where $\tilde{A}_c^{(r+3)}$ is a Hessenberg matrix. The vector $[\tilde{b}_{r+1}, 0, \cdots, 0]^T \in \mathbb{R}^{n-r}$ is reduced to $[\times, \tilde{b}_{r+2}, \tilde{b}_{r+3}, 0, \cdots, 0]^T$ and the complete controllability ensures that $\tilde{b}_{r+3} \neq 0$.

Now (12) may be used to determine the elements k_{r+1}, k_{r+2} of the transformed gain matrix $\tilde{k}Q_1 \cdots Q_{r+1,r+2}$. As a result one obtains

$$\tilde{b}_{r+2}k_{r+1} = a_{r+2,r+1} + q_1^2\tilde{y}_{r+2,1}/\tilde{x}_{r+1,1},$$
$$\tilde{b}_{r+2}k_{r+2} = a_{r+2,r+2} - p_1 - q_1^2\tilde{y}_{r+1,1}/\tilde{x}_{r+1,1} \tag{13}$$

and

$$\tilde{b}_{r+3}k_{r+1} = a_{r+3,r+1},$$
$$\tilde{b}_{r+2}k_{r+2} = a_{r+3,r+2} \tag{14}$$

where $a_{r+i,r+j}$; $i = 2, 3$, $j = 1, 2$ are elements of the transformed open-loop system matrix $Q_{r+1,r+2}^T \cdots Q_1^T\tilde{A}Q_1 \cdots Q_{r+1,r+2}$.

The equations (13) and (14) are algebraically consistent and may be solved as (7) and (8) in the real case.

It may be observed that at this step the real and the imaginary parts of the eigenvectors are obtained as a solution of a four-diagonal system of linear equations.

In this way the complex conjugate poles are treated in a similar manner as the real poles at the cost of a small increase in the number of the computational operations (an additional subdiagonal of the open-loop system matrix is used).

The next steps are performed in the same way. At steps $(n - 1)$, n the vector $x_m \in \mathbb{R}^2$ is transformed only once. No element of $y_m \in \mathbb{R}^2$ is to be annihilated. The elements k_{n-1}, k_n are obtained from equations of type (13) which cannot be zero identities since the closed-loop system must be completely controllable.

Finally, one obtains $\tilde{k} = [k_1, \cdots, k_n]Q^T$ and $k = \tilde{k}P^T$, where $Q = Q_1 \cdots Q_{r+1,r+2} \cdots Q_{n-1,n}$.

The algorithm presented has much in common with the deflation techniques [5] used to eliminate a known eigenvalue from an eigenvalue problem. For example, if an approximate eigenvector is known it is possible to construct an orthogonal transformation in order to produce a matrix of order one less than the original matrix that does not contain the corresponding eigenvalue. It is shown in [5] that this technique is very stable, although the approximate eigenvector may be far from the accurate one. This is because the errors in the transformed matrix depend not on the errors in the eigenvector v_i, but on the residual $Av_i - v_is_i$ which may be very small even if the eigenvector is not very accurate.

It may be shown that the algorithm proposed also has very good numerical properties due to the fact that the computation of an eigenvector, its transformation, and the determination of a gain matrix element correspond to a small residual in the equation for this eigenvector. In this way it is possible to prove that the subdiagonal elements of the triangular form obtained are negligible and since it is exact for a matrix which is close to the closed-loop system matrix, this ensures the numerical stability of the algorithm (the full proof is available from the authors).

The presentation of the algorithm will be concluded with an approximate operation count (as usual only the terms of order n^3 are considered).

	Operations
1) Row transformations of $\tilde{A}$	$2n^3/3$
2) Column transformations of $\tilde{A}$	$4n^3/3$
3) Accumulation of the transformations	$2n^3$
Total	$4n^3$

Adding to this figure the number of necessary operations for reducing the system into orthogonal canonical form one can find $17n^3/3$ operations. With respect to the array storage the algorithm requires $2n^2 + 6n$ words.

The algorithm is implemented as a Fortran program which is carefully tested with various problems of order up to 50.

III. CONCLUSIONS

An efficient computational algorithm for pole assignment of linear single-input systems based on orthogonal triangularization of the closed-loop system matrix is proposed. The algorithm is numerically stable with respect to the determination of the gain matrix and performs equally well with real and complex, distinct, and multiple desired poles. It is applicable to ill-conditioned and high-order problems and may be used for synthesis of continuous- as well as discrete-time systems.

ACKNOWLEDGMENT

The authors would like to thank Prof. A. J. Laub and Dr. M. J. Denham for their interest in this work.

REFERENCES

[1] A. Varga, "A Schur method for pole assignment," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 517-519, 1981.
[2] M. M. Konstantinov, P. Hr. Petkov, and N. D. Christov, "A Schur approach to pole assignment problem," in *Proc. 8th IFAC Congress*, Kyoto, Japan, Aug. 1981, vol. 3, pp. 1587-1592. Oxford, England: Pergamon, 1982.
[3] G. S. Miminis and C. C. Paige, "An algorithm for pole assignment of time invariant linear systems," *Int. J. Contr.*, vol. 35, pp. 341-354, 1982.
[4] M. M. Konstantinov, P. Hr. Petkov, and N. D. Christov, "Orthogonal invariants and canonical forms for linear controllable systems," in *Proc. 8th IFAC Congress*, Kyoto, Japan, Aug. 1981, vol. 1, pp. 49-54. Oxford, England: Pergamon, 1982.
[5] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*. Oxford, England: Clarendon, 1965.

Coefficient Assignability for Linear Systems with Delays

E. BRUCE LEE AND WU-SHENG LU

Abstract—**Using canonical forms for the linear delay systems with commensurate delays, an approach to coefficient assignment of the characteristic polynomial under feedback control of polynomial delay type is given. The results are achieved under quite weak requirements of controllability.**

I. INTRODUCTION

We present some new results on coefficient assignment by state feedback control for linear systems with commensurate time delays. Such a system may be characterized in state-space version by a pair $(A(z), B(z)) \in R^{n\times n}[z] \times R^{n\times m}[z]$ where $R[z]$ is the ring composed of all polynomials in the delay operator z with real coefficients. Define $X = \{(A, B) \in R^{n\times n}[z] \times R^{n\times m}[z] | \text{rank}\,[A|B] = n$ for all but finitely many $z \in C\}$ which will be labeled by X_1 when $m = 1$ where $[A|B] \equiv [B, AB, \cdots, A^{n-1}B]$. Also let $Y = \{(A, B) \in R^{n\times n}[z] \times R^{n\times m}[z] | \text{span}_{R[z]}[A|B] = R^n[z]\}$ which is denoted by Y_1 if $m = 1$. In the literature a pair $(A, B) \in X$ is a system controllable over $R(z)$, where $R(z)$ is the set of all rational functions in z with real coefficients, and $(A, B) \in Y$ means controllability over $R[z]$.

There have been quite a few publications related to coefficient assignment for delay systems. It seems that Morse's result given in [1] is still one of the best; being a claim that for each set of polynomials $\{\beta_i, 1 \leqslant i \leqslant n\}$ in $R[z]$ there exists $K \in R^{m\times n}[z]$ such that $\det\,[sI - (A + BK)] = \Pi_{i=1}^n (s - \beta_i)$ if and only if $(A, B) \in Y$. It is evident that for the single-input case the n coefficients of $\det\,[sI - (A + bk)]$ can be assigned arbitrary values in $R[z]$ if and only if $(A, b) \in Y_1$. However, for $(A, B) \in Y$, the question of coefficient assignability is still open [2].

Manuscript received January 5, 1984. This work was supported by the National Science Foundation under Grant ECS 8217375.

The authors are with the Department of Electrical Engineering, University of Minnesota, Minneapolis, MN 55455.

0018-9286/84/1100-1048$01.00 © 1984 IEEE

Using certain types of canonical forms of delay systems, we look at this question in a broader system class, i.e., in X. Canonical forms play a crucial role in establishing coefficient assignability for linear time-invariant systems. One may expect to get similar insight for the delay systems if various canonical forms in X are searched.

For single-input systems in X_1, this approach combined with a powerful lemma from [3, Lemma 6.6-1, p. 471] leads to a complete characterization of the possible coefficients of det $[sI - (A + bk)]$ when k varies in $R^{1\times n}[z]$. For multiinput systems in X, one may use a canonical form similar to that developed in [4] to reduce it to the coefficient assignment problem for m single-input $R(z)$-controllable subsystems. The use of the result obtained for the single-input case then gives a solution. In case $(A, B) \in Y$, a direct extension of Popov's canonical form makes it possible to obtain a sufficient condition yielding coefficient assignability.

II. The Case $(A, b) \in X_1$

Let $[A|b]$ be the controllability matrix of (A, b),

$$[A\,|\,b]^{-1} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \quad \text{and } T_0 = \begin{bmatrix} h_n \\ h_nA \\ \vdots \\ h_nA^{n-1} \end{bmatrix}$$

then

$$\hat{b} \equiv T_0 b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$\hat{A} \equiv T_0 A_0 T_0^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}$$

where a_i's $\in R[z]$ are determined by computing det $[sI - A(z)] = s^n + \Sigma_{i=1}^n a_is^{i-1}$. Notice that $T_0 \in R^{n\times n}(z)$, $\hat{A} \in R^{n\times n}[z]$, and the characteristic equation of the closed-loop system when applying a polynomial state feedback $u(t) = k(z)x(t)$ is $\Delta(s, z) \equiv$ det $[sI - (A(z) + b(z)k(z))] =$ det$[sI - (\hat{A} + \hat{b}\hat{k})]$ where $\hat{k}(z) = k(z)T_0^{-1} \equiv [\hat{k}_1(z), \cdots, \hat{k}_n(z)]$. Namely, $\Delta(s, z) = s^n + \Sigma_{i=1}^n (a_i - \hat{k}_i)s^{i-1} \equiv s^n + \Sigma_{i=1}^n \alpha_is^{i-1}$. To characterize all possible sets $\{\alpha_i(z), 1 \leqslant i \leqslant n\}$ obtained by applying a state feedback $u = k(z)x$ with $k(z) \in R^{1\times n}[z]$, we need the following lemma.

Lemma 2.1: Assume that $H(z) = D^{-1}(z)N(z)$ is an irreducible MFD for strictly proper $H(z) \in R^{n\times n}(z)$. Then, for $\hat{k}(z) \in R^{1\times n}[z]$, $\hat{k}(z)H(z)$ belongs to $R^{1\times n}[z]$ if and only if there exists an $f(z) \in R^{1\times n}[z]$ such that $\hat{k}(z) = f(z)D(z)$.

Proof: The "if part" is trivial. The "only if part" is [3, Lemma 6.6-1].

We now assume that $T_0(z) = D^{-1}(z)N(z)$ is an irreducible MFD. By the lemma, it is seen that $k(z) = \hat{k}T_0(z) = \hat{k}(z)D^{-1}(z)N(z)$ will be a $1 \times n$ polynomial matrix whenever $\hat{k}D^{-1}N \in R^{1\times n}[z]$, i.e., $\hat{k}(z) = f(z)D(z)$ for some $f(z) \in R^{1\times n}[z]$. Denoting

$$D(z) = \begin{bmatrix} d_1(z) \\ \vdots \\ d_n(z) \end{bmatrix}$$

and $\mathfrak{D} = \text{span}_{R[z]}\{d_i(z), 1 \leqslant i \leqslant n\}$, we observe that $\mathfrak{D}$ is a submodule in $R^n[z]$ and $k(z) = \hat{k}(z)T_0(z)$ will belong to $R^{1\times n}[z]$ iff $\hat{k}(z) \in \mathfrak{D}$.

Letting $a(z) = [a_i(z), \cdots, a_n(z)]$, $\alpha(z) = [\alpha_1(z), \cdots, \alpha_n(z)]$, one has the following result.

Theorem 2.2: Given a single-input $R(z)$-controllable delay system $(A(z), b(z))$, the coefficients of det $[sI - (A + bk)]$ can be assigned to be $\alpha(z) \in R^{1\times n}[z]$ by a polynomial state feedback $u(t) = k(z)x(t)$ if and only if $a(z) - \alpha(z) \in \mathfrak{D}$, where $\mathfrak{D}$ is a submodule (in $R^n[z]$) spanned by $d_i(z)$'s and the desired feedback gain is then given by $k(z) = (a(z) - \alpha(z))T_0(z)$.

Remark 1: By Morse's result [1], $\mathfrak{D} \neq R^n[z]$ unless $(A, b) \in Y_1$.

Remark 2: Denote $T_0(z) = P(z) + H(z)$ with $P(z) \in R^{n\times n}[z]$ and $H(z) \in R^{n\times n}(z)$ strictly proper, $H(z) = D_1^{-1}(z)N_1(z)$ is an irreducible MFD. It then is easy to show that Theorem 2.2 also holds if $\mathfrak{D}$ is replaced by $\mathfrak{D}_1$, where $\mathfrak{D}_1$ is a submodule generated by the rows of $D_1(z)$.

Example: Consider a time-delay pair

$$A(z) = \begin{bmatrix} a_1(z) & a_2(z) \\ z^2 & a_3(z) \end{bmatrix}, \quad b(z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with $a_3(0) < 0$. One can recognize that (A, b) might not be stable independent of delay if, e.g., $a_1a_3 \neq 0$ on $|z| = 1$ and $a_1(1) + a_3(1) > 0$ [5].

Note that

$$[A\,|\,b]^{-1} = \begin{bmatrix} 1 & -z^{-2}a_1 \\ 0 & z^{-2} \end{bmatrix}, \quad T_0(z) = \begin{bmatrix} 0 & z^{-2} \\ 1 & z^{-2}a_3 \end{bmatrix}.$$

Let $a_3 = g_0 + g_1z + \cdots + g_lz^l$, then

$$T_0(z) = \begin{bmatrix} 0 & 0 \\ 1 & g_2 + \cdots + g_lz^{l-2} \end{bmatrix} + \begin{bmatrix} 0 & z^{-2} \\ 0 & (g_0 + g_1z)z^{-2} \end{bmatrix}$$

where the strictly proper part has an irreducible MFD

$$\begin{bmatrix} 0 & z^{-2} \\ 0 & (g_0 + g_1)z^{-2} \end{bmatrix} = \begin{bmatrix} z^2 & 0 \\ -(g_0 + g_1z) & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which gives the structure of submodule $\mathfrak{D}_1$:$\mathfrak{D}_1 = \{[f_1(z)z^2 - f_2(z)(g_0 + g_1z), f_2(z)], f_1(z), f_2(z) \in R[z]\}$. Note that det $[sI - A(z)] = s^2 - (a_1 + a_3)\,z + (a_1a_3 - z^2a_2)$, so $a(z) = [a_1a_3 - z^2a_2 - (a_1 + a_3)]$. We now look at the possibility of having $\alpha = [a_3{}^2(0) - 2a_3(0)] = [g_0{}^2 - 2g_0]$, i.e., for the closed-loop system to have a stable characteristic polynomial $s^2 - 2g_0s + g_0{}^2$. To do this, one can try to equate

$$a(z) - \alpha = [a_1a_3 - z^2a_2 - g_0^2 \;\; -(a_1 + a_3) + 2g_0] = [f_1z^2 - f_2(g_0 + g_1z) \;\; f_2]$$

and find a solution $f = [f_1\, f_2] \in R^{1\times 2}[z]$. Actually, we have

$$f_2(z) = 2g_0 - a_1(z) - a_3(z),$$

and

$$f_1(z) = [a_1(z)a_3(z) - z^2a_2 - g_0^2 + (g_0 + g_1z)(2g_0 - a_1 - a_3)]/z^2 \equiv \hat{f}_1(z)/z^2 \quad (1)$$

in which one can verify that $\hat{f}_1(0) = 0$ and

$$\left[\frac{d}{dz}\hat{f}(z)\right]_{z=0} = 0$$

so that $\hat{f}_1(z)/z^2$ is also a polynomial. Returning to the original system, the desired state feedback by which the characteristic polynomial is assigned as $s^2 - 2a_3(0)s + a_3{}^2(0)$ is $u(t) = k(z)x(t)$ where

$$\begin{aligned} k(z) &= \hat{k}(z)T_0(z) = \hat{k}(z)P(z) + \hat{k}(z)D^{-1}(z)N(z) \\ &= \hat{k}(z)P(z) + f(z)N(z) = (a(z) - \alpha)P(z) + f(z)N(z) \\ &= [a_1a_3 - z^2a_2 - g_0^2 \;\; 2g_0 - a_1 - a_3] \begin{bmatrix} 0 & 0 \\ 1 & g_2 + \cdots + g_lz^{l-2} \end{bmatrix} \\ &\quad + [f_1 \;\; f_2] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= [2g_0 - a_1 - a_3 \;\; f_1 + (2g_0 - a_1 - a_3)(g_2 + \cdots + g_lz^{l-2})] \\ &= [2a_3(0) - a_1 - a_3 \;\; f_1 + (2a_3(0) - a_1 - a_3)(a_3 - a_3(0) - a_3'(0))] \end{aligned}$$

where f_1 is given in (1) and $a_3{}'(0)$ represents the coefficient of the first-order term in $a_3(z)$.

III. The Case $(A, B) \in X$

We begin our treatment by transforming a pair $(A, B) \in X$ to its canonical form, which is a slight modification of recent work [4].

Let $B = [b_1, \cdots, b_m]$ have rank m and $\tilde{Q}(z) = [b_m, \cdots, A^{p_m-1}b_m, \cdots, b_1, \cdots, A^{p_1-1}b_1]$ is a full rank square matrix such that, for $1 \leqslant i \leqslant m$, $A^{p_i}b_i$ is a linear combination of $b_m, Ab_m, \cdots, A^{p_i-1}b_i$ in $R(z)$. We now rearrange the columns $\tilde{Q}(z)$ as $Q(z) = [A^{p_1-1}b_1, \cdots, b_1, \cdots, A^{p_m-1}b_m, \cdots, b_m]$ for which there is a unimodular $T(z) \in R^{n\times n}[z]$ such that

$$T(z)Q(z) \equiv H(z) = \begin{bmatrix} h_{11} & & 0 & \\ h_{21} & h_{22} & & \\ \vdots & \vdots & \ddots & \\ h_{n1} & h_{n2} & \cdots & h_{nn} \end{bmatrix} \tag{2}$$

with $\deg h_{ij} < \deg h_{jj}$ for $1 \leqslant j < i \leqslant n$. Define $\hat{A} = TAT^{-1}$, $\hat{B} = TB$ and note that

$$H(z) = T(z)Q(z) = [\hat{A}^{p_1-1}\hat{b}_1 \cdots \hat{b}_1, \cdots, \hat{A}^{p_m-1}\hat{b}_m, \cdots, \hat{b}_m].$$

Thus,

$$\hat{B} = [\hat{b}_1, \cdots, \hat{b}_m] = \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \vdots & & \vdots \\ h_{l_1,l_1} & \vdots & & \vdots \\ x & 0 & \cdots & \cdot \\ \vdots & h_{l_2,l_2} & & \vdots \\ \vdots & x & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & 0 \\ x & x & & h_{nn} \end{bmatrix}$$

where $l_i \equiv \sum_{j=1}^{i} p_j$, and $\hat{A} = \hat{A}HH^{-1} = [\hat{A}^{p_1}\hat{b}_1, \cdots, \hat{A}\hat{b}_1, \cdots, \hat{A}^{p_m}\hat{b}_m, \cdots, \hat{A}\hat{b}_m]H^{-1}$

where $\hat{A}_i \in R^{p_i\times p_i}[z]$, $1 \leqslant i \leqslant m$. It is seen that for each $1 \leqslant i \leqslant m$, defining $\hat{b}_i = [0, \cdots, 0\ h_{l_i,l_i}]^T \in R^{p_i\times 1}[z]$, $(\hat{A}_i, \hat{b}_i)$ is a single-input p_i-dimensional $R(z)$-controllable subsystem. Thus, a block-diagonal state feedback $u(t) = \hat{K}(z)x(t)$ with

$$\hat{K}(z) = \begin{bmatrix} \hat{K}_1(z) & & & \\ & \hat{K}_2(z) & & 0 \\ & & \ddots & \\ 0 & & & \hat{K}_m(z) \end{bmatrix}$$

where $\hat{K}(z) \in R^{1\times p_i}[z]$ will lead to a closed-loop system with

$$\det\,[sI_n - (\hat{A} + \hat{B}\hat{K})] = \prod_{i=1}^{m} \det\,[sI_{p_i} - (\hat{A}_i + \hat{b}_i\hat{K}_i)]$$

where I_n and I_{p_i} are identity matrices with dimensions n and p_i, respectively.

Let $\det\,(sI_{p_i} - \hat{A}_i) = s^{p_i} + a^{(i)}_{p_i}s^{p_i-1} + \cdots + a^{(i)}_2 s + a^{(i)}_1$, and $a^{(i)}(z) = [a^{(i)}_{p_i}, \cdots, a^{(i)}_1]$. Also let

$$[\hat{A}_i | \hat{b}_i]^{-1} = \begin{bmatrix} h_1^{(i)} \\ \vdots \\ \vdots \\ h_{p_i}^{(i)} \end{bmatrix},$$

$$T_0^{(i)} = \begin{bmatrix} h_{p_i}^{(i)} \\ h_{p_i}^{(i)}\hat{A}_i \\ \vdots \\ h_{p_i}^{(i)}\hat{A}_i^{p_i-1} \end{bmatrix}, \quad T_0^{(i)} = P_i(z) + H_i(z)$$

where $P_i(z) \in R^{p_i\times p_i}[z]$, $H_i(z) \in R^{p_i\times p_i}(z)$ are strictly proper and $H_i(z) = D_i^{-1}(z)N_i(z)$ with D_i and N_i left coprime. Denote the submodules spanned by the rows of D_i by $\mathfrak{D}^{(i)}$ then, by Theorem 2.2, we have the following.

$$= H \begin{array}{c} \begin{array}{ccc} \overbrace{\qquad\qquad\qquad}^{p_1} & \overbrace{\qquad\qquad\qquad}^{p_2} & \qquad\qquad \overbrace{\qquad\qquad\qquad}^{p_m} \end{array} \\ \left[\begin{array}{ccccc:ccccc:c:ccccc} x & 1 & 0 & \cdots & 0 & & & & & & & & & & & \\ \cdot & 0 & 1 & \cdots & 0 & & & & & & & & & & & \\ \cdot & & & \ddots & & & & & & & & & & 0 & & \\ \cdot & & & & 1 & & & & & & & & & & & \\ x & 0 & \cdots & \cdots & 0 & & & & & & & & & & & \\ \hdashline x & & & & & x & 1 & 0 & \cdots & 0 & & & & & & \\ \cdot & & & & & \cdot & 0 & 1 & \cdots & 0 & & & & & & \\ \cdot & & & & & \cdot & & & \ddots & & & & & & & \\ \cdot & & & & & \cdot & & & & 1 & & & & & & \\ \cdot & & & & & x & 0 & \cdots & \cdots & 0 & & & & & & \\ \hdashline \cdot & & & & & \cdot & & & & & \ddots & & & & & \\ \hdashline \cdot & & & & & \cdot & & & & & & x & 1 & 0 & \cdots & 0 \\ \cdot & & & & & \cdot & & & & & & \cdot & 0 & 1 & \cdots & 0 \\ \cdot & & & & & \cdot & & & & & & \cdot & & & \ddots & \vdots \\ \cdot & & & & & \cdot & & & & & & \cdot & & & & 1 \\ x & & & & & x & & & & & & x & 0 & \cdots & \cdots & 0 \end{array}\right] \end{array} H^{-1} = \left[\begin{array}{c:c:c:c} \hat{A}_1 & & & \\ \hdashline & \hat{A}_2 & & 0 \\ \hdashline & & \ddots & \\ \hdashline * & & & \hat{A}_m \end{array}\right] \tag{3}$$

Theorem 3.1: Given a delay system $(A, B) \in X$ and $\alpha^{(i)}z = [\alpha_1^{(i)}(z), \cdots, \alpha_{p_i}^{(i)}(z)] \in R^{p_i}[z]$ $(1 \leqslant i \leqslant m)$, det $[sI - (A + BK)]$ can be assigned to be $\Pi_{i=1}^{m} q^{(i)}(s,z)$ with $q^{(i)}(s, z) = s^{p_i} + \alpha_{p_i}^{(i)} s^{p_i - 1} + \cdots + \alpha_2^{(i)} s + \alpha_1^{(i)}$ if $a^{(i)}(z) - \alpha^{(i)}(z) \in \mathfrak{D}^{(i)}$. The state feedback is then given by

$$K(z) = \hat{K}(z)T(z) = \begin{bmatrix} (a^{(1)}(z) - \alpha^{(1)}(z))T_0^{(1)}(z) & & & 0 \\ & (a^{(2)}(z) - \alpha^{(2)}(z))T_0^{(2)}(z) & & \\ & & \ddots & \\ 0 & & & (a^{(m)}(z) - \alpha^{(m)}(z))T_0^{(m)}(z) \end{bmatrix}$$

where $T(z)$ is given in (2) and $(a^{(i)} - \alpha^{(i)})T_0^{(i)} = (a^{(i)} - \alpha^{(i)})P_i + f^{(i)}N_i$ for some $f^{(i)} \in R^{p_i}[z]$.

IV. THE CASES $(A, b) \in Y_1$ AND $(A, B) \in Y$

In case $(A, b) \in Y_1$, $[A|b]^{-1}$ and $T_0(z)$ are polynomial matrices so that for any specified coefficient vector $\alpha(z)$, the corresponding state feedback $k(z) = (a(z) - \alpha(z))T_0(z)$ is always in $R^{1 \times n}[z]$. Also note that $(A, b) \in Y_1$ implied $\mathfrak{D} = R^n[z]$.

For the multiinput case, the coefficient assignability problem is difficult. Similar to the single-input case, however, one may look at the controller form [3] for a pair $(A, B) \in Y$.

Without loss of generality, we assume that span $R_{[z]}(b_1, \cdots, b_i) \subset \text{span}_{R[z]}(b_1, \cdots b_{i+1})$ for any $1 \leqslant i \leqslant m - 1$, where $S_1 \subset S_2$ means that S_1 is a proper subset of S_2, otherwise the input channel can be reduced. Since $(A, B) \in Y$, some columns from $[A|b]$ can be picked up, say

$$G = \{b_1, \cdots, b_m, Ab_{j1}, \cdots, Ab_{j_r}, A^2b_{k_1}, \cdots, A^2b_{k_s}, \cdots,$$
$$A^{p_{l_1}-1}b_{l_1}, \cdots, A^{p_{l_1}-1}b_{l_t}\} \tag{3}$$

such that i) $\text{span}_{R[z]}G = R^n[z]$, ii) $\text{span}_{R[z]}V_t \subset \text{span}_{R[z]}V_{t+1}$ where V_t is a proper subset of G composed of first t columns in G and V_{t+1} is composed by first $(t + 1)$ col[illegible] in G, iii) for each $1 \leqslant i \leqslant m$, if $A^{p_i-1}b_i \in G$ but $A^{p_i}b_i \notin G$ then $A^{p_i}b_i$ can be expressed as a linear combination of the columns in G previous to a position where $A^{p_i}b_i$ is supposed to be if $A^{p_i}b_i \in G$. Note that

$$\{1, 2, \cdots, m\} \supseteq \{j_1, \cdots, j_r\} \supseteq \{k_1, \cdots, k_s\}$$
$$\supseteq \cdots \supseteq \cdots \supseteq \{l_1, \cdots, l_t\}$$

and for each $1 \leqslant i \leqslant m$, there exit $\{\beta_\nu^{(i)}, \nu \in z^+\}$ in $R[z]$ such that

$$A^{p_i}b_i = \beta_1^{(i)}b_1 + \cdots + \beta_m^{(i)}b_m + \beta_{m+1}^{(i)}Ab_{j_1} + \cdots + \beta_{l_i}^{(i)}A^{q_i}b_{r_i}.$$

We then put all terms with nonzero powers of A on the left, and factor out A as

$$A(A^{p_i-1}b_i - \beta_{m+1}^{(i)}b_{j_1} - \cdots - \beta_{l_i}^{(i)}A^{q_i-1}b_{r_i}) = \beta_1^{(i)}b_1 + \cdots + \beta_m^{(i)}b_m.$$

It is evident that the procedure given above is the same as in Popov's canonical form [6], also see [3, ch. 6, pp. 435–436]. Keep doing this and define $e_{i1} = b_i$ for $1 \leqslant i \leqslant m$. We now have a set $\{e_{11}, \cdots, e_{1p_1}, \cdots, e_{m1}, \cdots, e_{mp_m}\}$, which may form a basis if $\Sigma_{i=1}^{m} p_i = n$. Indeed in such a case we have the following matrix:

$$\hat{A} = T^{-1}AT = \left[\begin{array}{cccc:ccccc:ccc} x & x & \cdots & x & x & \cdots & & & & & & x \\ 1 & & & 0 & & & & & & & & \\ & \ddots & & \vdots & & & & & & & & \\ & & 1 & 0 & & & & & & & & \\ \hdashline x & \cdots & & x & x & x & \cdots & x & x & \cdots & & x \\ & & & & 1 & & & 0 & & & & \\ & & & & & \ddots & & \vdots & & & & \\ & & & & & & 1 & 0 & & & & \\ \hdashline & & & & & & & & \ddots & & & \\ \hdashline x & \cdots & & & & & & & x & x & \cdots & x \\ & & & & & & & & 1 & & & 0 \\ & & & & & & & & & \ddots & & \vdots \\ & & & & & & & & & & 1 & 0 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}x\\1\\ \ddots\\1\end{array}}\right\} p_1 \\ \left.\vphantom{\begin{array}{c}x\\1\\ \ddots\\1\end{array}}\right\} p_2 \\ \\ \left.\vphantom{\begin{array}{c}x\\1\\ \ddots\\1\end{array}}\right\} p_m \end{array}$$

$$\text{and } \hat{B} = T^{-1}B = \left[\begin{array}{cccc} 1 & 0 & & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ \hdashline 0 & 1 & & 0 \\ \vdots & 0 & & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ \hdashline \vdots & & & \vdots \\ \vdots & & & \vdots \\ \hdashline 0 & 0 & & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{array}\right]$$

where $T^{-1} = \{e_{11}, \cdots, e_{1p_1}, \cdots e_{m,p_m}\} \in R^{n\times n}[z]$ and the possible nonzero elements x's are all polynomials. It is obvious that given a specified coefficient vector $\alpha(z) = [\alpha_1, \cdots, \alpha_n] \in R^{1\times n}[z]$, there exists a $\hat{K}(z) \in R^{n\times n}[z]$ such that $\det [sI - (\hat{A} + \hat{B}\hat{K})] = \det [sI - T^{-1}(A + B\hat{K}T^{-1})T] = \det [sI - (A + BK)] = s^n_{i=1} + \Sigma^n \alpha_i s^{i-1}$, where $K \equiv \hat{K}T^{-1} \in R^{n\times n}[z]$.

We now have the following.

Theorem 4.1: Coefficint assignability holds for $(A, B) \in Y$ if $\Sigma_{i=1}^m p_i = n$ where p_i's are integers in (3).

REFERENCES

[1] A. S. Morse, "Ring models for delay-differential systems," in *Proc. IFAC Symp. Multivariable Technol. Syst.*, Manchester, U.K., 1974, pp. 56.1-56.7.

[2] E. D. Sontag, "Linear systems over commutative rings: A survey," *Ricerche di Automatica*, vol. 7, no. 1, pp. 1-34, 1976.

[3] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[4] S. H. Żak, W.-S. Lu, and E. B. Lee, "Invariants and canonical forms for linear systems with delays," submitted for publication.

[5] E. Kamen, "Correction to "Linear systems wth commensurate time delays: Stability and stabilization independent of delay,"" *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 248-249, 1983.

[6] V. M. Popov, "Invariant description of linear, time-invariant controllable systems," *SIAM J. Contr.*, vol. 10, pp. 252-264, 1972.

Decentralized Control of Dynamically Interconnected Systems

ARNO LINNEMANN

Abstract—**Interconnected systems, where the subsystems are interconnected by some dynamic interaction system, are considered. It is shown that this type of system can be stabilized by decentralized dynamic output feedback, if the subsystems are stabilizable by (centralized) dynamic output feedback and the interaction system is stable. The relation to previous results is discussed.**

I. INTRODUCTION

Consider the systems described by the equations

$$\dot{x}_i = A_i x_i + B_i u_i$$
$$y_i = C_i x_i, \qquad i = 1, \cdots, k \tag{1}$$

which are interconnected by the interaction system given by

$$\dot{z} = Mz + \sum_{j=1}^{k} L_j y_j$$
$$u_i = N_i z + \sum_{j=1}^{k} P_{ij} y_j + v_i, \qquad i = 1, \cdots, k. \tag{2}$$

Here, $x_i \in \mathbb{R}^{n_i}(n_i \geqslant 1)$, $u_i \in \mathbb{R}^{m_i}(m_i \geqslant 1)$, $y_i \in \mathbb{R}^{p_i}(p_i \geqslant 1)$, $z \in \mathbb{R}^q(q \geqslant 0)$, $v_i \in \mathbb{R}^{m_i}$, and the matrices A_i, B_i, C_i, M, L_i, N_i, P_{ij} are of compatible sizes. With $q = 0$, (2) includes the special case where the subsystems are statically interconnected as follows:

$$u_i = \sum_{j=1}^{k} P_{ij} y_j + v_i, \qquad i = 1, \cdots, k. \tag{3}$$

Manuscript received December 5, 1983.

The author is with the Forschungsschwerpunkt Dyamische Systeme, Universität Bremen, Bremen, West Germany.

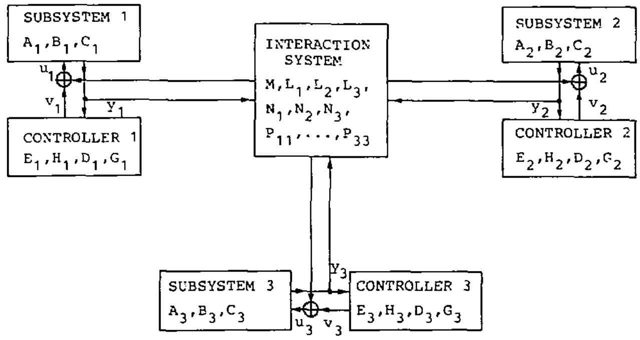

Fig. 1.

Recently, there has been some interest in controlling this type of interconnected system or special cases of it [1]-[4].

In this note, sufficient conditions for the interconnected system (1), (2) to be stabilizable by decentralized dynamic output feedback of the form

$$v_i = E_i w_i + H_i y_i$$
$$\dot{w}_i = D_i w_i + G_i y_i, \qquad i = 1, \cdots, k \tag{4}$$

where $w_i \in \mathbb{R}^{s_i}(s_i \geqslant 0)$ and the matrices E_i, H_i, D_i, G_i are of compatible sizes, will be derived. Fig. 1 shows the structure of the closed-loop system for $k = 3$.

II. MAIN RESULT

The main result of this note is as follows.

Theorem 1: Suppose that the subsystems (1) are stabilizable and detectable, i.e.,

$$\text{rank } [sI - A_i \quad B_i] = n_i; \qquad \text{rank} \begin{bmatrix} sI - A_i \\ C_i \end{bmatrix} = n_i$$

$$\text{for all } s \in \mathbb{C}_+ := \{s \in \mathbb{C} \mid \text{Re} s \geqslant 0\}, \quad i = 1, \cdots, k \tag{5}$$

and that the interaction system (2) is stable, i.e.,

$$\text{spectrum } (M) \subset \mathbb{C}_- := \mathbb{C} \setminus \mathbb{C}_+. \tag{6}$$

Then there exists decentralized dynamic output feedback of the form (4) such that the closed-loop system given by (1), (2), and (4) is stable.

The proof of this theorem is delegated to the next section. Now, some remarks concerning Theorem 1 will be stated and some conclusions will be derived.

Remark 1: The conditions (5) mean that the subsystems (1) can be stabilized by dynamic output feedback. Of course, they may be replaced by the stronger conditions that the subsystems are controllable and observable.

Remark 2: The sufficient conditions of Theorem 1 are independent of the coefficient matrices L_i, N_i, P_{ij} of the interaction system. Only the free motion of the interaction system is important.

Applying Theorem 1 to statically interconnected systems (1), (3), the following result of [3] is immediate.

Corollary 1: The statically interconnected system (1), (3) is stabilizable by decentralized dynamic output feedback, if the subsystems (1) are stabilizable by dynamic output feedback.

In [1], the following special cases of the interconnection structure (2) are studied.

Structure I:

$$M = \text{diag } (M_1, \cdots, M_k), \; M_i \in \mathbb{R}^{q_i \times q_i}$$

$$N_i = (0, \cdots, 0, \tilde{N}_i, 0, \cdots, 0), \; \tilde{N}_i \in \mathbb{R}^{m_i \times q_i}$$