3) Image of def: If $j\omega_0$ is a root of $p + p'$, it must be a root of *both* p and p'. Let $j\omega_0$ be a root of p such that $p(s) = (s - j\omega_0)^k q(s)$, $q(j\omega_0) \neq 0$. Then q is either quasi-real or quasi-imaginary. Further,

$$F(s) = 1 + \frac{k}{(s - j\omega_0)} + \frac{q'(s)}{q(s)}.$$

On def, $s - j\omega_0 = \epsilon e^{j\theta}$ with $\angle s: \pi/2 \xrightarrow{\pi\downarrow} -\pi/2$, and $q'(s)/q(s)$ will be nearly equal to $q'(j\omega_0)/q(j\omega_0)$ which is purely imaginary. Hence, the image of def will be close to a large semicircle in the RHP, traversed counterclockwise and it will not cross the negative-real axis.

From 1)-3) we conclude that $F(C)$ does not encircle the origin at all.

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Stabilization of Two-Dimensional Systems

E. BRUCE LEE AND WU-SHENG LU

Abstract—**Several new results on stabilization of discrete two-dimensional systems are presented. If the horizontal (or vertical) part of the system in the Roesser model is controllable, then the stabilizability question is the same as that for a related discrete delay system.**

I. INTRODUCTION

There is interest in the Roesser two-dimensional model, which was originally motivated by image processing, as a model for certain multipass processes (such as machining of metal). Recent results [1] have demonstrated that stability along the pass is equivalent to certain two-dimensional stability criteria. The interest here is in acquiring stability by the use of local feedback control.

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Stabilizing a discrete two-dimensional (2-D) system by state feedback or output feedback has long been of interest; see, e.g., [2]-[8]. It is now becoming clear [3]-[8] that stabilizing a discrete 2-D system by a constant state feedback is, in general, very difficult. Intuitively speaking, this is because the state in the Roesser model [12] is only a kind of local information of an infinite dimensional system.

In this paper, we present several new results dealing with stabilization of a discrete 2-D system having the following model:

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i, j) \equiv AX(i, j) + Bu(i, j)$$

$$y(i, j) = [C_1 \quad C_2]X(i, j) \equiv CX(i, j) \tag{1.1}$$

with boundary conditions $x^h(0, j) = h(j)$ and $x^v(i, 0) = v(i)$ for $i, j \geqslant 0$ by a polynomial state feedback

$$u(i, j) = [K_1(z, w)\ K_2(z, w)]X(i, j) \equiv K(z, w)X(i, j) \tag{1.2}$$

where $x^h \in R^n$, $x^v \in R^m$, $u \in R^p$, $y \in R^r$, i and j are integers, and z, w are delay operators in the horizontal and vertical directions, respectively. $K(z, w) \in R^{p\times(n+m)}[z, w]$, i.e., $K(z, w)$ is from the polynomial ring in two variables.

In light of an earlier work [7], the problem is reduced to the question of controllability of a 1-D pair (A_1, B_1), and then, if it is so controllable, checking stabilizability of a relevant discrete delay system. Consequently, it is seen that the stabilizability of a discrete 2-D system by polynomial state feedback is a generic property when $p > 1$. Moreover, if $p = 1$ and $n = 1$ (or $m = 1$), some well-known properties of an analytic function will immediately lead to a condition for stabilizing such a discrete 2-D system. Two examples illustrating the results are included in Section III.

II. MAIN RESULTS

A 2-D z transform applied to (1.1) gives the transfer function matrix corresponding to (1.1):

$$H(z, w) = \frac{Q(z, w)}{a(z, w)}$$

where

$$a(z, w) = \det \begin{bmatrix} I_n - zA_1 & -zA_2 \\ -wA_3 & I_m - wA_4 \end{bmatrix}. \tag{2.1}$$

It may be noted that the variables z and w in (1.2) and (2.1) are consistent so that (1.1) and (1.2) give a closed-loop system with characteristic polynomial as follows:

$$\hat{a}(z, w) = \det \begin{bmatrix} I_n - z(A_1 + B_1K_1) & -z(A_2 + B_1K_2) \\ -w(A_3 + B_2K_1) & I_m - w(A_4 + B_2K_2) \end{bmatrix}. \tag{2.2}$$

The main result on stabilizability (i.e., ensuring that $\hat{a}(z, w)$ is a Shanks' polynomial) is as follows.

Theorem 2.1: System (1.1) is stabilizable by state feedback $u(i, j) = KX(i, j)$ with $K = [K_1\ K_2(z)]$, $K_1 \in R^{p\times n}$, $K_2(z) R^{p\times m}[z]$ if: 1) the pair (A_1, B_1) is a 1-D controllable, and 2) the parametrized pair $(F(z), G(z))$ with $|z| \leqslant 1$ and

$$F(z) = A_4 + z^n(A_3 + B_2K_1)Adj[z^{-1}I_n - (A_1 + B_1K_1)]A_2 \tag{2.3}$$

$$G(z) = B_2 + z^n(A_3 + B_2K_1)Adj[z^{-1}I_n - (A_1 + B_1K_1)]B_1 \tag{2.4}$$

is stabilizable by polynomial state feedback $K_2(z) \in R^{p\times m}[z]$ where $K_1 \in R^{p\times n}$ satisfies

$$\det\ [z^{-1}I_n - (A_1 + B_1K_1)] = z^{-n}. \tag{2.5}$$

Proof: The polynomial $\hat{a}(z, w)$ in (2.2) may be written as [7]

$$\begin{aligned}\hat{a}(z, w) &= \det\,[I_n - z(A_1 + B_1K_1]\ \det\,\{I_m - w[(A_4 + B_2K_2)\\ &\quad + (A_3 + B_2K_1)(z^{-1}I_n - (A_1 + B_1K_1))^{-1}(A_2 + B_1K_2)]\}\\ &= \det\,[I_n - z(A_1 + B_1K_1)]\ \det\,[I_m - w(F_1(z) + G_1(z)K_2)]\end{aligned}$$

where

$$F_1(z) = A_4 + (A_3 + B_2K_1)[z^{-1}I_n - (A_1 + B_1K_1)]^{-1}A_2 \tag{2.6}$$

and

$$G_1(z) = B_2 + (A_3 + B_2K_1)[z^{-1}I_n - (A_1 + B_1K_1)]^{-1}B_1. \tag{2.7}$$

By virtue of [7, Corollary 4.2 and Theorem 4.3], one concludes that (1.1) is stabilizable by $K = [K_1\ K_2(z)]$ if $A_1 + B_1K_1$ is stable and $K_2(z)$ stabilizes (F_1, G_1) with $|z| \leqslant 1$. Note that the controllability of (A_1, B_1) implies the existence of a $K_1 \in R^{p\times n}$ such that (2.5) holds. Doing this, the rational matrices $F_1(z)$ and $G_1(z)$ become $F(z)$ and $G(z)$ in (2.3) and (2.4), respectively; condition 2) then completes the proof of the theorem. □

Notice that $F_1(z)$ and $G_1(z)$ defined in (2.6) and (2.7) are, in general, rational matrices, which makes the stabilization question of $(F_1(G), G_1(z))$ by a polynomial feedback difficult. However, by choosing constant feedback gain K_1 such that (2.5) holds, the resulting matrix pair $(F(z), G(z))$ given in (2.3) and (2.4) now are polynomial matrices in z so that the well-known theorem of Morse [9] leads to the following conclusion.

Corollary 2.2: System (1.1) is stabilizable by (1.2) with $K_1 \in R^{p\times n}$, $K_2 \in R^{p\times m}[z]$ if (A_1, B_1) is a 1-D controllable pair and $(F(z), G(z))$ is controllable over $R[z]$.

Remark 1: Using a similar argument, it is seen that Theorem 2.1 is also valid under the following conditions: 1′) the pair (A_4, B_2) is 1-D controllable, and 2′) the parametrized pair $(P(w), Q(w))$ with $|w| \leqslant 1$ and

$$P(w) = A_1 + w^m(A_2 + B_1K_2)Adj[w^{-1}I_m - (A_4 + B_2K_2)]A_3$$
$$Q(w) = B_1 + w^m(A_2 + B_1K_2)Adj[w^{-1}I_m - (A_4 + B_2K_2)]B_2$$

is stabilizable by polynomial state feedback $K_1(w) \in R^{p\times n}[w]$ where $K_2 \in R^{p\times m}$ satisfies

$$\det\,[w^{-1}I_m - (A_4 + B_2K_2)] = w^{-m}.$$

Corollary 2.3: In the case $p > 1$, the stabilizability of the 2-D system (1.1) by polynomial state feedback (1.2) is a generic property.

Proof: It is known [13] that the controllability of the 1-D pair (A_1, B_1) in (1.1) is generic. Further, for a given controllable pair (A_1, B_1), once the feedback gain K_1 is chosen such that (2.5) holds, the entries of A_2, A_3, A_4, and B_2 which appear in $F(z)$ and $G(z)$ relate the pair $(F(z), G(z))$ to a point of a Euclidean parameter space in a natural way. Now repeat the second part of the proof of [14, Theorem 2] (also using [14, Lemmas 1 and 2]) where the continuous functions f_i, $i = 1, \cdots, m + p - 1$ of [14] are now given by

$$q[w^{-1}I_m - F(z)] = 0 \text{ and } qG(z) = 0$$

with $q = (q_1, \cdots, q_m) \neq 0$ to conclude the genericity for the controllability of the pair $(F(z), G(z))$. This, along with Corollary 2.2, completes the proof. □

We now look at a special case of a single-input 2-D system, namely, we consider the case $m = 1$. Denoting the pair given in (2.3) and (2.4) by $(f(z), g(z))$, one may ask when there exists a polynomial $k_2(z)$ such that $\zeta(z) \equiv f(z) + g(z)k_2(z)$ maps the closed unit disk into the open unit disk in the ζ plane. In case $g(z)$ has no zeros in $|z| \leqslant 1$, $1/g(z)$ could be expressed as a convergent (uniformly in $|z| \leqslant 1$) power series $\Sigma_{i=0}^{\infty} \beta_i z^i$, $|z| \leqslant 1$ since $|f(z)|$ and $|g(z)|$ are bounded in $|z| \leqslant 1$; by taking $k_2(z) = -f(z)\ \Sigma_{i=0}^{N} \beta_i z^i$ with a sufficiently large N, one has

$$|\zeta(z)| = |g(z)|\left|\frac{f(z)}{g(z)} + k_2(z)\right| = |f(z)g(z)|\left|\sum_{i=N+1}^{\infty} \beta_i z^i\right| < 1 \text{ for } |z| \leqslant 1.$$

If $g(z)$ has some zeros in $|z| \leqslant 1$, but $f(z)$ also vanishes at these points, the same argument implies that there exists a $k_2(z) \in R[z]$ such that $|\zeta(z)| < 1$ for $|z| \leqslant 1$. We thus have the following.

Corollary 2.4: In the case $p = 1$, $m = 1$, system (1.1) is stabilizable by $K = [K_1, k_2(z)]$ with $K_1 \in R^{1\times n}$, $k_2(z) \in R[z]$ if (A_1, B_1) is controllable and $f(z)/g(z)$ (after possible cancellation) is analytic in $|z| \leqslant 1$ where $f(z)(\equiv F(z))$ and $g(z)(\equiv G(z))$ are given by (2.3) and (2.4), respectively.

Remark 2: A similar assertion to Corollary 2.4 holds in the case $p = 1$ and $n = 1$.

Remark 3: If $g(z)$ has some zeros, say z_1, in the unit disk, and $f(z_1) \neq 0$, then $\zeta(z_1) = f(z_1) + g(z_1)k_2(z_1) = f(z_1)$; the maximum modules theorem thus gives $|f(z_1)| \leqslant \max_{|z|\leqslant 1} |\zeta(z)|$. Therefore, in order to have a $k_2(z) \in R[z]$ such that $|\zeta(z)| < 1$ for $|z| \leqslant 1$, it is necessary to have $|f(z_1)| < 1$. This necessary condition may be useful for checking the possibility of having such a polynomial $k_2(z)$.

III. Examples

Example 3.1

Consider an unstable 2-D system (1.1) with

$$A = \left[\begin{array}{cc:cc} 1 & 0 & 0 & 1\\ 0 & -1 & -1 & 0\\ \hdashline 1 & -1 & 1 & -1\\ 0 & 0 & 0 & 1 \end{array}\right], \quad B = \left[\begin{array}{cc} 0 & -1\\ 1 & 0\\ \hdashline 0 & 1\\ -1 & 0 \end{array}\right]. \tag{3.1}$$

The instability of the system could be easily checked by a necessary condition developed in [7]. Notice that (A_1, B_1) here is controllable in a 1-D sense, and using $K_1 = I_2$ gives $\det\,[z^{-1}I_2 - (A_1 + B_1K_1)] = z^{-2}$. The pair $(F(z), G(z))$ in (2.3) and (2.4) thus becomes

$$F(z) = \begin{bmatrix} 1+z^2 & -1+z+z^2\\ -z^2 & 1-z-z^2 \end{bmatrix}, \quad G(z) = \begin{bmatrix} -z^2 & 1-z-z^2\\ -1+z^2 & z+z^2 \end{bmatrix}$$

for which

$$\begin{aligned}&\text{rank } \langle F(z) | G(z)\rangle\\ &= \text{rank}\begin{bmatrix} -z^2 & 1-z-z^2 & 1-z-3z^2+z^3 & 1-2z+z^3\\ -1+z^2 & z+z^2 & -1+z+2z^2-z^3 & z-z^2-z^3 \end{bmatrix}\\ &= 2 \quad \text{for all } z \in C,\end{aligned}$$

i.e., $(F(z), G(z))$ is $R[z]$ controllable. Therefore, for any desired poles s_1 and s_2 in $R[z]$, there exists a $k_2(z) \in R^{2\times 2}[z]$ such that $\det\,[sI - F(z) + G(z)K_2(z))] = (s - s_1)(s - s_2)$. A procedure to construct such a polynomial state feedback matrix is available in [10] (also see [11]). For instance, if $s_1 = -0.5$, $s_2 = 0.5$, some straightforward manipulations yield

$$K_2(z) = \begin{bmatrix} -5z^5 - 13z^4 - 14.5z^3 - 7.5z^2 - 6z - 0.5 & -5z^5 - 13z^4 - 4.5z^3 + 3.5z^2 - z + 5\\ 20z^7 + 52z^6 + 33z^5 + 2.5z^4 - 6z^3 + 11.75z^2 - 7.75z - 2 & 20z^7 + 52z^6 + 23z^5 - 18.5z^4 - 7z^3 - 5.75z^2 - 5.75z^2 - 5.75z + 6.25 \end{bmatrix}$$

Thus, state feedback $u(i, j) = KX(i, j)$ with $K = [I_2\ K_2(z)]$ stabilizes 2-D system (3.1).

Example 3.2: Consider an unstable single-input 2-D system with $m = 1$ as follows:

$$A = \left[\begin{array}{cc:c} -2 & 0 & 3\\ -1 & 1 & -2\\ \hdashline -2 & -4 & 2 \end{array}\right], \quad b = \left[\begin{array}{c} 1\\ 0\\ \hdashline 3 \end{array}\right]. \tag{3.2}$$

Note that (A_1, b_1) is controllable and $K_1 = [1\ 1]$ makes $\det [z^{-1}I_2 - (A_1 + b_1K_1)] = z^{-2}$, and $f(z)(=F(z))$ and $g(z)(=G(z))$ in (2.3), (2.4) are $f(z) = z + 2$ and $g(z) = z + 3$. To find $k_2(z)$, we estimate $|g(z)| = |z + 3| \leq 4$ for $|z| \leq 1$ and expand

$$\frac{f(z)}{g(z)}=\frac{z+2}{z+3}=\frac{z+2}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-\cdots\right).$$

Note that for $|z| \leq 1$,

$$\left|\frac{1}{(l+1)}\left(\frac{1}{g(z)}\right)^{(l+1)} z^{l-1}\right| \leq \left|\frac{1}{(3+z)^{l+2}}\right| \leq \frac{1}{2^{l+2}},$$

and $|f(z)| = |z + 2| \leq 3$. Thus, by taking $l = 2$ and

$$k_2(z)=-\frac{z+2}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}\right)=-\frac{1}{27}(z^3-z^2+3z+14)$$

we have

$$|f(z)+g(z)k_2(z)|=|g(z)|\left|\frac{f(z)}{g(z)}+k_2(z)\right|\leq 4|f(z)|\frac{1}{2^4}<1,$$

namely, state feedback $u(i, j) = [K_1\ k_2(z)] = [1\ \ 1\ \ -1/27(z^3 - z^2 + 3z + 14)]$ will stabilize system (3.2).

IV. Conclusions

Theorem 2.1 and the corollaries indicate that stabilizing a 2-D system will become easier if one uses certain past history of the local states instead of static state feedback. Moreover, it has been seen that such a task can almost always be done if $p > 1$. For the single-input case, $R[z]$ controllability and thus the present way of stabilizing a 2-D system is clearly nongeneric. On the other hand, some recent observations on the coefficient assignability question for retarded delay systems have made it possible to deal with the problem in a larger system class. The interested reader may peruse the recent paper [15].

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A Simplified Derivation of the Zeheb-Walach 2-D Stability Test with Applications to Time-Delay Systems

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Abstract—**Zeheb and Walach gave a stability test for *N-D* systems. For 2-D systems, a simpler derivation is presented here using the results of DeCarlo *et al.* It is then shown how the method may also be used to test for stability (independent of delay) of retarded time-delay systems.**

I. Stability of 2-D Polynomials

Let

$$a(z_1, z_2)=\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} a_{ij}z_1^iz_2^j,$$

with $a_{ij} \in R\ \forall\ i, j$. Note $n_1 = \deg_{z_1} a(z_1, z_2)$, and $n_2 = \deg_{z_2} a(z_1, z_2)$. The 2-D polynomial $a(z_1, z_2)$ is said to be *stable* iff $a(z_1, z_2) \neq 0\ |z_1| \leq 1$, $|z_2| \leq 1$, $(z_1, z_2) \in C \times C$. For convenience, we assume that $a(z_1, z_2)$ is irreducible. Now, by DeCarlo *et al.* [2] (a similar criterion is given in [3]) it follows that $a(z_1, z_2)$ is stable iff

$$a(z_1, z_2)\neq 0\,\forall (z_1, z_2) \in T^2 \triangleq \{(z_1, z_2) \in C\times C\ |z_1|=|z_2|=1\} \tag{1}$$

and

$$a(z, z)\neq 0\,\forall z \in C \text{ such that } |z|\leq 1. \tag{2}$$

This result of DeCarlo *et al.* was apparently first noted by Rudin [14] and is quoted by Bose [15, p. 173]. Condition (1) is still, however, a two-variable problem. We can simplify this by using the ideas given in [1, Theorem 4]. Define

$$\tilde{a}(z_1, z_2) \triangleq z_1^{n_1}z_2^{n_2}\, a(1/z_1, 1/z_2) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} a_{n_1-i,n_2-j}\, z_1^iz_2^j.$$

Now we note that $a(z_1^0, z_2^0) = 0$ for $(z_1^0, z_2^0) \in T^2$ iff $a(1/z_1^0, 1/z_2^0) = 0$ for $(1/z_1^0, 1/z_2^0) = (\bar{z}_1^0, \bar{z}_2^0) \in T^2$ where $\bar{z}$ denotes the complex conjugate of z, i.e., $a(z_1^0, z_2^0) = 0$ on T^2 iff $\tilde{a}(z_1^0, z_2^0) = 0$ on T^2.

We have thus established the following.

Theorem 1: $a(z_1, z_2)$ is stable iff

$$a(z_1, z_2) \text{ and } \tilde{a}(z_1, z_2) \text{ have no common zeros on the unit bidisk } T^2 \tag{1'}$$

and

$$a(z, z)\neq 0 \qquad |z|\leq 1,\ z \in C. \tag{2}$$

Zeheb and Walach [1] proposed a test consisting of condition (1′) above along with

$$a(z_1^0, z_2)\neq 0\ |z_2|\leq 1,\ z_1^0 \text{ any element of } C \text{ satisfying } |z_1^0|=1, \text{ and} \tag{2'}$$

$$a(z_1, z_2^0)\neq 0\ |z_1|\leq 1,\ z_2^0 \text{ any element of } C \text{ satisfying } |z_2^0|=1.$$

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