Hence, the U-type hyperbolic m-periodic point cannot exist, and in consequence, possible bifurcations are of the T_1-, D_1-, D_3-, I_1-, and I_3-types. In the numerical results shown below, however, only the T_1-, D_1-, and I_1-types are actually observed.

All the terms on the right-hand side of (7) are odd with respect to x_1 and x_2, and the input term satisfies the following condition:

$$\sin\omega\left(t+\frac{2k+1}{2}\tau\right)=-\sin\omega t \qquad (k=0,1,2,\cdots). \tag{11}$$

Therefore, the D-type bifurcation can exist, and in addition, the following properties hold.

1) Symmetric $(2k+1)\tau$-periodic solutions $(k=0,1,2,\cdots)$ whose trajectories are symmetric with respect to the origin of the state space can exist, while symmetric $2k\tau$-periodic solutions $(k=1,2,\cdots)$ cannot.

2) If there exists an asymmetric $m\tau$-periodic solution $(m=1,2,\cdots)$, then there must exist another asymmetric $m\tau$-periodic solution whose trajectory is symmetric to that of the former with respect to the origin.

The Y versus ω curve determined numerically is shown in Fig. 1, together with the one determined previously by use of the describing-function method [4]. As expected, there exists a great discrepancy between these two results in the low-frequency range. In Fig. 2, a topological structure is sketched in order to make the fine structure of the complicated Y versus ω curve easy to see.

The main branch $A'ABC_1C_1'D_1E_1C_2C_2'D_2E_2\cdots$ contains symmetric τ-periodic solutions of the S-type $(BC_1, C_1'D_1, E_1C_2, C_2'D_2, E_2C_3,\cdots)$, the D-type $(AB, C_1C_1', D_1E_1, C_2C_2', D_2E_2,\cdots)$, and the nonhyperbolic type (AA'). Therefore, $B, D_1, E_1, D_2, E_2,\cdots$ are the T_1-type bifurcation points where a jump must occur with a change in ω, while $C_1, C_1', C_2, C_2',\cdots$ are the D_1-type bifurcation points from which bypass branches of twin asymmetric τ-periodic solutions emanate. The character of the subbranch AA' is rather singular. For each value of ω, there exists an infinite number of twin asymmetric τ-periodic solutions, in addition to a symmetric τ-periodic solution. The trajectories of these periodic solutions are ellipses of the same size centered on the x_1 axis and are entirely contained within the domain $-1\leqslant x_1\leqslant 1$ where the state equation is linear. The periodic point corresponding to each of these solutions is nonhyperbolic because $\mu_1=1$ and $\mu_2=e^{-m\tau}$. Thus, the bifurcation point A where a jump must occur is of the tangential type, but it is not of the T_1-type. The advent of such a singular bifurcation might be due to the fact that the $\psi(x_1)$ has no derivative at $x_1=\pm 1$.

The bypass branch $C_1F_1G_1H_1K_1L_1M_1C_1'$ contains twin asymmetric τ-periodic solutions of the S-type $(C_1F_1, G_1H_1, K_1L_1, M_1C_1')$, the D-type (H_1K_1), and the I-type (F_1G_1, L_1M_1). At two T_1-type bifurcation points H_1 and K_1, a jump must occur. From four I_1-type bifurcation points F_1, G_1, L_1, and M_1, bypass branches of twin asymmetric 2τ-periodic solutions emanate.

The bypass branch $F_1f_1g_1G_1$ contains twin asymmetric 2τ-periodic solutions of the S-type (F_1f_1, g_1G_1) and the I-type (f_1g_1). From two I_1-type bifurcation points f_1 and g_1, a bypass branch $f_1f_1'g_1'g_1$ of the twin asymmetric 4τ-periodic solutions emanates. In a like manner, bypass branches of the twin asymmetric $2^k\tau$-periodic solutions $(k=1,2,\cdots)$ must follow in order. A series of the I_1-type bifurcation points $F_1, f_1, f_1',\cdots$ tends to a limiting point f_1^∞ which is followed by a chaotic branch $f_1^\infty f_1^J$. The branch $f_1^\infty f_1^J$ contains the so-called chaotic solutions, but some twin asymmetric $(2k+1)\tau$-solutions $(k=1,2,\cdots)$ are observed in the windows which are narrow frequency ranges scattered in the chaotic region. At the point f_1^J, which is the extreme point of the chaotic region, a jump must occur with a change in ω. A jump of this type is not contained in the classification stated before.

The state of the system must change frequently in its character with continuous and slow change in ω, and therefore many jumps must occur. Further, there exist branches of stable states such as $H_1G_1g_1g_1'g_1^\infty g_1^J$ and $D_1C_1'M_1m_1m_1'm_1^\infty m_1^J$ to which no jump can occur unless some violent disturbance is applied.

IV. Conclusion

The true frequency response of nonlinear feedback control systems could be surprisingly complicated relative to the approximate one determined by use of the describing-function method. Many bifurcations with and without jumps could appear, and the system might sometimes be brought into a chaotic state. It should be kept in mind that a simple nonlinear feedback control system might behave in a quite strange manner subject to some ill-starred disturbances.

Acknowledgment

The authors are greatly indebted to R. Hayami for performing tedious numerical calculations.

References

[1] J. E. Gibson, *Nonlinear Automatic Control*. New York: McGraw-Hill, 1963.
[2] H. Kawakami and K. Kobayashi, "Periodic oscillations and their bifurcations in nonlinear circuits with periodically forcing terms" (in Japanese), *Trans. Inst. Electron. Commun. Eng. Japan*, vol. J63-A, pp. 753–760, Nov. 1980.
[3] N. Levinson, "Transformation theory of non-linear differential equations of the second order," *Ann. Math.*, vol. 45, pp. 723–737, Oct. 1944.
[4] A. Fukuma and M. Matsubara, "Jump resonance in nonlinear feedback systems—Part I: Approximate analysis by the describing-function method," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 891–896, Oct. 1978.

Linear Compensator Design for Bounded Input–Bounded Output Stability of Nonlinear Systems

WU-SHENG LU AND K. S. P. KUMAR

Abstract—**A method of determining a cascade compensator in a nonlinear feedback system to provide BIBO stability is discussed. The procedure is based on the Popov locus. Two examples are included to illustrate the method.**

I. Introduction

It is of considerable interest to engineers to design nonlinear systems such that the output remains bounded and to be able to prescribe the bound on the output. For nonlinear feedback systems, various sufficient conditions exist for BIBO stability [1]–[4]. These tests, however, are not adequate by themselves if the designer wants to prescribe tolerable bounds on the output. Moreover, in many of these criteria, it is necessary to put a bound on the derivative of the input to prove bounded output. This proves to be a limitation if noise exists in the input channel.

This paper describes a simple method for choosing a cascade attenuator or a first-order stable compensator so that the output of the compensated system is confined to a prescribed region without placing any requirement of boundedness on the derivative of the input.

II. The Effect of a Cascade Compensator

Consider a nonlinear feedback system (Fig. 1) composed of a linear time-invariant plant with a proper transfer function $G(s)$ which has all its poles in the left half of the s plane, a nonlinear element N characterized by a piecewise continuous function $\Phi(\cdot)$ defined on $(-\infty,\infty)$ such that $0\leqslant\Phi(\sigma)/\sigma\leqslant k<\infty$, $\forall\sigma\neq 0$ and $\Phi(0)=0$. In Fig. 1, $z_0(t)$ is the zero input response of the linear plant. Also, assume that there exists some real $q\geqslant 0$ such that

$$\operatorname{Re}(1+jq\omega)G(j\omega)+\frac{1}{k}\geqslant\delta>0, \qquad \forall\omega\in R^+. \tag{2.1}$$

If $G(s)$ is strictly proper, then (2.1) yields

$$\operatorname{Re}G(j\omega)+qv_1+\frac{1}{k}\geqslant\delta>0$$

Manuscript received November 2, 1982. This work was supported in part by the National Science Foundation under Grant NSF/INT 8120366.

The authors are with the Department of Electrical Engineering, University of Minnesota, Minneapolis, MN 55455.

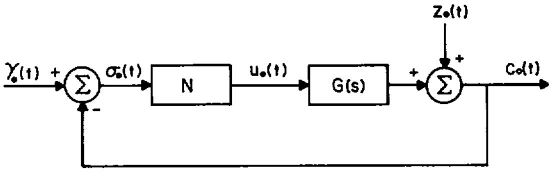

Fig. 1. Nonlinear feedback system.

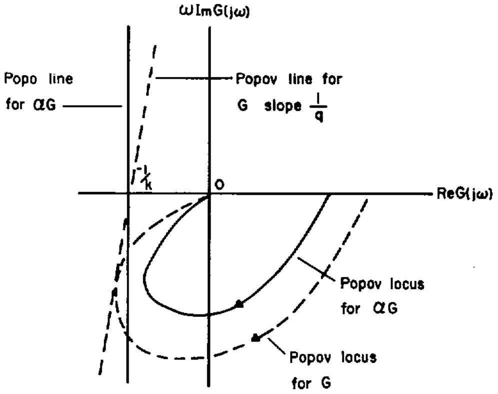

Fig. 2. Popov locus.

where

$$\nu_1 = \max_\omega |\omega \operatorname{Im} G(j\omega)| \tag{2.2}$$

or

$$\operatorname{Re} \alpha G(j\omega) + \frac{1}{k} \geqslant \alpha\delta > 0 \tag{2.3}$$

where

$$\alpha = \frac{1}{1 + kq\nu_1}. \tag{2.4}$$

Graphically, (2.1) means that there exists a Popov line that passes through the point $(-(1/k),0)$ and has a slope $1/q$ such that the Popov locus for $G(j\omega)$ lies strictly to the right of it, while (2.3) implies that the Popov locus for $\alpha G(j\omega)$ will shrink so that the corresponding Popov line can be replaced by a perpendicular one passing through the same point $(-(1/k),0)$ (see Fig. 2).

In case $G(s)$ is not strictly proper, we note that, with positive β_1 and p to be determined later,

$$\begin{aligned} &\operatorname{Re}\left[(1+jq\omega)\frac{\beta_1 G(j\omega)}{j\omega+p}\right] + \frac{1}{k} \\ &= \frac{\beta_1 p}{\omega^2+p^2}\operatorname{Re}(1+jq\omega)G(j\omega) + \frac{\beta_1\omega}{\omega^2+p^2}\operatorname{Im}(1+jq\omega)G(j\omega) + \frac{1}{k} \\ &\geqslant \beta_1\left[\frac{p\left(\delta-\frac{1}{k}\right)}{\omega^2+p^2} - \nu_2\right] + \frac{1}{k} \\ &\geqslant -\beta_1\left(\frac{\left|\delta-\frac{1}{k}\right|}{p} + \nu_2\right) + \frac{1}{k} \end{aligned}$$

where

$$\nu_2 = \max_\omega \left|\frac{\omega \operatorname{Im}(1+jq\omega)g(j\omega)}{\omega^2+p^2}\right|. \tag{2.5}$$

Thus, if

$$\beta_1 < \frac{p}{|\delta k - 1| + kp\nu_2},$$

then

$$\operatorname{Re}\left[(1+jq\omega)\frac{\beta_1 G(j\omega)}{j\omega+p}\right] + \frac{1}{k} \geqslant \delta_1 > 0 \tag{2.6}$$

where

$$\delta_1 = \beta_1 - \frac{p}{|\delta k - 1| + kp\nu_2}.$$

Further,

$$\operatorname{Re}\left[(1+jq\omega)\frac{\beta_1 G(j\omega)}{j\omega+p}\right] \leqslant \operatorname{Re}\frac{\beta_1 G(j\omega)}{j\omega+p} + \beta_1 q\nu_3 \tag{2.7}$$

where

$$\nu_3 = \max_\omega \left|\operatorname{Im}\frac{\omega G(j\omega)}{j\omega+p}\right|.$$

Equations (2.6) and (2.7) thus lead to

$$\operatorname{Re}\frac{\beta G(j\omega)}{j\omega+p} + \frac{1}{k} \geqslant \beta_2\delta_1 > 0$$

where

$$\beta_2 = \frac{1}{1+kq\nu_3} \tag{2.8}$$

and

$$\beta = \beta_1\beta_2. \tag{2.9}$$

As a summary, we have the following.

Lemma: For the nonlinear system given above, there exists a feedforward compensator $B(s)$ defined by

$$B(s) = \begin{cases} \alpha, & \text{for } G(s) \text{ strictly proper} \\ \dfrac{\beta}{s+p}, & \text{for } G(s) \text{ not strictly proper} \end{cases} \tag{2.10}$$

such that

$$\operatorname{Re} W(j\omega) + \frac{1}{k} \geqslant \delta_2 > 0, \qquad \forall \omega \in R^+ \tag{2.11}$$

where

$$W(s) = B(s)G(s) \tag{2.12}$$

and

$$\delta_2 = \begin{cases} \alpha\delta, & \text{for } G(s) \text{ strictly proper} \\ \beta_2\delta_1, & \text{for } G(s) \text{ not strictly proper.} \end{cases} \tag{2.13}$$

III. An Estimate of the Bound of the Output

Bergen, Iwens, and Rault [1] proved that for the system shown in Fig. 1, the following inequality holds for sufficiently small $\eta > 0$:

$$\begin{aligned} &\left[\int_0^t e^{2\eta\tau}u_0^2(\tau)\,d\tau\right]^{1/2} \\ &\quad\leqslant \left[\frac{1}{\delta^2}\int_0^t e^{2\eta\tau}[\gamma_0(\tau) - z_0(\tau) + q(\dot\gamma_0(\tau) - \dot z_0(\tau))]^2\,d\tau \right. \\ &\quad\quad \left. + \frac{2q}{\delta}\int_0^{\sigma(0)}\Phi(\sigma)\,d\sigma\right]^{1/2}, \qquad \forall t \geqslant 0. \end{aligned} \tag{3.1}$$

Thus, by the lemma, for the compensated system with $B(S)$ given by (2.10) shown in Fig. 3, (3.1) becomes

$$\left[\int_0^t e^{2\eta\tau}u^2(\tau)\,d\tau\right]^{1/2} \leqslant \frac{1}{\delta_2}\left[\int_0^t e^{2\eta\tau}[\gamma(\tau) - z(\tau)]^2\,d\tau\right]^{1/2}, \qquad \forall t \geqslant 0. \tag{3.2}$$

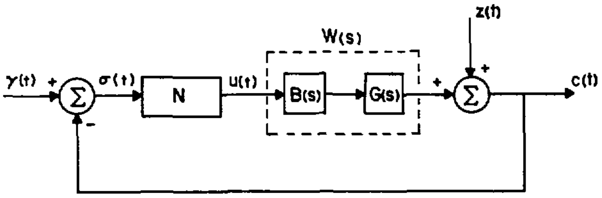

Fig. 3. Example 4.1.

In addition, by checking the proof of the main lemma in [1], it may be observed that the real η in (3.2) could be any positive constant.

Inequality (3.2) implies that no bounds are required on the input derivative for BIBO stability of the compensated system. In fact, denoting the impulse response corresponding to $W(s)$ by $h(t)$ and using (3.2), we have

$$\begin{aligned} |c(t)| &\leqslant |z(t)| + \int_0^t |h(t-\tau)u(\tau)|\,d\tau \\ &\leqslant |z(t)| + e^{-\eta t}\left[\int_0^t e^{2\eta\tau}h^2(\tau)\,d\tau \int_0^t e^{2\eta\tau}u^2(\tau)\,d\tau\right]^{1/2} \\ &\leqslant |z(t)| + \frac{e^{-\eta t}}{\delta_2}\left[\int_0^t e^{2\eta\tau}h^2(\tau)\,d\tau \int_0^t e^{2\eta\tau}[\gamma(\tau)-z(\tau)]^2\,d\tau\right]^{1/2} \\ &\leqslant |z(t)| + \frac{\sqrt{2}\,e^{-\eta t}}{\delta_2}\left[\int_0^t e^{2n\tau}h^2(\tau)\,d\tau \int_0^t e^{2\eta\tau}\left[\gamma^2(\tau)+z^2(\tau)\right]d\tau\right]^{1/2} \end{aligned} \tag{3.3}$$

where $z(t)$ is the zero input response of $W(s)$.

Now let $p_1,\cdots,p_n$ be the poles of $G(S)$ and define

$$\eta_0 = \max_i |\mathrm{Re}\,p_i|. \tag{3.4}$$

In (2.10), we choose

$$p > \eta_0 \tag{3.5}$$

so that there exist constants K_1, K_2, and η_1 such that

$$|h(t)| \leqslant K_1 e^{-\eta_1 t} \tag{3.6}$$

and

$$|z(t)| \leqslant K_2 e^{-\eta_1 t}. \tag{3.7}$$

Also, suppose that the input is bounded, i.e., there exists a constant b such that

$$|\gamma(t)| \leqslant b. \tag{3.8}$$

Thus, choosing $\eta = \eta_1/2$, (3.3) gives

$$\begin{aligned} |c(t)| &\leqslant K_2 e^{-\eta_1 t} + \frac{\sqrt{2}\,K_1}{\delta_2}\left[e^{-\eta_1 t}\int_0^t e^{-\eta_1\tau}\,d\tau \right. \\ &\qquad \left.\cdot \int_0^t \left[b^2 e^{\eta_1\tau} + K_2^2 e^{-\eta_1\tau}\right]d\tau\right]^{1/2} \\ &\leqslant K_2 e^{-\eta_1 t} + \frac{\sqrt{2}\,K_1 b}{\eta_1\delta_2}\left(1 + \frac{K_2^2 e^{-\eta_1 t}}{b^2}\right)^{1/2} \end{aligned} \tag{3.9}$$

which means that the output has an asymptotic bound $\sqrt{2}\,K_1 b/\eta_1\delta_2$, i.e., $c(t) \to \sqrt{2}\,K_1 b/\eta_1\delta_2$ as $t \to \infty$. Moreover, we note that if one replaces α or β in $B(s)$ defined by (2.10) by $\rho\alpha$ or $\rho\beta$ with $\rho \leqslant 1$, respectively, the resulting output $c(t)$ would have an asymptotic bound $\sqrt{2}\,\rho K_1 b/\eta_1\delta_2$. This implies that, for a bounded input $\gamma(t)$, the designer can confine the output to a prescribed region by inserting a feedforward compensator $\bar{B}(s)$ defined as

$$\bar{B}(s) = \rho B(s) \tag{3.10}$$

with an appropriate $\rho \leqslant 1$. We thus have proved the following.

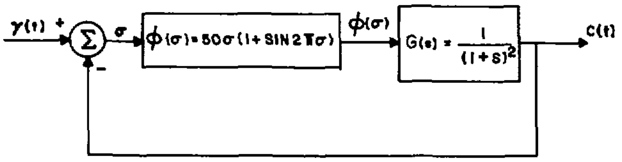

Fig. 4. Example 4.2.

Theorem: For the nonlinear system described in Section II and shown in Fig. 1, a simple linear compensator $B(s)$ given by (2.10) will cause the output to have an asymptotic bound $(\sqrt{2}\,K_1/\eta_1\delta_2)b$. Moreover, an extra attenuator factor ρ causes the output to have an asymptotic bound $(\sqrt{2}\,K_1\rho/\eta_1\delta_2)b$ so that the output can asymptotically be constrained to a desired region by adjusting the factor ρ.

Given a desired output bound M, the following steps for determining a compensator $\bar{B}(s)$ are followed.

1) Depending on if $G(s)$ is strictly proper or not, constants ν_1 and α or constants ν_2, ν_3, β_1, β_2, β, and δ_1 are chosen. Then, by (2.13), δ_2 is calculated.

2) Choose the constants η_0, η_1, ρ, and K_1.

3) Take

$$\rho = \begin{cases} 1, & \text{if } M \geqslant \dfrac{\sqrt{2}\,K_1}{\eta_1\delta_2}b \\[2ex] \dfrac{M\eta_1\delta_2}{\sqrt{2}\,K_1 b}, & \text{if } M < \dfrac{\sqrt{2}\,K_1}{\eta_1\delta_2}b. \end{cases}$$

IV. Illustrative Examples

Example 4.1: Consider a system with a nonlinear element $\Phi(\sigma) = u(t) = 0.6(\sigma(t) + \sigma(t)\cos\sigma(t))$ and a linear plant $G(s) = (1-s)/(1+s)^2$. The input is $\gamma(t) = 0.5\sin\omega_0 t$. We would like to put a feedforward compensator $\bar{B}(s) \equiv \rho B(s) = \rho\alpha$ in the system such that the output has an asymptotic bound 4.

Note that $0 \leqslant \phi(\sigma)/\sigma \leqslant 1.2$. It can be verified that

$$\mathrm{Re}\left(1 + j\frac{\omega}{3}\right)G(j\omega) + \frac{5}{6} \geqslant \frac{1}{3} > 0,$$

which means that $q = 1/3$ and $\delta = 1/3$. A straightforward calculation leads to $\nu_1 = 1$, $\alpha = 0.6$, $\eta_0 = 1$, $\delta_2 = 0.2$, and $h(t) = 0.6(2t-1)e^{-t}$, so $|h(t) \leqslant 0.72e^{-0.5t}$, i.e., $K_1 = 0.72$ and $\eta_1 = 0.5$; hence, $\rho = 4/5.1$. Thus, an attenuator $\bar{B}(s) = \rho\alpha \approx 0.47$ will constrain the output to be in the region $|c(t)| \leqslant 4$ for sufficiently large t.

Example 4.2: Baker and Bergen [2] have given an example (see Fig. 4) which indicated that the output might be unbounded even though Popov inequality (2.1) was satisfied and the input was bounded. It means that the boundedness of the first derivative of the input is necessary for the boundedness of the output. By the analysis given in Sections II and III, however, it is possible to put a compensator in the system such that the output is bounded without the requirement of boundedness of the first derivative of the input.

To do this, note that $\Phi \in [0,100]$, i.e., $k = 100$ and

$$\mathrm{Re}\,\frac{1+j\omega}{(1+j\omega)^2} + \frac{1}{100} \geqslant \frac{1}{100} > 0,$$

which implies that $q = 1$ and $\delta = 1/100$. Also, we have $\nu_1 = 2$ and $\alpha = 0.0049$. Note that $h(t) = 0.0049te^{-t}$ so that $|h(t)| \leqslant 0.0037e^{-0.5t}$, i.e., $K_1 = 0.0037$, $\eta_1 = 0.5$; thus, $(\sqrt{2}\,K_1/\eta_1\delta_2)b \approx 214$ provided $|\gamma(t)| \leqslant 1$. Therefore, with an attenuator $B(s) = 0.0049$, the output will be bounded and $|c(t)| \leqslant 214$ for sufficiently large t.

V. Conclusion

If the linear plant $G(s)$ satisfies the Popov inequality (2.1) and N is a sector nonlinearity $[0, k]$, it is possible to get some very simple feedforward compensator (it may be an attenuator or a first-order stable compensator according to whether $G(s)$ is strictly proper or not) to constrain the output in a specified region as long as the input is bounded. The compensator parameters can be determined easily.

REFERENCES

[1] A. R. Bergen, R. P. Iwens, and A. J. Rault, "On input-output stability of nonlinear feedback systems," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 742–744, Oct. 1966.
[2] R. A. Baker and A. R. Bergen, "Remark on the conditions for bounded output," *IEEE Trans. Automat. Contr.*, vol. AC-12, p. 224, Apr. 1967.
[3] I. W. Sandberg, "Some stability result related to those of V. M. Popov," *Bell Syst. Tech. J.*, vol. 44, pp. 2133–2148, Nov. 1965.
[4] P. Varaiya and R. Liu, "Bounded-input, bounded-output stability of the problem of Lur'e," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 745–746, Oct. 1966.

Simultaneous Stabilization of Nonlinear Systems

C. A. DESOER AND C. A. LIN

Abstract—**We study the problem of simultaneous stabilization of a given set of *nonlinear* plants by *one* nonlinear, not necessarily stable compensator. We obtain a necessary and sufficient condition under which there is a single compensator which stabilizes a given set of n *nonlinear* plants. This note emphasizes the importance of the following problem: when is a nonlinear unstable plant stabilizable by an incrementally stable compensator.**

I. INTRODUCTION

The problem of simultaneous stabilization of a given set of plants by one compensator arises frequently in practice, due to plant uncertainty, plant variation (failure modes, etc.), or plants with several modes of operation. Therefore, it is of interest to know the conditions under which there exists a solution to this problem.

For the linear case, Saeks and Murray [3] obtained a necessary and sufficient condition which guarantees simultaneous stabilization of a given set of *linear* plants by one *linear* compensator. Vidyasagar and Viswanadham [4] showed that the problem of simultaneously stabilizing n linear plants by a linear compensator is equivalent to the problem of simultaneously stabilizing $n-1$ linear plants by a *stable linear compensator*.

In this paper, we study the problem of simultaneous stabilization of a given set of *nonlinear* plants by one nonlinear, not necessarily stable compensator. We obtain a necessary and sufficient condition under which there is a single (nonlinear) compensator which stabilizes a given set of n *nonlinear* plants. The problem of two-step compensation of nonlinear unstable plants is treated in [7].

II. DEFINITIONS AND NOTATIONS

Let $(\mathscr{L}, \|\cdot\|)$ be a normed space of "time functions" $\mathrm{T} \to V$ where T is the time-set (typically $\mathbb{R}_+$ or $\mathbb{N}$), V is a normed space (typically $\mathbb{R}, \mathbb{R}^n, \mathbb{C}^n, \cdots$), and $\|\cdot\|$ is the chosen norm on $\mathscr{L}$. Let $\mathscr{L}_e$ be the corresponding extended space [6], [1], [5]. A nonlinear causal *map* P: $\prod_{j=1}^m \mathscr{L}_e^{n_j} \to \prod_{k=1}^l \mathscr{L}_e^{m_k}$ is said to be *finite-gain* (*f.g.*) *stable* iff $\exists \gamma(P) < \infty$ s.t. $\forall T > 0$, $\forall (u_1, u_2, \cdots, u_m) \in \prod_{j=1}^m \mathscr{L}_e^{n_j}$,

$$\|P(u_1, u_2, \cdots, u_m)\|_T \leqslant \gamma(P)(\|u_1\|_T + \|u_2\|_T + \cdots + u_m\|_T).$$

We shall use repeatedly the fact that the sum and the composition of f.g. stable maps are f.g. stable.

P is said to be *incrementally* (*inc.*) *stable* iff

1) P is f.g. stable, 2) $\exists \tilde{\gamma}(P) < \infty$ s.t. $\forall T > 0$

$$\forall (u_1, u_2, \cdots, u_m), (\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_m) \in \prod_{j=1}^m \mathscr{L}_e^{n_j},$$

$$\|P(u_1, u_2, \cdots, u_m) - P(\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_m)\|_T \leqslant \tilde{\gamma}(P)(\|u_1 - \bar{u}_1\|_T + \|u_2 - \bar{u}_2\|_T + \cdots + \|u_m - \bar{u}_m\|_T).$$

Manuscript received December 7, 1982. This work was supported by the National Science Foundation under Grant ECS-8119763 and the Joint Services Electronics Program under Contract F49620-79-C-0178.

The authors are with the Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, CA 94720.

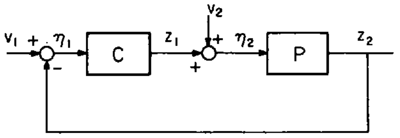

Fig. 1. Shows the system ${}^1S(P, C)$.

A nonlinear *system* N with input $(u_1, \cdots, u_m) \in \prod_{j=1}^m \mathscr{L}_e^{n_j}$ and output $(z_1, z_2, \cdots, z_l) \in \prod_{k=1}^l \mathscr{L}_e^{m_k}$ is said to be *f.g. stable* iff $\exists \gamma(N) < \infty$ s.t. $\forall T > 0$, $\forall (u_1, u_2, \cdots, u_m) \in \prod_{j=1}^m \mathscr{L}_e^{n_j}$ for any corresponding output $(z_1, z_2, \cdots, z_l) \in \prod_{k=1}^l \mathscr{L}_e^{m_k}$,

$$\|z_1\|_T + \|z_2\|_T + \cdots + \|z_l\|_T \leqslant \gamma(N)(\|u_1\|_T + \|u_2\|_T + \cdots + \|u_m\|_T).$$

We say that a feedback system is *well formed* iff the relation between the inputs of interest and the outputs of interest is a well-defined causal map between suitable extended spaces. The f.g. stability of a well-formed feedback system is equivalent to the f.g. stability of its input–output map. We *assume* throughout that each system under consideration is well-formed. System ${}^1S(P, C)$ is defined by Fig. 1: its inputs, outputs, and "errors" are (v_1, v_2), (z_1, z_2), and (η_1, η_2), respectively.

III. MAIN RESULTS

The main result of this note is a theorem. A simplified version of the theorem can be described as follows. Consider two nonlinear plants described by nonlinear causal input–output maps P_1 and P_2, where P_1 *is inc. stable*, the theorem shows that there exists a compensator C which stabilizes both P_1 and P_2 (i.e., the systems ${}^1S(P_1, C)$ and ${}^1S(P_2, C)$ are f.g. stable) if and only if there exists an *f.g. stable* Q such that the system ${}^1S(P_2 - P_1, Q)$ is f.g. stable. The theorem is preceded by a reduction lemma which is used to replace the condition that P_1 be inc. stable by the condition that P_1 be stabilizable by an *inc. stable compensator*.

Lemma 1: Let $\overline{P}_i$: $\mathscr{L}_e^{n_i} \to \mathscr{L}_e^{n_o}$ and C, F: $\mathscr{L}_e^{n_o} \to \mathscr{L}_e^{n_i}$ be nonlinear causal maps. Let $P_i := \overline{P}_i(I - F(-\overline{P}_i))^{-1}$. Under these conditions, assuming that F is inc. stable,

$${}^1S(\overline{P}_i, C + F) \text{ is f.g. stable} \Leftrightarrow {}^1S(P_i, C) \text{ is f.g. stable.}$$

Comments: 1) None of the maps $\overline{P}_i$, P_i, and C are required to be stable. 2) Contrary to some popular arguments based on block diagram manipulations, it is a fact that F must be inc. stable. Consider the following example. Let $\overline{P}_i = (s-1)/(s+3) =: \bar{n}/\bar{d}$, $F = 3/(s-1)$, and $C = 3/1 =: n_c/d_c$. By calculation, $C + F = 3s/(s-1) =: n_{c+f}/d_{c+f}$ and $P_i = \overline{P}_i(1 - F(-\overline{P}_i))^{-1} = (s-1)/(s+6) =: n/d$. The system ${}^1S(P_i, C)$ is stable, since its characteristic polynomial is $nn_c + dd_c = 4s + 3$. However, the system ${}^1S(\overline{P}_i, C + F)$ is unstable since its characteristic polynomial is $\bar{n}n_{c+f} + \bar{d}d_{c+f} = (s-1)(4s+3)$.

Proof: ($\Rightarrow$) Consider the system ${}^1S(P_i, C)$ shown in Fig. 2, write the equations defining $\tilde{e}_1$ and $\tilde{e}_2$.

$$\tilde{e}_1 = u_1 - \overline{P}_i \tilde{e}_2 \tag{1}$$

$$\tilde{e}_2 = u_2 + C\tilde{e}_1 + F(-\overline{P}_i \tilde{e}_2) = u_2 + C\tilde{e}_1 + F(\tilde{e}_1 - u_1). \tag{2}$$

By adding and subtracting $F\tilde{e}_1$ to (2) and rearranging terms, we have

$$\tilde{e}_2 = u_2 + F(\tilde{e}_1 - u_1) - F(\tilde{e}_1) + (C + F)\tilde{e}_1. \tag{3}$$

Let

$$\tilde{u}_1 := u_1,$$

$$\tilde{u}_2 := u_2 + F(\tilde{e}_1 - u_1) - F(\tilde{e}_1),$$

and rewrite (1) and (3) as

$$\tilde{e}_1 = \tilde{u}_1 - \overline{P}_i \tilde{e}_2 \tag{6}$$

$$\tilde{e}_2 = \tilde{u}_2 + (C + F)\tilde{e}_1. \tag{7}$$

Note that (6) and (7) describe the system ${}^1S(\overline{P}_i, C + F)$ with input $(\tilde{u}_1, \tilde{u}_2)$; hence by assumption, the map $(\tilde{u}_1, \tilde{u}_2) \mapsto (\tilde{e}_1, \tilde{e}_2)$ is f.g. stable.

0018-9286/84/0500-0455$01.00 ©1984 IEEE