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Leon O. Chua (S'60–M'62–SM'70–F'74), for a photograph and biography please see page 000 of this issue.

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Stability Analysis for Two-Dimensional Systems via a Lyapunov Approach

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***Abstract*—Some necessary and sufficient conditions are given for stability analysis of two-dimensional (2-D) systems based on a Lyapunov approach. The study was carried out using the Roesser state-space model, which when combined with the Lyapunov theory provides the new checkable tests for stability. Also, the results lead to techniques for selecting stabilizing state feedback gain matrices for the 2-D systems.**

I. Introduction

An IMPORTANT aspect of a 2-D digital filter is its BIBO (Bounded-Input Bounded-Output) stability (meaning that a bounded input always yields a bounded output). A 2-D filter cannot be adopted in practice unless its stability is guaranteed. Many publications have appeared in this area [1], most of which were carried out in terms of the stability of a related polynomial in two complex variables. On the other hand, recent progress on internal descriptions of the 2-D systems has provided the possibility to describe the stability question in a state-space version. We mention [2], where an appropriate state-space model for the 2-D filters was suggested known as Roesser model; [3] where a simple scheme is given to get a minimal realization of a 2-D transfer function of the form $1/a(z_1^{-1}, z_2^{-1})$ so that one could discuss the stability of the 2-D polynomial $a(z_1^{-1}, z_2^{-1})$ in a state-space version without loss of generality; and [4], [5] which set forth a counterpart of the 1-D Lyapunov stability theorem for the 2-D case.

In [6], some observations on the stability issue in state space form have been given. This paper is its continuation with emphasis on the Lyapunov approach.

Consider Roesser's state-space model [2] for a SISO 2-D system:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i,j)$$

$$\equiv A \begin{bmatrix} x^h \\ x^v \end{bmatrix} + Bu$$

$$y(i,j) = [C_1, C_2] \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \equiv C \begin{bmatrix} x^h \\ x^v \end{bmatrix} \tag{1.1}$$

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where x^h and x^v are n_1-dimensional and n_2-dimensional vectors, respectively. The 2-D z-transform yields the system transfer function

$$H(z_1^{-1}, z_2^{-1}) = \frac{b(z_1^{-1}, z_2^{-1})}{a(z_1^{-1}, z_2^{-1})} \tag{1.2}$$

Equation (1.2) where z_1^{-1}, z_2^{-1} are delay operators and

$$a(z_1^{-1}, z_2^{-1}) = \det \begin{bmatrix} I_{n1} - z_1^{-1}A_1 & -z_1^{-1}A_2 \\ -z_2^{-1}A_3 & I_{n2} - z_2^{-1}A_4 \end{bmatrix} \tag{1.3}$$

Throughout this paper, the polynomials $a(z_1^{-1}, z_2^{-1})$ and $b(z_1^{-1}, z_2^{-1})$ are assumed to be factor coprime and there are no nonessential singularities of second kind [7]. A polynomial $a(z_1^{-1}, z_2^{-1})$ is said to be a Shanks' polynomial whenever $a(z_1^{-1}, z_2^{-1}) \neq 0$ in the closed bidisc $U^2 \equiv \{(z_1^{-1}, z_2^{-1}) \mid |z_1^{-1}| \leqslant 1,\ |z_2^{-1}| \leqslant 1\}$. It is well known that the system (1.1) is BIBO stable if and only if $a(z_1^{-1}, z_2^{-1})$ is a Shanks' polynomial [1].

In 1-D case (state-space version) the Lyapunov theory has been playing a crucial role in stability analysis. The stability question for a 2-D system is reduced to the existence of positive definite Hermitian (P.D.H.) solution of a 1-D Lyapunov equation with a complex parameter in the next section. As a result, a new stability criterion is given. In the particular cases when $n_1 = 1$ or $n_2 = 1$ this criterion becomes quite simple and computationally attractive when certain conditions are satisfied. A couple of sufficient conditions are also given.

A 2-D Lyapunov theorem as established in [4] and [5] can be stated as follows.

Theorem 1.1 [4], [5]: $a(z_1^{-1}, z_2^{-1})$ in (1.3) is a Shanks' polynomial if and only if there exists a positive definite (P.D.) matrix $G = G_1 \oplus G_2$ such that the matrix

$$W \equiv G - A^T G A \tag{1.4}$$

is P.D., where $\oplus$ denotes the direct sum of matrices $G_1 \in R^{n_1 \times n_1}$ and $G_2 \in R^{n_2 \times n_2}$.

Differing from 1-D case, one may have difficulty in using this theorem to check the stability of a given 2-D system since G in (1.4) must have a special structure. However, the theorem has been shown to be a good starting point to get some insight into the stability properties. In Section III, necessary conditions as well as a sufficient condition are derived from the 2-D Lyapunov equation (1.4).

Moreover, some stabilization could also be obtained by the 2-D Lyapunov approach. Actually, a related algebraic Riccati equation is derived and a stabilizability criterion, using some results in [10], is given.

II. 1-D Lyapunov Equation with a Complex Parameter

In [6], the following theorem was given:

Theorem 2.1 [6]: The following statements are equivalent

1) System (1.1) is BIBO stable;

2) (i) A_1 is stable,[1]
   (ii) $A_4 + A_3(z_1 I_{n1} - A_1)^{-1} A_2$ with $|z_1| = 1$ stable;

3) (i) A_4 is stable,
   (ii) $A_1 + A_2(z_2 I_{n2} - A_4)^{-1} A_3$ with $|z_2| = 1$ is stable.

Define

$$P_1(z) = A_4 + A_3(zI_{n1} - A_1)^{-1} A_2 \tag{2.1}$$

$$P_2(z) = A_1 + A_2(zI_{n2} - A_4)^{-1} A_3. \tag{2.2}$$

Note that for each fixed $z \in T \equiv \{z \mid |z| = 1\}$, $P_1(z)$ is in general a complex matrix. The following lemma gives a condition for $P_1(z)$ to be a stable matrix for each $z \in T$.

Lemma 2.2: A matrix $F(z)$ with a complex parameter $z \in T$ is stable if and only if for any given P.D.H. matrix $W(z)$ with $z \in T$, there exists a unique P.D.H. matrix $G(z)$ such that

$$G - F^* G F = W. \tag{2.3}$$

Proof: The proof is similar to that of [8] where a Lyapunov theorem for a complex matrix related to a continuous system was shown.

1) Sufficiency: For any fixed $z \in T$, let λ and v be any eigenvalue and eigenvector of F, *respectively*, i.e., $Fv = \lambda v$, $v^* F^* = \bar{\lambda} v^*$ where $v^* \equiv \bar{v}^T$, $F^* \equiv \bar{F}^T$. Premultiplying (2.3) by v^* and post-multiplying (2.3) by v yields

$$v^* G v - v^* F^* G F v = v^* G v (1 - |\lambda|^2) = v^* W v.$$

Hence,

$$1 - |\lambda|^2 = \frac{v^* W v}{v^* G v} > 0$$

which shows F is stable.

2) Necessity: Suppose that for any fixed $z \in T$, F is stable and a P.D.H. matrix W is given. Set

$$G = \sum_{k=0}^{\infty} (F^*)^k W F^k. \tag{2.4}$$

Note that G in (2.4) is well defined since F is stable and G is clearly a P.D.H. matrix. Moreover, we have

$$G - F^* G F = \sum_{k=0}^{\infty} (F^*)^k W F^k - \sum_{k=0}^{\infty} (F^*)^{k+1} W F^{k+1} = W.$$

Assume now that G_1 is another solution of (2.3), then

$$G = \sum_{k=0}^{\infty} (F^*)^k W F^k = \sum_{k=0}^{\infty} (F^*)^k (G_1 - F^* G_1 F) F^k$$
$$= \sum_{k=0}^{\infty} (F^*)^k G_1 F^k - \sum_{k=0}^{\infty} (F^*)^{k+1} G_1 F^{k+1} = G_1$$

By this lemma, Theorem 2.1 has following equivalent form.

Theorem 2.3: System (1.1) is BIBO stable if and only if

1) (i) A_1 is stable,
   (ii) The matrix equation

$$G_1(z_1) - P_1^*(z_1) G_1(z_1) P_1(z_1) = W_1(z_1) \tag{2.5}$$

[1] A square matrix A is stable in case all its eigenvalues lie in the interior of the unit circle in the complex plane.

has a P.D.H. solution $G_1(z_1)$ for any given $n_2 \times n_2$ P.D.H. matrix $W_1(z_1)$ and any $z_1 \in T$,

or

2) (i) A_4 is stable,
(ii) The matrix equation

$$G_2(z_2) - P_2^*(z_2) G_2(z_2) P_2(z_2) = W_2(z_2) \tag{2.6}$$

has a P.D.H. solution $G_2(z_2)$ for any given $n_1 \times n_1$ P.D.H. matrix $W_2(z_2)$ and any $z_2 \in T$.

For a given $W_1(z_1)$, say $W_1(z_1) = I_{n_2}$, one may solve the linear matrix equation (2.5) to obtain a rational matrix solution $G_1(z_1)$ which is obviously a Hermitian matrix. Representing $z_1 = e^{j\theta}$ with $0 \leqslant \theta \leqslant 2\pi$ and denoting the kth-order principal minor of $G_1(z_1)$ by $g_k(\theta)$, we note that $g_k(\theta)$ $(1 \leqslant k \leqslant n_2)$ are real rational functions of one real variable θ over the closed interval $[0, 2\pi]$. We thus have

Corollary 2.4: System (1.1) is BIBO stable if and only if

1) (i) A_1 is stable,
(ii) $g_k(\theta) > 0$ for $0 \leqslant \theta \leqslant 2\pi$, $1 \leqslant k \leqslant n_2$;

or

2) (i) A_4 is stable
(ii) $h_k(\theta)$ for $0 \leqslant \theta \leqslant 2\pi$, $1 \leqslant k \leqslant n_1$ where $h_k(\theta)$ is the kth-order principal minor of $G_2(e^{j\theta})$ which is the solution of (2.6) for a given P.D.H. matrix $W_2(e^{j\theta})$.

Example 2.5: Consider the case $n_1 = n_2 = 1$, (1.1) then becomes

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(i,j). \tag{2.7}$$

Taking $W_1(e^{j\theta}) \equiv 1$, (2.5) gives

$$g_1\left(1 - \left| a_4 + \frac{a_2 a_3}{z - a_1} \right|^2\right) = 1 \qquad \text{with } z = e^{j\theta}$$

thus $g_1 > 0$ if and only if

$$\left| a_4 + \frac{a_2 a_3}{z - a_1} \right| < 1 \qquad \text{for } z = e^{j\theta}. \tag{2.8}$$

Note that $\zeta(z) \equiv a_4 + (a_2 a_3/(z - a_1))$ is a bilinear transformation which maps the unit circle in the z-plane onto a circle in the ζ-plane which is symmetric with respect to the real axis. Therefore, the minimum and the maximum of $|\zeta|$ may be achieved only when z is real. That is, (2.8) is equivalent to

$$\max\left\{ \left| a_4 + (a_2 a_3/(1 - a_1)) \right|, \left| a_4 - (a_2 a_3/(1 + a_1)) \right| \right\} < 1. \tag{2.9}$$

Namely, system (2.7) is BIBO stable if and only if $|a_1| < 1$ and (2.9) hold. This example motivated us to pursue the following stability criterion for the special class of 2-D systems with $n_1 = 1$ or $n_2 = 1$.

In case $n_1 = 1$,

$$A = \begin{bmatrix} a_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix}$$

where $a_1 \in R$, $A_2 \in R^{1 \times n_2}$, $A_3 \in R^{n_2 \times 1}$, $A_4 \in R^{n_2 \times n_2}$. Assume that A_4 is a diagonalizable matrix with eigenvalues $\{\mu_i\}$ which are all real, i.e., there exist nonsingular T_2 such that $\tilde{A}_4 \equiv T_2^{-1} A_4 T_2 = \operatorname{diag}\{\mu_i\}$. Thus the transformation $T \equiv 1 \oplus T_2$ gives

$$\tilde{A} \equiv T^{-1} A T = \begin{bmatrix} a_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

where

$$\tilde{A}_2 = A_2 T_2 \equiv \begin{pmatrix} d_1 & \cdots & d_{n_2} \end{pmatrix}$$
$$\tilde{A}_3 = T_2^{-1} A_3 \equiv \begin{pmatrix} f_1 & \cdots & f_{n_2} \end{pmatrix}.$$

Define

$$\delta_i = d_i f_i, \qquad 1 \leqslant i \leqslant n_2. \tag{2.10}$$

We claim the following

Theorem 2.6: Assume that $n_1 = 1$, A_4 is a diagonalizable matrix with real eigenvalues $\{\mu_i, 1 \leqslant i \leqslant n_2\}$ and all δ_i given in (2.10) have the same sign. Then the system (1.1) is BIBO stable if and only if

(i) $|\mu_i| < 1$, $1 \leqslant i \leqslant n_2$.
(ii) $\operatorname{Max}\{|a_1 + A_2(I_{n_2} - A_4)^{-1}A_3|, |a_1 - A_2(I_{n_2} + A_4)^{-1}A_3|\} < 1$.

Proof: The necessity is obvious. To show the sufficiency, we note that

$$\begin{aligned} |\zeta(z_2)| &\equiv \left| a_1 + A_2(z_2 I_{n_2} - A_4)^{-1} A_3 \right| \\ &= \left| a_1 + \tilde{A}_2(z_2 I_{n2} - \tilde{A}_4)^{-1} \tilde{A}_3 \right| \\ &= \left| a_1 + \sum_{i=1}^{n_2} \frac{\delta_i}{z_2 - \mu_i} \right| = \left| a_1 + \operatorname{sgn}(\delta_1) \sum_{i=1}^{n_1} \frac{|\delta_i|}{z_2 - \mu_i} \right| \\ &= \left| a_1 + \operatorname{sgn}(\delta_1) \sum_{i=1}^{n_2} \zeta_i(z_2) \right| \end{aligned}$$

where we have defined

$$\zeta_i(z_2) = \frac{|\delta_i|}{z_2 - \mu_i}, \qquad 1 \leqslant i \leqslant n_2$$

which maps the unit circle in the z_2-plane onto the circle with real center. Thus $\zeta(z_2)$ could achieve its maximum only when $z_2 = 1$ or $z_2 = -1$. □

Remark: A similar result could be obtained in case $n_2 = 1$.

In case δ_i in (2.10) have different signs, we have the following sufficient condition.

Corollary 2.7: Assume $n_1 = 1$, A_4 is a diagonalizable matrix with real eigenvalues $\{\mu_i, 1 \leqslant i \leqslant n_2\}$, δ_i $(1 \leqslant i \leqslant t)$ in (2.10) have same sign and so do δ_i $(t+1 \leqslant i \leqslant n_2)$. The system (1.1) is then BIBO stable if (i) $|\mu_i| < 1$, $1 \leqslant i \leqslant n_2$, and (ii) $\max\{\alpha_1 + \beta_1, \alpha_1 + \beta_2, \alpha_2 + \beta_1, \alpha_2 + \beta_2\} < 1$ where

$$\alpha_1 = \left| a_1 + \sum_{i=1}^{t} \frac{|\delta_i|}{1 - \mu_i} \right|, \qquad \alpha_2 = \left| a_1 - \sum_{i=1}^{t} \frac{|\delta_i|}{1 + \mu_i} \right|$$

$$\beta_1 = \left| \sum_{i=t+1}^{n_2} \frac{|\delta_i|}{1 - \mu_1} \right|, \qquad \beta_2 = \left| \sum_{i=t+1}^{n_2} \frac{|\delta_i|}{1 + \mu_i} \right|.$$

Proof: Note that

$$\left|a_1 + A_2\left(z_2 I_{n_2} - A_4\right)^{-1} A_3\right|$$
$$= \left|a_1 + \sum_{i=1}^{t} \frac{\delta_i}{z_2 - \mu_i} + \sum_{i=t+1}^{n_2} \frac{\delta_i}{z_2 - \mu_i}\right|$$
$$\leqslant \max_{|z_2|=1} \left|a_1 + \operatorname{sgn}(\delta_1) \sum_{i=1}^{t} \frac{|\delta_i|}{z_2 - \mu_i}\right| + \max_{|z_2|=1} \left|\sum_{i=t+1}^{n_2} \frac{|\delta_i|}{z_2 - \mu_i}\right|.$$

Remark: One may tighten the condition (ii) in Corollary 2.7 by combining each two terms having opposite signs in δ_i to obtain a more accurate estimate. Suppose, for in stance, $\operatorname{sgn}\delta_1 = 1$, $\operatorname{sgn}\delta_2 = -1$, we consider

$$\zeta(z) = \frac{\delta_1}{z - \mu_1} + \frac{\delta_2}{z - \mu_2} = \frac{(\delta_1 + \delta_2)z - (\delta_2\mu_1 + \delta_1\mu_2)}{(z - \mu_1)(z - \mu_2)}$$
$$\equiv \zeta_1(z)\zeta_2(z)$$

where

$$\zeta_1(z) = \frac{(\delta_1 + \delta_2)z - (\delta_2\mu_1 + \delta_1\mu_2)}{z - \mu_1}, \quad \zeta_2(z) = \frac{1}{z - \mu_2}.$$

Note that $\zeta_1(z)$ is a bilinear transformation which maps the unit circle on the z-plane onto a circle in the ζ_1-plane with real center. Thus $|\zeta_1(z)|$ may reach its maximum at $z = 1$ or $z = -1$ only. Therefore, for $z \in T$,

$$\left|\frac{\delta_1}{z - \mu_1} + \frac{\delta_2}{z - \mu_2}\right|$$
$$\leqslant \max|\zeta_1(z)|\max|\zeta_2(z)|$$
$$= \frac{1}{1 - |\mu_2|} \max\left\{\left|\frac{\delta_1 + \delta_2 - \delta_2\mu_1 - \delta_1\mu_2}{1 - \mu_1}\right|, \left|\frac{\delta_1 + \delta_2 - \delta_2\mu_1 - \delta_1\mu_2}{1 + \mu_1}\right|\right\}. \tag{2.11}$$

Interchanging the roles of μ_1 and μ_2, we also have

$$\left|\frac{\delta_1}{z - \mu_1} + \frac{\delta_2}{z - \mu_2}\right|$$
$$\leqslant \frac{1}{1 - |\mu_1|} \max\left\{\left|\frac{\delta_1 + \delta_2 - \delta_2\mu_1 - \delta_1\mu_2}{1 - \mu_2}\right|, \left|\frac{\delta_1 + \delta_2 + \delta_2\mu_1 + \delta_1\mu_2}{1 + \mu_1}\right|\right\}. \tag{2.12}$$

Equations (2.11) and (2.12) may be useful in testing for stability as exhibited in the following example.

Example 2.8: Consider a 2-D system with

$$A = \left[\begin{array}{c:cc} 0.7 & 1.0 & 5.0 \\ \hdashline -9.97 & 0.50 & 0 \\ 2.0 & 0 & 0.495 \end{array}\right].$$

Condition (ii) in Corollary 2.7 is not satisfied, however, (2.11) gives

$$\max_{z_2 \in T}\left|0.7 + \frac{-9.97}{z_2 - 0.5} + \frac{10}{z - 0.495}\right|$$
$$\leqslant 0.7 + \max_{z_2 \in T}\left|\frac{-9.97}{z_2 - 0.5} + \frac{10}{z - 0.495}\right|$$
$$= 0.7 + 0.141 = 0.841$$

which with $a_1 = 0.7$ implies stability of the system.

III. Stability Results Related to a 2-D Lyapunov Theorem

An important result associated with the 2-D Lyapunov equation (1.4) has been described in theorem 1.1, from which some necessary conditions as well as some sufficient conditions could be derived for a 2-D systems without restriction of the dimension of its state space.

To begin with, we rewrite the right-hand side of (1.4) in detail:

$$G - A^T G A$$
$$= \begin{bmatrix} G_1 - A_1^T G_1 A_1 - A_3^T G_2 A_3 & -A_1^T G_1 A_2 - A_3^T G_2 A_4 \\ -A_2^T G_1 A_1 - A_4^T G_2 A_3 & G_2 - A_4^T G_2 A_4 - A_2^T G_1 A_2 \end{bmatrix}. \tag{3.1}$$

Taking

$$T = \begin{bmatrix} I_{n_1} & \left(G_1 - A_1^T G_1 A_1 - A_3^T G_2 A_3\right)^{-1}\left(A_1^T G_1 A_2 + A_3^T G_2 A_4\right) \\ 0 & I_{n_2} \end{bmatrix} \tag{3.2}$$

for which the involved inverse matrix is assumed to exist, we then have

$$T^T(G - A^T G A)T =$$
$$\begin{bmatrix} G_1 - A_1^T G_1 A_1 - A_3 G_2^T A_3 & 0 \\ 0 & G_2 - A_4^T G_2 A_4 - A_2^T G_1 A_2 - Q \end{bmatrix} \tag{3.3}$$

where

$$Q = \left(A_1^T G_1 A_2 + A_3^T G_2 A_4\right)^T Q_1^{-1}\left(A_1^T G_1 A_2 + A_3^T G_2 A_4\right) \tag{3.4}$$

and

$$Q_1 = G_1 - A_1^T G_1 A_1 - A_3^T G_2 A_3. \tag{3.5}$$

Theorem 3.1: The following conditions are necessary for BIBO stability of the system (1.1):

(i) A_1 is stable,
(ii) A_4 is stable,
(iii) A is stable.

Proof: By Theorem 1.1, if (1.1) is BIBO stable, then $W_1 = G_1 - A_1^T G_1 A_1 - A_3^T G_2 A_3 > 0$ which implies $G_1 - A_1^T G_1 A_1 = W_1 + A_3^T G_2 A_3 > 0$ i.e., A_1 is stable. Similarly, $G_2 - A_4^T G_2 A_4 = W_4 + A_2^T G_1 A_2 > 0$ has a positive definite solution G_2 so that A_4 is stable. Finally, the fact that $G - A^T G A = W$ for $W > 0$ has positive definite solution G leads to the stability of A. □

In [6] we obtained the same necessary conditions using a different approach.

Theorem 1.1 and (3.3) immediately lead to the following:

Theorem 3.2: The system (1.1) is BIBO stable if and only if there exist positive definite matrices G_1 and G_2 such that Q_1 in (3.5) and $G_2 - A_4^T G_2 A_4 - A_2^T G_1 A_2 - Q$ with Q in (3.4) are positive definite.

We now partition W in (1.4) as

$$W = \begin{bmatrix} W_1 & W_2 \\ W_2^T & W_4 \end{bmatrix}$$

where $W_1 \in R^{n_1 \times n_1}$, $W_4 \in R^{n_2 \times n_2}$, (1.4) then gives

$$W_1 = G_1 - A_1^T G_1 A_1 - A_3^T G_2 A_3 \tag{3.6}$$

$$W_4 = G_2 - A_4^T G_2 A_4 - A_2^T G_1 A_2 \tag{3.7}$$

$$W_2 = -A_1^T G_1 A_2 - A_3^T G_2 A_4. \tag{3.8}$$

One may reduce solving the 2-D Lyapunov equation (1.4) to asking the following questions. 1) Given $W_1 > 0$ and $W_4 > 0$, do (3.3) and (3.4) have positive definite solutions G_1 and G_2 under certain conditions? 2) If they do, under which conditions does the matrix W_2 obtained by substituting the positive definite solutions G_1 and G_2 of (3.6) and (3.7), make $W > 0$?

The way we answer the first question is to construct two iterative P.D. solution sequences $\{G_1^{(k)}\}$ and $\{G_2^{(k)}\}$ which could be determined from the following recursive equations:

$$G_1^{(k)} - A_1^T G_1^{(k)} A_1 - A_3^T G_2^{(k-1)} A_3 = W_1 \tag{3.9}$$

$$G_2^{(k)} - A_4 G_2^{(k)} A_4 - A_2^T G_2^{(k-1)} A_2 = W_4 \tag{3.10}$$

for $k = 1, 2, \cdots$, with initial data $G_1^{(0)} = I_{n_1}$ and $G_2^{(0)} = I_{n_2}$, provided that A_1 and A_4 are stable matrices.

Define the error sequence

$$E^{(k)} = \left(G_1^{(k+1)} - G_1^{(k)}\right) \oplus \left(G_1^{(k+1)} - G_2^{(k)}\right).$$

By (3.9) and (3.10) we have

$$E^{(k)} - \Lambda^T E^{(k)} \Lambda = \Gamma^T E^{(k-1)} \Gamma \tag{3.11}$$

where

$$\Lambda = A_1 \oplus A_4 \quad \text{and} \quad \Gamma = \begin{bmatrix} 0 & A_2 \\ A_3 & 0 \end{bmatrix}.$$

To estimate $E^{(k)}$, let $F = \frac{1}{2}(\Lambda - I_n)(\Lambda + I_n)^{-1}$ and $H = \Gamma(\Lambda + I_n)^{-1}$ with $n = n_1 + n_2$, then (3.8) becomes

$$E^{(k)} F + F^T E^{(k)} = -H^T E^{(k-1)} H. \tag{3.12}$$

In [9], some results involving estimate of a Lyapunov-type operator were given. Let $B \in C^{l \times l}$, $C \in C^{m \times m}$, define a linear operator T: $C^{m \times l} \to C^{m \times l}$ by $TP = PB - CP$. The separation of B and C is then defined by

$$\operatorname{sep}_d(B, C) = \inf_{\|P\|_d = 1} \|TP\|_d = \inf_{\|P\|_d = 1} \|PB - CP\|_d$$

where $d = 2$ or $d = F$ which refer to the spectral norm and Frobenius norm, respectively.

Theorem 3.3 [9]: Let the columns of X and Y form the complete system of eigenvectors for B and C, respectively, then

$$\operatorname{sep}_F(B, C) \geqslant \frac{\min|\lambda(B) - \lambda(c)|}{D(X) D(Y)} \tag{3.13}$$

$$\operatorname{sep}_2(B, C) \geqslant \frac{\min|\lambda(B) - \lambda(c)|}{\sqrt{\min\{l, m\}}\, D(X) D(Y)} \tag{3.14}$$

where $\lambda(B)$ and $\lambda(C)$ are spectra of B and C, respectively, $|\lambda(B) - \lambda(C)| \equiv \{|\lambda - \lambda'| \,|\, \lambda \in \lambda(B),\ \lambda' \in \lambda(C)\}$, $D(\cdot)$ is the condition number of the involved matrix with Frobenius norm.

In our case (3.13) and (3.14) mean that

$$\operatorname{sep}_F(F, -F^T) \geqslant \frac{\min|\lambda(F) - \lambda(-F)|}{D^2(X)} \tag{3.15}$$

$$\operatorname{sep}_2(F, -F^T) \geqslant \frac{\min|\lambda(F) - \lambda(-F)|}{n^{1/2} D^2(Y)}. \tag{3.16}$$

By (3.15) and (3.12), we obtain

$$\|E^{(k)}\|_F \leqslant \frac{\|H\|_F^2 D^2(X)}{\min|\lambda(F) - \lambda(-F)|} \|E^{(k-1)}\|_F \tag{3.17}$$

similarly,

$$\|E^{(k)}\|_2 \leqslant \frac{n^{1/2} \|H^2\|_2 D^2(X)}{\min|\lambda(F) - \lambda(-F)|} \|E^{(k-1)}\|_2. \tag{3.18}$$

The following theorem gives an answer to question 1).

Theorem 3.4: Assume that A_1 and A_4 are stable matrices. For given $W_1 > 0$, $W_4 > 0$, (3.6) and (3.7) have unique P.D. solution G_1 and G_2 if

$$\epsilon_F \equiv \frac{\|H\|_F^2 D^2(X)}{\min|\lambda(F) - \lambda(-F)|} < 1 \tag{3.19}$$

where X is the complete system of eigenvectors for F.

Proof: Equations (3.19) and (3.17) imply $\|E^{(k)}\|_F \leqslant \epsilon_F^k \|E^{(0)}\|_F$ and $\|E^{(k)}\|_F \to 0$ as $k \to \infty$. Namely $\{G_1^{(k)}\}$ and $\{G_2^{(k)}\}$ are two convergent sequences and $G_1 \equiv \lim_{k \to \infty} G_1^{(k)} \geqslant 0$, $G_2 \equiv \lim_{k \to \infty} G_2^{(k)} \geqslant 0$. Letting $k \to \infty$ in (3.9) and (3.10) yield (3.6) and (3.7), respectively. Furthermore, $G_1 = W_1 + A_1^T G_1 A_1 + A_3^T G_2 A_3 > 0$ and $G_2 = W_4 + A_4 G_2 A_4^T + A_2^T G_1 A_2 > 0$. To show the uniqueness, suppose that $\{G_1, G_2\}$ and $\{\tilde{G}_1, \tilde{G}_2\}$ are the solutions of (3.6) and (3.7). Same procedure as done for $E^{(k)}$ gives

$$\|(G_1 - \tilde{G}_1) \oplus (G_2 - \tilde{G}_2)\|_F \leqslant \epsilon_F \|(G_1 - \tilde{G}_1) \oplus (G_2 - \tilde{G}_2)\|_F$$

which would be impossible unless $G_1 = \tilde{G}_1$ and $G_2 = \tilde{G}_2$. □

Now we are in a position to deal with question 2). We first define a linear operator L: $C^{n \times n} \to C^{n \times n}$ by

$$L\begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} K + K^T \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}$$

where

$$K = \tfrac{1}{2}(A - I_n)(A + I_n)^{-1} \tag{3.20}$$

provided that A is stable. Equations (3.6) and (3.7) could now be written as

$$L\begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} = -(A+I_n)^{-T}\begin{bmatrix} W_1 & 0 \\ 0 & W_4 \end{bmatrix}(A+I_n)^{-1}$$

Thus Theorem 3.3 yields

$$\left\|\begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}\right\|_2 \leqslant \frac{n^{1/2}D^2(Z)}{\min|\lambda(K)-\lambda(-K)|} \cdot \left\|(A+I_n)^{-T}(W_1 \oplus W_4)(A+I_n)^{-1}\right\|_2 \tag{3.21}$$

where Z is the complete system of eigenvectors for K. We also note that (3.8) may be written as

$$W_2 = -\left[A_1^T \quad A_3^T\right](G_1 \oplus G_2)\left[A_2^T \quad A_4^T\right]^T$$

which with (3.21) lead to

$$\|W_2\|_2 \leqslant \left\|\begin{bmatrix} A_1 \\ A_3 \end{bmatrix}\right\|_2 \left\|\begin{bmatrix} A_2 \\ A_4 \end{bmatrix}\right\|_2 \frac{n^{1/2}D^2(Z)}{\min|\lambda(K)-\lambda(-K)|} \cdot \left\|(A+I_n)^{-T}(W_1 \oplus W_4)(A+I_n)^{-1}\right\|_2 \equiv \delta(W_1, W_4). \tag{3.22}$$

For any nonzero $x \in R^n$, let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{with } x_1 \in R^{n_1},\ x_2 \in R^{n_2},$$

$$\begin{aligned} x^T W x &= x_1^T W_1 x_1 + x_2 W_4 x_2^T + 2x_1^T W_2 x_2 \\ &\geqslant x^T(W_1 \oplus W_4)x - \|W_2\|_2 \|x\|_2^2 \\ &\geqslant x^T\left[(W_1 \oplus W_4) - \delta(W_1, W_4)I_n\right]x \end{aligned} \tag{3.23}$$

which indicates that the matrix W will be positive definite if $[(W_1 \oplus W_2) - \delta(W_1, W_4)I_n]$ is a P.D. matrix. We now choose $W_1 = I_{n_1}$, $W_4 = I_{n_2}$, (3.23) then becomes

$$x^T W x \geqslant x^T\left(1 - \delta(I_{n_1}, I_{n_2})\right)x \tag{3.24}$$

with

$$\delta(I_{n_1}, I_{n_2}) = \left\|\begin{bmatrix} A_1 \\ A_3 \end{bmatrix}\right\|_2 \left\|\begin{bmatrix} A_2 \\ A_4 \end{bmatrix}\right\|_2 \frac{n^{1/2}D^2(Z)}{\min|\lambda(K)-\lambda(-K)|} \cdot \left\|(A+I_n)^{-T}(A+I_n)^{-1}\right\|_2 \tag{3.25}$$

Thus $x^T W x$ will be strictly greater than zero if

$$\delta(I_{n_1}, I_{n_2}) < 1. \tag{3.26}$$

The main result in this section could now be stated as

Theorem 3.5: Assume that A, A_1, and A_4 are stable, then the system (1.1) is BIBO stable if $\epsilon_F < 1$ and $\delta(I_{n_1}, I_{n_2}) < 1$ where ϵ_F and $\delta(I_{n_1}, I_{n_2})$ are defined by (3.19) and (3.25), respectively.

As an example to illustrate the use of the theorem, we consider a 2-D system with

$$A = \begin{bmatrix} 0.4 & -0.2 \\ 0.2 & 0.4 \end{bmatrix}$$

for which

$$F = \begin{bmatrix} -0.21 & 0 \\ 0 & -0.21 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & -0.042 \\ 0.042 & 0 \end{bmatrix} \quad \text{and} \quad X = I_2.$$

Thus

$$\epsilon_F = \frac{2 \times 0.042^2 \times 2}{2 \times 0.21} = 0.017 < 1.$$

To check the condition (3.26), we compute

$$K = \frac{1}{4}\begin{bmatrix} -0.8 & -0.4 \\ 0.4 & -0.8 \end{bmatrix}$$

$$(A + I_2)^{-1} = \begin{bmatrix} 0.7 & 0.1 \\ -0.1 & 0.7 \end{bmatrix}.$$

Also note that A is normal so that the matrix Z in (3.25) is orthogonal, i.e., $D^2(Z) = 2$. Hence

$$\delta(I_{n_1}, I_{n_2}) = (0.4^2 + 0.2^2) \times \frac{2^{1/2} \times 2 \times 0.5}{0.4} = 0.71 < 1.$$

It is also seen that A_1, A_4, and A are stable. Theorem 3.5 now implies the stability of the system.

IV. Stabilization of a 2-D System by State Feedback

In case the system (1.1) is unstable, one may ask whether a state feedback

$$u(i,j) = -K\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \tag{4.1}$$

exists such that the closed-loop system

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = (A - BK)\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$$

is stable. The same question for MIMO 2-D systems may also be considered. In the rest of this section we assume that $B \in R^{(n_1+n_2) \times p}$ and $K \in R^{p \times (n_1+n_2)}$.

Replacing matrix A in (1.4) by $A - BK$ gives

$$(B^T G A)^T K + K^T(B^T G A) - K^T B^T G B K + (G - A^T G A - W) = 0 \tag{4.2}$$

or

$$\hat{A}^T K + K^T \hat{A} - K^T D K + Q = 0 \tag{4.3}$$

where

$$\hat{A} = B^T G A \qquad D = B^T G B \quad \text{and} \quad Q = G - A^T G A - W. \tag{4.4}$$

Notice that, in case $p = n_1 + n_2$, (4.3) is a matrix Riccati equation with $D \geqslant 0$ and $Q^T = Q$.

The following lemma is now an immediate consequence of Theorem 1.1.

Lemma 4.1: A 2-D system $(A, B) \in R^{(n_1+n_2)\times(n_1+n_2)} \times R^{(n_1+n_2)\times(n_1+n_2)}$ is stabilizable by state feedback (4.1) if and only if there exist two (n_1+n_2)-dimensional positive definite matrices W and $G = G_1 \oplus G_2$ with $G_1 \in R^{n_1 \times n_1}$, $G_2 \in R^{n_2 \times n_2}$ such that algebraic Riccati equation (4.3) has a real solution K.

Concerning the solvability of (4.3), one may construct the following $2(n_1+n_2)\times 2(n_1+n_2)$ matrix:

$$M = \begin{bmatrix} \hat{A} & -D \\ -Q & -\hat{A}^T \end{bmatrix} = \begin{bmatrix} B^TGA & -B^TGB \\ -(G - A^TGA - W) & -A^TGB \end{bmatrix}. \tag{4.5}$$

We now use the notation

$$a_i = \begin{bmatrix} b_i \\ c_i \end{bmatrix}, \qquad 1 \leqslant i \leqslant n_1 + n_2$$

for the $2(n_1+n_2)$-dimensional eigenvector of M corresponding to the eigenvalue λ_i in which a_i is partitioned into two (n_1+n_2)-dimensional vectors b_i and c_i. Following [10] we then have

Theorem 4.2 [10]: Let $a_1, \cdots, a_{n_1+n_2}$ be eigenvectors of M corresponding to eigenvalues $\lambda_1, \cdots, \lambda_{n_1+n_2}$ and assume $[b_1, \cdots, b_{n_1+n_2}]^{-1}$ exits. If $\bar{\lambda}_j \neq -\lambda_k$, $1 \leqslant j, k \leqslant n_1 + n_2$, then

$$K = \begin{bmatrix} c_1 & \cdots & c_{n_1+n_2} \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_{n_1+n_2} \end{bmatrix}^{-1} \tag{4.6}$$

is a solution of (4.3). Moreover, if all eigenvectors $a_1, \cdots, a_{n_1+n_2}$ are real, then the matrix K given in (4.6) is a real solution of (4.3).

Lemma 4.1 and Theorem 4.2 enable one to choose a feedback matrix K so that the resulting closed-loop system is stable:

Corollary 4.3: A 2-D system $(A, B) \in R^{(n_1+n_2)\times(n_1+n_2)} \times R^{(n_1+n_2)\times(n_1+n_2)}$ is stabilizable if $W > 0$ and $G = G_1 \oplus G_2 > 0$ could be chosen such that M in (4.5) has $n_1 + n_2$ real eigenvectors $a_1, \cdots, a_{n_1+n_2}$ corresponding to the eigenvalues $\lambda_1, \cdots, \lambda_{n_1+n_2}$ with $\bar{\lambda}_j \neq -\lambda_k$ for $1 \leqslant j, k \leqslant n_1 + n_2$ and $[b_1, \cdots, b_{n_1+n_2}]^{-1}$ exists.

Example 4.4: Consider an unstable 2-D system with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where $n_1 = n_2 = 1$. One may choose $G = \alpha I_2$, $W = \beta I_2$ with $\alpha > 0$ and $\beta > 0$ so that

$$M = -\alpha \begin{bmatrix} -1 & -2 & 2 & -2 \\ 1 & 2 & -2 & 2 \\ q-1 & -1 & 1 & -1 \\ -1 & q-2 & 2 & -2 \end{bmatrix} \equiv -\alpha \hat{M}$$

where $q = 1 - (\beta/\alpha)$.

Note that $\det[\lambda I - \hat{M}] = \lambda^2[\lambda^2 - (4q - 1)]$, so choosing $\alpha = 1$, $\beta = 0.5$ yields the eigenvalues of M:

$$\lambda_{1,3} = 0, \qquad \lambda_{2,4} = \pm 1$$

for $\lambda_1 = 0$ and $\lambda_2 = 1$, one could find the corresponding eigenvectors are

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} \quad \text{and} \quad a_2 = \begin{bmatrix} 1 \\ -1 \\ 0.5 \\ 0.5 \end{bmatrix}.$$

Thus

$$K = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \end{bmatrix}$$

which is a real symmetric solution of (4.3). In fact the resulting 2-D system matrix is

$$A - BK = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$

which is stable [6].

Remark: In case $p < n_1 + n_2$, let the matrix K in (4.1) be of the form $K = \Gamma P$ with $\Gamma \in R^{p \times (n_1+n_2)}$ and $P \in R^{(n_1+n_2)\times(n_1+n_2)}$, then equation Riccati (4.3) becomes

$$\hat{A}_1^TP + P^T\hat{A}_1 - P^TD_1P + Q = 0 \tag{4.7}$$

where

$$\hat{A}_1 = \Gamma^TB^TGA \qquad D_1 = \Gamma^TB^TGB\Gamma. \tag{4.8}$$

Thus the stabilization procedure could be carried out for a general 2-D system.

As an illustrative example, we consider a SISO unstable 2-D system (A, b) with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

where $n_1 = n_2 = 1$. It is seen that if we choose $\Gamma = [1 \quad -1]$, then

$$b\Gamma = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and in the light of the previous example, we know that (4.7) has a real solution

$$P = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \end{bmatrix}$$

that is, the state feedback with

$$K = \Gamma P = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

will stabilize the system (A, b).

V. Conclusions

Based on the 1-D Lyapunov theory with a complex parameter and a 2-D Lyapunov theorem, several stability results have been developed for the 2-D systems in state-space form. The results given in Section II indicate that to obtain a more accurate and checkable sufficient condition one needs to perform a deeper study of the geometric properties of the bilinear mapping. The results given in Section III could be applied to general 2-D systems, while the condition (3.26) seems to be quite restrictive. However, this condition might be improved by choosing the matrices W_1 and W_4 in (3.23) in an "optimal way" which may be accomplished by means of, for example, a programming approach. It is known that stabilizing a 2-D system by a local state feedback is very difficult. The results presented in Section IV shows that one could reduce the stabilization problem to a solvability problem of an algebraic Riccati equation in which the matrix Q may not be nonnegative and there are two "parameter matrices" W and G, the choice of which are directly related to the resolution of the stabilization problem.

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Explicit Formulas for Lattice Wave Digital Filters

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***Abstract*—Explicit formulas are derived for designing lattice wave digital filters of the most common filter types, for Butterworth, Chebyshev, inverse Chebyshev, and Cauer parameter (elliptic) filter responses. Using these formulas a direct top down design method is obtained and most of the practical design problems can be solved without special knowledge of filter synthesis methods. Since the formulas are simple enough also in the case of elliptic filters, the design process is sufficiently simple to serve as basis in the first part (filter design from specs to algorithm) of silicon compilers or applied to high level programmable digital signal processors.**

I. Introduction

WAVE DIGITAL filters (WDF's) [1] have some notable advantages [2]: excellent stability properties even under nonlinear operating conditions resulting from overflow and roundoff effects, low coefficient wordlength requirements, inherently good dynamic range, etc. All these properties are essentially a consequence of the fact that WDF's, if properly designed, behave completely like passive circuits.

For a proper design the full apparatus of the classical filter synthesis techniques (including those for microwave filters) can be made use of, which guarantees a solid mathematical basis of the WDF's. This fact, however, could be a serious hindrance when the designer is not familiar with the intricate techniques of the classical network theory (e.g., in the case of signal processing applications in medical, seismic, image, speech area etc, where the companies and institutions may not have available for this purpose specialized filter design groups, as well as programming and computer facilities).

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