time moment is preserved), taking into account the extra computation involved. Nevertheless, further work is needed to form more definite conclusions.

REFERENCES

[1] M. F. Hutton and B. Friedland, "Routh approximations for reducing order of linear, time-invariant systems," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 329-337, June 1975.

[2] A. S. Rao, S. S. Lamba, and S. Vittal Rao, "Routh-approximant time-domain reduced-order models for single-input single-output systems," in *Proc. IEE*, vol. 125, pp. 1059-1063, Oct. 1978.

[3] Y. Shamash, "Stable biased reduced order models using the Routh method of reduction," *Int. J. Syst. Sci.*, vol. 11, pp. 641-654, 1980.

[4] Y. Bistritz and U. Shaked, "Stable linear systems simplification via Padé approximants to Hurwitz polynomials," *J. Dynam. Syst. Meas. Contr.*, vol. 103, pp. 279-284, Sept. 1981.

[5] G. Hwang and K. Y. Wang, "Optimal Routh approximations for continuous-time systems," *Int. J. Syst. Sci.*, vol. 15, pp. 249-259, 1984.

[6] Y. Shamash, "Critical review of methods for deriving stable reduced-order models," in *IFAC Symp. Proc. Identif. Syst. Param. Estimat.*, Washington, DC, 1982, pp. 1519-1523.

[7] Y. Shamash, "Failure of the Routh-Hurwitz method of reduction," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 314-315, Apr. 1980.

[8] G. J. Lastman, N. K. Sinha, and P. Rózsa, "On the selection of states to be retained in a reduced-order model," in *IEE Proc. Part D*, vol. 131, pp. 15-22, Jan. 1984.

[9] T. N. Lucas and A. M. Davidson, "Frequency-domain reduction of linear systems using Schwarz approximation," *Int. J. Contr.*, vol. 37, pp. 1167-1178, 1983.

Model Reduction Via a Quasi-Kalman Decomposition

WU-SHENG LU AND E. BRUCE LEE

Abstract—**A system theoretic model reduction algorithm is proposed which appears to be simple to work with. The algorithm is based on a quasi-Kalman decomposition.**

I. INTRODUCTION

Approximating linear systems by simple models has been of concern for engineers and system-theorists for a long time; see [1]-[7] and the references therein.

Here we deal with model reduction via an explicit structural property of linear time-invariant systems, the so-called quasi-Kalman decomposition (QKD) which was initiated in a recent study [8] and will be further developed now.

One may recall the well-known Kalman controllability decomposition which is a claim that for an uncontrollable system $S = (A, B, C)$ of dimension n, there exists a nonsingular T such that the algebraically equivalent triple $(\hat{A}, \hat{B}, \hat{C})$ has the form

$$\hat{A} = TAT^{-1} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}, \quad \hat{B} = TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \hat{C} = CT^{-1} = [C_1 \quad C_2]$$

where (A_1, B_1) is a controllable pair of dimension $r < n$. Thus, the system has an rth-order external description $C_1(zI - A_1)^{-1}B_1$ rather than an nth-order one. A similar decomposition holds for an unobservable pair (A, C). Further, one notices that, in spite of the genericity of controllability and observability for finite-dimensional linear systems, the performance of a controlled system depends heavily on its representative from among the set of all controllable systems. At this point, it is natural to consider the distance between a given controllable and observable system (A, B, C) and the nearest uncontrollable (and/or unobservable) system, denoted by $\mu(A, B)$ (and $\mu(A, B, C)$ or $\mu(A, C)$, respectively). Roughly speaking, the QKD for a given minimal triple (A, B, C) (a system is called minimal if it is both controllable and observable) is an algebraic equivalent triple with appropriate partitioning

$$\hat{A} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad \hat{C} = [C_1 \quad C_2]$$

where $\|A_2\|$, $\|A_3\|$, $\|B_2\|$, and $\|C_2\|$ are closely related to the distance $\mu(A, B, C)$. In other words, in the QKD of a given minimal triple (A, B, C) the distance $\mu(A, B, C)$ is structurally "visible." Concerning the model reduction issue, it will become clearer [2], [9] that a system can

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successfully be approximated by a lower order model if it is near an uncontrollable and unobservable one in the same state space. It seems obvious tc consider the QKD to provide a simple tool to deal with the model reduction question.

In the next section, the QKD and some related ideas are described. As an application, a model reduction algorithm along with its L^∞-error bound are given in Section III. Several examples then follow in Section IV to illustrate the approach.

In what follows, we shall denote the largest singular value of a matrix F by $\bar{\sigma}(F)$, and the corresponding norm $\|F\|$ is assigned to be the 2-norm, i.e., $\|F\| = \bar{\sigma}(F)$.

II. The Quasi-Kalman Decomposition of a Minimal (A, B, C)

Let $S = (A, B, C)$ be a minimal realization of the transfer function $G(z) \in R^{p\times m}(z)$ with $A \in R^{n\times n}$.

Definition 2.1: The distance between S and the nearest uncontrollable/observable, the nearest controllable/unobservable, the nearest uncontrollable/unobservable system are defined by

$$\mu(A, B) = \min_{\delta A,\, \delta B} \|[\delta A, \delta B]\| \tag{2.1}$$

$$\mu(A, C) = \min_{\delta A,\, \delta C} \left\| \begin{bmatrix} \delta A \\ \delta C \end{bmatrix} \right\| \tag{2.2}$$

$$\mu(A, B, C) = \min_{\delta A,\, \delta B,\, \delta C} \left\| \begin{bmatrix} \delta A & \delta B \\ \delta C & 0 \end{bmatrix} \right\| \tag{2.3}$$

respectively, where $\delta A \in R^{n\times n}$, $\delta B \in R^{n\times m}$, $\delta C \in R^{p\times n}$ such that $(A + \delta A, B + \delta B)$, $(A + \delta A, C + \delta C)$, $(A + \delta A, B + \delta B, C + \delta C)$ are uncontrollable, unobservable, and uncontrollable/unobservable, respectively.

In [10], [11], [8], the computation of $\mu(A, B)$ as well as $\mu(A, C)$ has been described. It turns out that $\mu(A, B, C)$ is a model-reduction-related quantity whose tight estimate can be found through a quasi-Kalman decomposition of the system S which will be defined later.

Let

$$P = [B \; AB \cdots A^{n-1}B], \quad Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

and $H = QP$. Notice that rank P = rank Q = rank $H = n$, hence, the right inverse of P, denoted by P_r^{-1}, and the left inverse of Q, denoted by Q_l^{-1}, exist. Next obtain the singular value decomposition (SVD) of H

$$H = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V \tag{2.4}$$

with U and V orthogonal and $\Sigma = \operatorname{diag}\{\sigma_1 \cdots \sigma_n\}$, $\sigma_1 \geqslant \cdots \geqslant \sigma_n > 0$. Observe that the matrix equation

$$TP = [\Sigma^{1/2} \quad 0]V \tag{2.5}$$

has solution

$$T = [\Sigma^{1/2} \; 0]\, VP_r^{-1} \tag{2.6}$$

and that T is an $n \times n$ nonsingular matrix. Thus, (2.4) gives

$$H = QT^{-1}TP = QT^{-1}[\Sigma^{1/2} \; 0]\, V = U[\Sigma^{1/2} \; 0]^T[\Sigma^{1/2} \; 0]\, V.$$

So

$$QT^{-1} = U[\Sigma^{1/2} \; 0]^T \tag{2.7}$$

and

$$T^{-1} = Q_l^{-1}U[\Sigma^{1/2} \; 0]^T. \tag{2.8}$$

Definition 2.2: For a given minimal triple (A, B, C), its quasi-Kalman decomposition is the algebraically equivalent triple (TAT^{-1}, TB, CT^{-1}) where T is given by (2.6) [or (2.8)].

A remarkable property of the QKD is described in the following.

Theorem 2.3: For a fixed integer $0 < r < n$, let

$$\hat{A} = TAT^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

$$\hat{B} = TB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \hat{C} = CT^{-1} = [C_1 \; C_2]$$

where $A_1 \in R^{r\times r}$, $A_4 \in R^{(n-r)\times(n-r)}$, $B_1 \in R^{r\times m}$, $C_1 \in R^{p\times r}$, then

$$\|A_2\| \leqslant \beta_r \sigma_{r+1}^{1/2} \tag{2.9}$$

$$\|A_3\| \leqslant \beta_r \sigma_{r+1}^{1/2} \tag{2.10}$$

$$\|B_2\| \leqslant \sigma_{r+1}^{1/2} \tag{2.11}$$

$$\|C_2\| \leqslant \sigma_{r+1}^{1/2} \tag{2.12}$$

where $\beta_r = \bar{\sigma}(A_o)/\sigma_r^{1/2}$, A_o is the observer form of the matrix A [12, p. 51], and T is given by (2.6).

Proof: Note that $\hat{B} = TP[I_m 0]^T$, (2.5) then gives

$$\hat{B} = \begin{bmatrix} \Sigma_1^{1/2} & 0 & 0 \\ 0 & \Sigma_2^{1/2} & 0 \end{bmatrix} V \begin{bmatrix} I_m \\ 0 \end{bmatrix} = \begin{bmatrix} \Sigma_1^{1/2} & 0 \\ 0 & \Sigma_2^{1/2} \end{bmatrix} V_1$$

$$\equiv \begin{bmatrix} \Sigma_1^{1/2} & 0 \\ 0 & \Sigma_2^{1/2} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} \Sigma_1^{1/2} V_{11} \\ \Sigma_2^{1/2} V_{21} \end{bmatrix} \tag{2.13}$$

where V_1 is the upper left $n \times m$ submatrix of V. Since V_{21} is a submatrix of the orthogonal matrix V, we have $\|V_{21}\| \leqslant 1$, and therefore $\|B_2\| \leqslant \|\Sigma_2^{1/2}\| = \sigma_{r+1}^{1/2}$. Similarly, $\hat{C} = [I_p \; 0]QT^{-1}$, (2.7) then yields

$$\hat{C} = [I_p \; 0]\, U \begin{bmatrix} \Sigma_1^{1/2} & 0 \\ 0 & \Sigma_2^{1/2} \\ 0 & 0 \end{bmatrix} = U_1 \begin{bmatrix} \Sigma_1^{1/2} & 0 \\ 0 & \Sigma_2^{1/2} \end{bmatrix}.$$

$$\equiv [U_{11} \; U_{12}] \begin{bmatrix} \Sigma_1^{1/2} & 0 \\ 0 & \Sigma_2^{1/2} \end{bmatrix} = [U_{11}\Sigma_1^{1/2} \; U_{12}\Sigma_2^{1/2}] \tag{2.14}$$

where U_1 is the upper left $p \times n$ submatrix of U, which gives the estimate (2.12). To prove (2.10) set $TP = \hat{P}$ and observe that

$$\hat{A}\hat{P} = \hat{P}(A_{co} \otimes I_m) = \begin{bmatrix} \Sigma_1^{1/2} & 0 & 0 \\ 0 & \Sigma_2^{1/2} & 0 \end{bmatrix} V(A_{co} \otimes I_m)$$

where A_{co} is the controllability form of A [12, p. 51] and $\otimes$ represents the Kronecker product; (2.5) then modifies the above expression to be

$$\hat{A} = \begin{bmatrix} \Sigma_1^{1/2} & 0 & 0 \\ 0 & \Sigma_2^{1/2} & 0 \end{bmatrix} V(A_{co} \otimes I_m)V^T \begin{bmatrix} \Sigma_1^{-1/2} & 0 & 0 \\ 0 & \Sigma_2^{-1/2} & 0 \end{bmatrix}^T.$$

Let the upper left $n \times n$ submatrix of $V(A_{co} \otimes I_m)V^T$ be $F = (F_{ij})$, $1 \leqslant i, j \leqslant 2$ with $F_{11} \in R^{r\times r}$ and $F_{22} \in R^{(n-r)\times(n-r)}$; one then rewrites $\hat{A}$ as

$$\hat{A} = \begin{bmatrix} \Sigma_1^{1/2}F_{11}\Sigma_1^{-1/2} & \Sigma_1^{1/2}F_{12}\Sigma_2^{-1/2} \\ \Sigma_2^{1/2}F_{21}\Sigma_1^{-1/2} & \Sigma_2^{1/2}F_{22}\Sigma_2^{-1/2} \end{bmatrix}. \tag{2.15}$$

Using some elementary properties of the Kronecker product [13, p. 415], it can be seen that $\|A_{co} \otimes I_m\| = \|A_{co}\|$. Thus, by (2.15)

$$\|A_3\| = \|\Sigma_2^{1/2}F_{21}\Sigma_1^{-1/2}\| \leqslant \|\Sigma_2^{1/2}\|\,\|V(A_{co} \otimes I_m)\|\,\|\Sigma_1^{-1/2}\|$$

$$\leqslant \frac{\|A_{co}\|}{\sigma_r^{1/2}}\sigma_{r+1}^{1/2} = \beta_r \sigma_{r+1}^{1/2}$$

where the last equality is because $A_{co} = A_o^T$. To show (2.9), let $\hat{Q} =$

QT^{-1} and note that

$$\hat{Q}\hat{A}=(A_0\otimes I_p)\hat{Q}=(A_0\otimes I_p)U\begin{bmatrix}\Sigma_1^{1/2} & 0 & 0\\ 0 & \Sigma_2^{1/2} & 0\end{bmatrix}^T$$

which gives

$$\hat{A}=\begin{bmatrix}\Sigma_1^{-1/2} & 0 & 0\\ 0 & \Sigma_2^{-1/2} & 0\end{bmatrix}U^T(A_o\otimes I_p)U\begin{bmatrix}\Sigma_1^{1/2} & 0 & 0\\ 0 & \Sigma_2^{1/2} & 0\end{bmatrix}^T$$

$$=\begin{bmatrix}\Sigma_1^{-1/2}G_{11}\Sigma_1^{1/2} & \Sigma_1^{-1/2}G_{12}\Sigma_2^{1/2}\\ \Sigma_2^{-1/2}G_{21}\Sigma_1^{1/2} & \Sigma_2^{-1/2}G_{22}\Sigma_2^{1/2}\end{bmatrix} \tag{2.16}$$

where (G_{ij}), $1 \leqslant i, j \leqslant 2$, with $G_{11} \in R^{r\times r}$, $G_{22} \in R^{(n-r)\times(n-r)}$ is the upper left $n \times n$ submatrix of $U^T(A_o \otimes I_p)U$. Therefore,

$$\|A_2\|=\|\Sigma_1^{-1/2}G_{12}\Sigma_2^{1/2}\|\leqslant\|\Sigma_1^{-1/2}\|\|U^T(A_o\otimes I_p)U\|\|\Sigma_2^{1/2}\|$$

$$\leqslant\frac{\|A_o\|}{\sigma_r^{1/2}}\sigma_{r+1}^{1/2}=\beta_r\sigma_{r+1}^{1/2}. \qquad \Box$$

For a minimal realization $S = (A, B, C)$ define the equivalence class $E_s = \{(\tilde{A}, \tilde{B}, \tilde{C})|$ nonsingular L such that $\tilde{A} = LAL^{-1}$, $\tilde{B} = LB$, $\tilde{C} = CL^{-1}\}$. Then we have the following.

Theorem 2.4: If the singular values of H are distinct, then every system in E_s has the same QKD.

Proof: It is known that when the singular values of H are distinct, the matrix V_1 in (2.13) is uniquely determined by the sets of orthogonal eigenvectors for the nonzero eigenvalues of H^TH, and hence U_1 in (2.14) is also uniquely determined (see, e.g., [15, p. 6]). The expressions (2.13), (2.14), and (2.15) thus show that the QKD of the given S is unique. Further, let $\tilde{P}$ and $\tilde{Q}$ be the controllability and observability matrices for $\tilde{S} = (\tilde{A}, \tilde{B}, \tilde{C}) \in E_s$, respectively. Thus, $\tilde{P} = LP$, $\tilde{Q} = QL^{-1}$, $\tilde{P}_r^{-1} = P_r^{-1}L^{-1}$, $\tilde{Q}_l^{-1} = LQ_l^{-1}$, and $\tilde{H} = \tilde{Q}\tilde{P} = QP = H$. So the transformation matrix which brings $\tilde{S}$ to its QKD is $\tilde{T} = [\Sigma^{1/2}\ 0]V\tilde{P}_r^{-1} = [\Sigma^{1/2}\ 0]VP_r^{-1}L^{-1} = TL^{-1}$. Therefore, the QKD of the system $\tilde{S}$ is $(\tilde{T}\tilde{A}\tilde{T}^{-1}, \tilde{T}\tilde{B}, \tilde{C}\tilde{T}^{-1})$ which is the QKD of the system S. $\Box$

The QKD of a given minimal (A, B, C) also provides an appropriate framework to obtain an upper bound on $\mu(A, B, C)$. First rewrite A by mixing its two expressions (2.15) and (2.16) as

$$\hat{A}=\begin{bmatrix}\Sigma_1^{1/2}F_{11}\Sigma_1^{-1/2} & \Sigma_1^{-1/2}G_{12}\Sigma_2^{1/2}\\ \Sigma_2^{1/2}F_{21}\Sigma_1^{-1/2} & \Sigma_2^{-1/2}G_{22}\Sigma_2^{1/2}\end{bmatrix} \tag{2.17}$$

which, in the special case $r = n - 1$, becomes

$$\hat{A}=\begin{bmatrix}\hat{\Sigma}_1^{1/2}\hat{F}_{11}\hat{\Sigma}_1^{-1/2} & \sigma_n^{1/2}\hat{\Sigma}_1^{-1/2}\hat{G}_{12}\\ \sigma_n^{1/2}\hat{F}_{21}\hat{\Sigma}_1^{-1/2} & \hat{G}_{22}\end{bmatrix} \tag{2.18}$$

where $\hat{\Sigma}_1 = \operatorname{diag}\{\sigma_1, \cdots, \sigma_{n-1}\}$, $\hat{G}_{12} \in R^{(n-1)\times 1}$, $\hat{F}_{21} \in R^{1\times(n-1)}$, and $\hat{G}_{22} \in R$. Correspondingly, by (2.13) and (2.14),

$$\hat{B}=\begin{bmatrix}\hat{\Sigma}_1^{1/2}\hat{V}_{11}\\ \sigma_n^{1/2}\hat{V}_{21}\end{bmatrix},\quad \hat{C}=[\hat{U}_{11}\hat{\Sigma}_1^{1/2}\ \ \sigma_n^{1/2}\hat{U}_{12}] \tag{2.19}$$

where $\hat{V}_{21} \in R^{1\times m}$ and $\hat{U}_{12} \in R^{p\times 1}$. Thus, $(\hat{A} + \delta\hat{A}, \hat{B} + \delta\hat{B}, \hat{C} + \delta\hat{C})$ with

$$\delta\hat{A}=\begin{bmatrix}0 & -\sigma_n^{1/2}\hat{\Sigma}_1^{-1/2}\hat{G}_{12}\\ -\sigma_n^{1/2}\hat{F}_{21}\hat{\Sigma}_1^{-1/2} & 0\end{bmatrix},$$

$$\delta\hat{B}=\begin{bmatrix}0\\ -\sigma_n^{1/2}\hat{V}_{21}\end{bmatrix} \text{ and } \delta\hat{C}=[0\ \ -\sigma_n^{1/2}\hat{U}_{12}]$$

is neither controllable nor observable, so that for the triple $(\hat{A}, \hat{B}, \hat{C})$ given in (2.18) and (2.19)

$$\mu(\hat{A}, \hat{B}, \hat{C})\leqslant\left\|\left[\begin{array}{cc|c}0 & \hat{\Sigma}_1^{-1/2}\hat{G}_{12} & 0\\ \hat{F}_{21}\hat{\Sigma}_1^{-1/2} & 0 & \hat{V}_{21}\\ \hline 0 & \hat{U}_{12} & 0\end{array}\right]\right\|\sigma_n^{1/2}.$$

Further notice that $\|\hat{F}_{21}\| \leqslant \bar{\sigma}(A_o)$, $\|\hat{G}_{12}\| \leqslant \bar{\sigma}(A_o)$, $\|\hat{U}_{12}\| \leqslant 1$, and $\|\hat{V}_{21}\| \leqslant 1$, hence

$$\left\|\begin{bmatrix}0 & \hat{\Sigma}_1^{-1/2}\hat{G}_{12} & 0\\ \hat{F}_{21}\hat{\Sigma}_1^{-1/2} & 0 & \hat{V}_{21}\\ 0 & \hat{U}_{12} & 0\end{bmatrix}\right\|\leqslant\left\|\begin{bmatrix}0 & \hat{\Sigma}_1^{-1/2}\hat{G}_{12}\\ \hat{F}_{21}\hat{\Sigma}_1^{-1/2} & 0\end{bmatrix}\right\|+\left\|\begin{bmatrix}0 & \hat{V}_{21}\\ \hat{U}_{12} & 0\end{bmatrix}\right\|$$

$$\leqslant\max(\|\hat{\Sigma}_1^{-1/2}\hat{G}_{12}\|, \|\hat{F}_{21}\hat{\Sigma}_1^{-1/2}\|)+\max(\|\hat{U}_{12}\|, \|\hat{V}_{21}\|)\leqslant 1+\beta_{n-1}$$

where $\beta_{n-1} = \bar{\sigma}(A_o)/\sigma_{n-1}^{1/2}$. Therefore,

$$\mu(\hat{A}, \hat{B}, \hat{C})\leqslant(1+\beta_{n-1})\sigma_n^{1/2}. \tag{2.20}$$

Based on the above estimate, we now have the following.

Theorem 2.5: Given a minimal realization (A, B, C), then

$$\mu(A, B, C)\leqslant\operatorname{cond}(T_1)(1+\beta_{n-1})\sigma_n^{1/2} \tag{2.21}$$

where cond (T_1) is the condition number of diag $\{T, 1\}$ with T given in (2.6).

Proof: Since

$$\begin{bmatrix}A & B\\ C & 0\end{bmatrix}=\begin{bmatrix}T^{-1} & 0\\ 0 & I_p\end{bmatrix}\begin{bmatrix}\hat{A} & \hat{B}\\ \hat{C} & 0\end{bmatrix}\begin{bmatrix}T & 0\\ 0 & I_m\end{bmatrix}$$

so if $(\delta\hat{A}\ \delta\hat{B}\ \delta\hat{C})$ is a disturbance of $(\hat{A}\ \hat{B}\ \hat{C})$ such that $(\hat{A} + \delta\hat{A}, \hat{B} + \delta\hat{B}, \hat{C} + \delta\hat{C})$ is neither controllable nor observable, then

$$\begin{bmatrix}\delta A & \delta B\\ \delta C & 0\end{bmatrix}\equiv\begin{bmatrix}T^{-1} & 0\\ 0 & I_p\end{bmatrix}\begin{bmatrix}\delta\hat{A} & \delta\hat{B}\\ \delta\hat{C} & 0\end{bmatrix}\begin{bmatrix}T & 0\\ 0 & I_m\end{bmatrix}$$

will be the same kind of disturbance for (A, B, C). The estimate (2.20) then immediately yields (2.21). $\Box$

We conclude this section with a remark on the relationship between QKD and the well-known balanced realization [2], [3]. By (2.5) one has

$$\sum_{i=0}^{n-1}\hat{A}^i\hat{B}\hat{B}^T(\hat{A}^T)^i=TP(TP)^T=\Sigma. \tag{2.22}$$

Similarly, (2.7) gives

$$\sum_{i=0}^{n-1}(\hat{A}^T)^i\hat{C}^T\hat{C}(\hat{A})^i=(QT^{-1})^TQT^{-1}=\Sigma. \tag{2.23}$$

In other words, the transformation T defined in (2.6) that gives the QKD for a given discrete-time system (A, B, C) is the one that makes the "truncated" controllability and observability Gramians be equal and diagonal.

III. A Model Reduction Algorithm and Its L^∞-Error Bound

Based on the QKD theory as developed above, a suggested model reduction algorithm for a given multivariable discrete-time system $S = (A, B, C)$ is the following.

Algorithm 3.1:

1) Form the controllability and observability matrices P and Q;

2) Form the Hankel matrix $H = QP$ and take its singular value decomposition (2.4);

3) Calculate the transform matrix T by (2.6);

4) Compute the QKD of the system S and partition it as

$$\hat{A}=TAT^{-1}=\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix},\ \hat{B}=TB=\begin{bmatrix}B_1\\ B_2\end{bmatrix} \text{ and } \hat{C}=CT^{-1}=[C_1\ \ C_2] \tag{3.1}$$

where $A_1 \in R^{r\times r}$, $B_1 \in R^{r\times m}$, and $C_1 \in R^{p\times r}$. Take as the rth-order reduced model of S the subsystem $S_r = (A_1, B_1, C_1)$.

One of the most important issues for model approximation is stability. If the considered triple is associated with a discrete-time system, then we have the following.

Theorem 3.2: Assume that i) the discrete-time system (A, B, C) is (BIBO) stable, ii) $BB^T - A^nBB^T(A^T)^n \geq 0$, or, ii') $C^TC - (A^T)^nC^TCA^n \geq 0$ then the subsystems $S_r = (A_1, B_1, C_1)$ and $S_{\bar r} = (A_4, B_2, C_2)$ are stable.

Proof: Denote $BB^T - A^nBB^T(A^T)^n = RR^T$, $C^TC - (A^T)^nC^TCA^n = W^TW$ if the condition ii) and ii') hold. By means of (2.22) and (2.23), one has

$$\hat{A}\Sigma\hat{A}^T - \Sigma = -(TR)(TR)^T \text{ and } \hat{A}^T\Sigma\hat{A} - \Sigma = -(WT^{-1})^T(WT^{-1})$$

which means that Σ is the controllability Gramian (if ii) holds) or the observability Gramian (if ii') holds) of the triple $(\hat{A}, TR, WT^{-1})$. By [14, Theorem 3.2], both systems S_r and $S_{\bar r}$ are stable. □

In the rest of this section it will be assumed that the triple $S = (A, B, C)$ is associated with a stable, discrete-time system and that the matrices A_1 and A_4 obtained by Algorithm 3.1 are stable, i.e., the eigenvalues of A_1 and A_4 are in the unit circle of the complex plane.

Let $G(z) = C(zI - A)^{-1}B$, let W_c and W_o be the controllability and the observability Gramians, respectively. The Hankel-singular-values [7] of $G(z)$ are defined by

$$\sigma_i(G) = [\lambda_i(W_cW_o)]^{1/2}, \qquad 1 \leqslant i \leqslant n \tag{3.3}$$

where $\{\lambda_i(W_cW_o)\}$ are eigenvalues of W_cW_o. The L^∞-norm of $G(z)$ is defined by $\|G(e^{j\omega})\|_\infty = \sup_{\omega\in R} \bar\sigma(G(e^{j\omega}))$. In a recent paper by Glover [7, Corollary 9.3], it has been shown that

$$\|G(e^{j\omega})\|_\infty \leqslant 2(\sigma_1 + \cdots + \sigma_n) \tag{3.4}$$

where σ_i's are those given in (3.3). We will use this estimate in the following error analysis.

Let $G_1(z) = C_1(zI - A_1)^{-1}B_1$ and $R(z) = G(z) - G_1(z)$. Simple manipulation gives

$$R(z) = G(z) - G_1(z) = [C_2 + C_1(zI - A_1)^{-1}A_2][zI - A_4 - A_3(zI - A_1)^{-1}A_2]^{-1} \cdot [B_2 + A_3(zI - A_1)^{-1}B_1]. \tag{3.5}$$

By estimates (2.9)–(2.12), an intuitive inspection of expression (3.5) indicates that one has no difficulty obtaining the estimate $\|G(z) - G_1(z)\| \leqslant \gamma\sigma_{r+1}$ by a routine norm-estimation technique, where γ is a constant relevant to the system data $(\hat{A}, \hat{B}, \hat{C})$. On the other hand, one may observe that $R(z)$ given in (3.5) is itself a strictly proper rational matrix so that there exists a minimal triple (F, G, H) of dimension l such that $R(z) = H(zI - F)^{-1}G$. Therefore, by (3.4), a better error bound is

$$\|Gj\omega) - G_1(j\omega)\|_\infty \leqslant 2\sum_1^l \sigma_i(R) \tag{3.6}$$

where the Hankel-singular-values $\{\sigma_i(R),\ 1 \leqslant i \leqslant l\}$ can be computed through triple $\{F, G, H\}$.

IV. Examples and Concluding Remarks

Bearing in mind the relations (2.22) and (2.23), it is expected that the QKD reduced model will be quite close to the one obtained by the balanced-realization-based technique. For instance, the discrete-time triple (A, b, c) with

$$A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c^T$$

is stable, minimal, and balanced. It is easy to check that the above (A, B, C) is already in its QKD form. Therefore, both approaches give the same first-order reduced model $G_1(z) = 1/z$ which is close to the optimal (in L^∞ sense) first-order approximation $(16/15)\, z$ [3].

The following examples also indicate the closeness of the two approaches.

Example 4.1 [3]: Given a third-order system with transfer function $G(z) = z^{-2} + z^{-3}$ and the following minimal realization:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c = [0 \quad 1 \quad 1]$$

for which

$$W_c = I \text{ and } W_o = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The standard computation procedure [5] gives the balanced realization of $G(z)$ as (A_b, b_b, c_b) with

$$A_b = \begin{bmatrix} 0.6773 & 0.5236 & 0.09633 \\ -0.5236 & -0.04788 & 0.4046 \\ 0.09633 & -0.4046 & -0.6294 \end{bmatrix}, \quad b_b = \begin{bmatrix} 0.7933 \\ 0.8230 \\ 0.2188 \end{bmatrix},$$

and $c_b = [0.7933 \quad -0.8230 \quad 0.2188]$.

On the other hand, from $G(z) = z^{-2} + z^{-3}$, one can write

$$H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The rest of the Algorithm 3.1 then gives its QKD as

$$\hat{A} = \begin{bmatrix} 0.6773 & 0.4356 & 0.09633 \\ -0.6294 & -0.04788 & 0.4864 \\ 0.09633 & -0.3366 & -0.6294 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 0.7933 \\ 0.9893 \\ 0.2188 \end{bmatrix},$$

and $\hat{c} = [0.7933 \quad -0.6846 \quad 0.2188]$.

Hence, both approaches yield the same first-order approximation

$$G_1(z) = \frac{0.6293}{z - 0.6733}$$

and the same second-order approximation

$$G_2(z) = \frac{-0.048z + 1.1726}{z^2 - 0.6294z + 0.2417}.$$

Example 4.2: Consider the second-order discrete-time system described by $G(z) = (z + 0.1)/(z^2 + 0.1z - 0.3)$ with minimal realization (A, b, c) where

$$A = \begin{bmatrix} -0.1 & 0.3 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } c = [1 \quad 0.1].$$

One now computes

$$Q = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.3 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -0.1 \\ 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

So

$$\Sigma^{1/2} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{0.3} \end{bmatrix}, \quad V = I, \text{ and } T = \begin{bmatrix} 1 & 0.1 \\ 0 & \sqrt{0.3} \end{bmatrix}.$$

Thus,

$$\hat{A} = \begin{bmatrix} 0 & \sqrt{0.3} \\ \sqrt{0.3} & -0.1 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{c} = [1 \quad 0]$$

and the first-order approximation due to the QKD approach is

$$\hat{G}_1(z) = \frac{1}{z}.$$

In order to use the balanced-realization-based approach, one calculates

$$W_c=\begin{bmatrix} 1.2281 & -0.1754 \\ -0.1754 & 1.2281 \end{bmatrix} \text{ and } W_0=\begin{bmatrix} 1.1010 & 0.09567 \\ 0.09567 & 0.1091 \end{bmatrix}$$

and then the algorithm given in [5] yields the balanced realization of as

$$A_b=\begin{bmatrix} -0.03779 & -0.5498 \\ -0.5498 & -0.06221 \end{bmatrix}, \; b_b=\begin{bmatrix} -0.9708 \\ 0.03442 \end{bmatrix},$$

$$c_b=[-0.9708 \quad 0.03442]$$

which gives the first-order reduced model

$$G_1(z)=\frac{0.9425}{z+0.03779}.$$

Clearly, $G_1(z)$ is quite close to $\hat{G}_1(z)$.

Remark: One may compare the error bound given in (3.6) and the error bound for a reduced model via a balanced realization. For instance, in Example 4.2 the error function $R(z)$ has a minimal realization (F, G, H) with

$$F=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0.3 & -0.1 \end{bmatrix}, \; G=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } H=[0.3 \quad 0 \quad 0]$$

which yields the Hankel-singular-values $\sigma_1 = 0.3741$, $\sigma_2 = 0.3741$, and $\sigma_3 = 0.2$; thus (3.6) becomes $\|G(j\omega) - \hat{G}_1(j\omega)\|_\infty \leqslant 1.8964$. On the other hand, it is known [6], [7] that the L^∞-error bound for balanced-realization-based reduction is given by $\|G(j\omega) - G_1(j\omega)\|_\infty \leqslant 2\sigma_2(G) = 0.7023$ which is considerably smaller than the former even though $\hat{G}_1(z)$ is fairly close to $G_1(z)$. An improvement of the error bound for the QKD-based reduction and a further study on the comparison to the balanced-realization-based method is therefore needed.

It can now be concluded that the QKD of a linear time-invariant system gives a way to structurally separate the poorly controllable and poorly observable subspace from the well controllable and well observable part. We have also shown that this decomposition provides for discrete-time systems the possibility to reduce a high-order mathematical model to a lower order one with good response characteristics. The reduction algorithm appears to be simple to work with since it does not require solving Lyapunov type equations and at the same time the reduced model is close to the corresponding one obtained by the balanced realization approach. This approach is also promising in simplifying some distributive systems, for example, delay systems and two-dimensional systems. In this regard a further study is now underway and will be reported on later.

References

[1] E. J. Davison, "A method for simplifying linear dynamic systems," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 93-101, 1966.
[2] B. C. Moore, "Principal component analysis in linear systems: Controllability, observability, and model reduction," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 17-32, 1981.
[3] L. M. Silverman and M. Bettayeb, "Optimal approximation of linear systems," in *Proc. 1980 JACC.*
[4] S.-Y. Kung and D. W. Lin, "Optimal Hankel-norm model reductions: Multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 832-852, 1981.
[5] A. J. Laub, "Computation of balancing transformations," in *Proc. 1980 JACC.*
[6] D. Enns, "Model reduction based on frequency response for control system design," in *Workshop on Identification Contr. of Flexible Space Structures*, San Diego, CA, 1984.
[7] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds," *Int. J. Contr.*, vol. 39, pp. 1115-1193, 1984.
[8] D. Boley and W.-S. Lu, "Quasi-Kalman decomposition and its relationship to state feedback," 1984.
[9] P. Harshavardhana, E. A. Jonckeere, and L. M. Silverman, "Open and closed loop approximation techniques—An overview," in *Proc. IEEE 1983 Int. Symp. Circuits and Syst.*, 1983, pp. 126-129.
[10] G. S. Miminis, "Numerical algorithms for controllability and eigenvalue allocation," M.S. thesis, School Comput. Sci., McGill Univ., Montreal, Canada, 1981.
[11] R. Eising, "The distance between a system and the set of uncontrollable systems," Eindhoven Univ. Technol., Eindhoven, Memo. COSOR 82-19, 1982.
[12] T. Kailath, *Linear Systems.* Englewood-Cliffs, NJ: Prentice-Hall, 1980.
[13] S. Barnett, *Polynomials and Linear Control Systems.* New York: Marcel-Dekker, 1983.
[14] L. Pernebo and L. M. Silverman, "Model reduction via balanced state space representations," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 382-387, 1982.
[15] F. M. Callier and C. A. Desoer, *Multivariable Feedback Systems.* New York: Springer-Verlag, 1982.

Singular Perturbation for the Dynamic Interaction Measure

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Abstract—**The singular perturbation technique is used for the dynamic extension of the RGA (relative gain array) based on the state-space model. The dynamic interaction measure derived serves as a means of pairing input and output variables for the systems having substantially different time scales.**

I. Introduction

One of the powerful tools that has appreciated in chemical industry is the relative gain array (RGA) which was originally proposed by Bristol [1] as a means of measuring control loop interactions [7]. The RGA gives a steady-state measure of interaction and, in many cases, the steady-state RGA analysis is useful for the problem of pairing between the input and output variables.

However, there is a need in some cases to account for the dynamics, and several attempts have been made for the dynamic extension of the RGA [2], [3], [8], [9]. Among these is the approach taken by Tung and Edgar [8] based on the state-space model. State-space formulation is useful in getting insight into the control problem of many chemical processes such as distillation, extraction, etc. (see [6], for example).

On the other hand, it has long been recognized that the flow, level, and pressure control loops are substantially faster than the composition control loops, and in many cases the dynamics of the former are neglected for the analysis of the latter. The singular perturbation technique is known to be a useful tool for treating such systems having substantially different time scales (see [4] and [5], for example).

In this correspondence, we apply the singular perturbation technique to the dynamic extension of the RGA proposed by Tung and Edgar [8] for the analysis of such processes whose time constants are substantially different.

II. Singular Perturbation

Consider a singularly perturbed linear time-invariant system

$$\dot{x}=A_{11}x+A_{12}p+B_1u \tag{1}$$

$$\epsilon\dot{p}=A_{21}x+A_{22}p+B_2u \tag{2}$$

$$y=C_1x+C_2p \tag{3}$$

where ϵ is a small positive parameter, x and p are $n \times 1$ and $m \times 1$ state vectors, u is an $l \times 1$ input vector, y is an $l \times 1$ output vector, and A_{ij}, B_i, C_i are constant matrices of appropriate dimensions. It can be shown that the system equations (1) and (2) have n slow eigenvalues and m fast

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