from the Polytechnic Institute of Brooklyn, Brooklyn, NY, in 1955, 1956, and 1968, respectively (Westinghouse Fellow 1956, Bell Telephone Laboratories Fellow 1957–1958).

Since 1958 he has been employed by Bell Laboratories, Murray Hill, NJ, where he presently is a Member of the Mathematical Studies Department. He has been concerned with analysis of radar systems for military defense, synthesis and analysis of active and time-varying networks, with several studies of properties of nonlinear systems, and with some problems in communication theory and numerical analysis. His more recent interests have included compartmental models, the theory of digital filtering, global implicit-function theorems, and functional expansions for nonlinear systems. He is a former Vice Chairman of the IEEE Group on Circuit Theory, and a former Guest Editor of the IEEE TRANSACTIONS ON CIRCUIT THEORY Special Issue on Active and Digital Networks.

Dr. Sandberg is a member of the American Association for the Advancement of Science, Eta Kappa Nu, Sigma Xi, Tau Beta Pi, and the National Academy of Engineering.

Stability Analysis for Two-Dimensional Systems

WU-SHENG LU AND E. BRUCE LEE, FELLOW, IEEE

Abstract—**The state-space versions for several Bounded-Input Bounded-Output (BIBO) stability criteria of 2-D systems are given. Several checkable sufficient conditions are also described. For the special cases when $n=1$ or $m=1$ a criterion which is related to DeCarlo's criterion is reported. Some results on stabilizability based on the stability results are included.**

I. Introduction

CONSIDER a shift-invariant causal SISO 2-D system with transfer function

$$H(z^{-1},w^{-1})=\frac{b(z^{-1},w^{-1})}{a(z^{-1},w^{-1})}=\frac{\sum_{i=0}^{n}\sum_{j=0}^{m}b_{ij}z^{-i}w^{-j}}{\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}z^{-i}w^{-j}},$$

$$a_{00}=1 \tag{1.1}$$

where $a(z^{-1},w^{-1})$ and $b(z^{-1},w^{-1})$ are coprime and there are no nonessential singularities of the second kind [1], i.e., there are no points (z^{-1},w^{-1}) such that $b(z^{-1},w^{-1})=a(z^{-1},w^{-1})=0$. The system is said to be Bounded-Input Bounded-Output (BIBO) stable whenever a bounded input always produces a corresponding bounded output.

Rewriting (1.1) as

$$H(z^{-1},w^{-1})=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}h_{lk}z^{-l}w^{-k} \tag{1.2}$$

the BIBO stability of the system is equivalent to

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}|h_{lk}|<\infty. \tag{1.3}$$

During the last decade, several BIBO stability criteria and some sufficient conditions for instability have been obtained ([2]–[5], [16], [17] etc.,) for example, we have

C1 ([2]): System (1.1) is BIBO stable if and only if

$$a(\overline{U}^2)\triangleq a(\overline{U},\overline{U})\neq 0.$$

C2 ([3]): System (1.1) is BIBO stable if and only if

(i) $a(\overline{U},0)\neq 0$

(ii) $a(T,\overline{U})\neq 0.$

C3 ([4]): System (1.1) if BIBO stable if and only if there exist α and β such that $|\alpha|\leqslant 1$, $|\beta|=1$ and

(i) $a(\alpha,\overline{U})\neq 0$

(ii) $a(\overline{U},\beta)\neq 0$

(iii) $a(T^2)\triangleq a(T,T)\neq 0.$

C4 ([5]): System (1.1) is BIBO stable if and only if

(i) $a(z^{-1},z^{-1})\neq 0, \quad z^{-1}\in\overline{U}$

(ii) $a(T^2)\neq 0$

where U is an open unit disc, $U^2=\{(z^{-1},w^{-1}),\ |z^{-1}|<1,\ |w^{-1}|<1\}$ is the open bidisc, T is an unit circle, $T^2=\{(z^{-1},w^{-1}),\ |z^{-1}|=|w^{-1}|=1\}$, $\overline{U}$ and $\overline{U}^2$ are the closures of U and U^2, respectively. The notation $a(\overline{U},\overline{U})$ represents the value of $a(z^{-1},w^{-1})$ for any $z^{-1}\in\overline{U}$ and $w^{-1}\in\overline{U}$, and so on.

Recently, a nice unified treatment for these stability theorems has been given by Delsarte *et al.* [6]. On the other hand, since some significant state-space models for 2-D

Manuscript received February 16, 1982; revised September 14, 1982 and January 7, 1983. This work was supported in part by the National Science Foundation under Grant ECS77-09388-A02.

The authors are with the Department of Electrical Engineering, University of Minnesota, Minneapolis, MN 55455.

systems and related theoretical works have appeared (e.g., [7]–[9]), the stability analysis carried in the 2-D state space might be useful for analysis and design of 2-D systems.

In the next section, after describing the state-space version of the criteria mentioned above, some simple necessary conditions which may be used to test the instability of a 2-D system are given. In addition, two other sufficient conditions are described. In Section III, for the special cases $n=1$ or $m=1$, some checkable sufficient conditions and a criterion which is closely related to DeCarlo's criterion are reported. As an application of the stability results, we discuss the stabilization of a 2-D system by state feedback or output feedback in Section IV.

II. Some Stability Results in the 2-D State Space

Consider Roesser's model [7] for a SISO 2-D system:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Bu(i,j)$$

$$y(i,j) = C\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \tag{2.1}$$

where x^h and x^v are n-dimensional and m-dimensional vectors, respectively.

The 2-D z-transform of (2.1) gives

$$\frac{Y(z^{-1},w^{-1})}{U(z^{-1},w^{-1})} = C\begin{bmatrix} I_n - z^{-1}A_1 & -z^{-1}A_2 \\ -w^{-1}A_3 & I_m - w^{-1}A_4 \end{bmatrix}^{-1} B = \frac{C\operatorname{adj}\begin{bmatrix} I_n - z^{-1}A_1 & -z^{-1}A_2 \\ -w^{-1}A_3 & I_m - w^{-1}A_4 \end{bmatrix} B}{\begin{vmatrix} I_n - z^{-1}A_1 & -z^{-1}A_2 \\ -w^{-1}A_3 & I_m - w^{-1}A_4 \end{vmatrix}}. \tag{2.2}$$

Set

$$a(z^{-1},w^{-1}) = \begin{vmatrix} I_n - z^{-1}A_1 & -z^{-1}A_2 \\ -w^{-1}A_3 & I_m - w^{-1}A_4 \end{vmatrix}$$

$$b(z^{-1},w^{-1}) = C\operatorname{adj}\begin{bmatrix} I_n - z^{-1}A_1 & -z^{-1}A_2 \\ -w^{-1}A_3 & I_m - w^{-1}A_4 \end{bmatrix} B \tag{2.3}$$

and assume that $a(z^{-1},w^{-1})$ and $b(z^{-1},w^{-1})$ are coprime and there are no nonessential singularities of second kind, then the transfer function of this system is

$$H(z^{-1},w^{-1}) = \frac{b(z^{-1},w^{-1})}{a(z^{-1},w^{-1})}.$$

The polynomial $a(z^{-1},w^{-1})$ given in (2.3) can be rewritten as

$$\begin{aligned} a(z^{-1},w^{-1}) &= |I_n - z^{-1}A_1|\left|I_m - w^{-1}\left[A_4 + A_3(zI_n - A_1)^{-1}A_2\right]\right| \\ &= |I_m - w^{-1}A_4|\left|I_n - z^{-1}\left[A_1 + A_2(wI_m - A_4)^{-1}A_3\right]\right| \end{aligned} \tag{2.4}$$

if the involved inverses exist.

A square matrix is said to be stable, if all of its eigenvalues lie in the interior of the unit circle in the complex plane. It is now easy to see that Huang's criterion and DeCarlo's criterion lead immediately to the following results:

Theorem 2.1. The following statements are equivalent:

1) System (2.1) is BIBO stable;
2) (i) A_1 is stable,
   (ii) $A_4 + A_3(zI_n - A_1)^{-1}A_2$ with $|z|=1$ is stable;
3) (i) A_4 is stable,
   (ii) $A_1 + A_2(wI_m - A_4)^{-1}A_3$ with $|w|=1$ is stable;
4) (i) $$A \triangleq \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$ is stable,
   (ii) A_1 has no eigenvalues on the unit circle,
   (iii) $A_4 + A_3(zI_n - A_1)^{-1}A_2$ with $|z|=1$ has no eigenvalues on the unit circle;
5) (i) A is stable,
   (ii) A_4 has no eigenvalues on the unit circle,
   (iii) $A_1 + A_2(wI_m - A_4)^{-1}A_3$ with $|w|=1$ has no eigenvalues on the unit circle.

Note that 2) in theorem 2.1 has appeared in a similar fashion in theorem 1 of [10]. However, their condition (b) seems to be redundant.

To verify the condition given in Theorem 2.1, e.g., 2) (ii) and 3) (ii), is not as easy as in 1-D case. However, in the simple case $n=m=1$, conditions 2)(i) and 2)(ii) of Theorem 2.1 become

(a) $|A_1| < 1$

(b) $\max_{|z|=1}\left|A_4 + \frac{A_2A_3}{z - A_1}\right| < 1.$

Thus a straightforward calculation yields a checkable stability criterion:

Corollary 2.1: If $n=m=1$ and $A_2A_3 \neq 0$ (in case $A_2A_3 = 0$, we simply have Corollary 2.3 which follows), the System (2.1) is BIBO stable iff

(i) $|A_1| < 1$

(ii) $\max\left\{\left|A_4 + \frac{A_2A_3}{1-A_1}\right|, \left|A_4 - \frac{A_2A_3}{1+A_1}\right|\right\}.$

Corollary 2.2. The following three conditions are necessary for BIBO stability of the system (2.1):

1) A is stable;
2) A_1 is stable;
3) A_4 is stable.

This corollary might be useful for checking the instability of a 2-D system. For example, a system with

$$A = \begin{bmatrix} 0.5 & 0.25 \\ -0.25 & 1 \end{bmatrix}$$

is unstable because A_4 is unstable even though A and A_1 are stable. Similarly, the system with

$$A = \begin{bmatrix} -0.5 & 0.75 \\ 1 & 0.5 \end{bmatrix}$$

is unstable because A is unstable even though A_1 and A_4 are stable. However, we have the following:

Corollary 2.3. If $A_2 = 0$ or $A_3 = 0$, then System (2.2) is BIBO stable if and only if A_1 and A_4 are stable.

On the other hand, the conditions given in Corollary 2.2 are not sufficient in general case. In fact, we have

Example 2.1.

$$A = \begin{bmatrix} -0.4 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$$

has eigenvalues $\lambda_1 = \lambda_2 = 0.2$, so that A is stable and so are A_1 and A_4. However, we have

$$A_4 + A_3(z - A_1)^{-1} A_2 = 0.8 - \frac{0.36}{z + 0.4}$$

which has value 1.4 at $z = -1$. Hence the system is unstable.

Therefore, it seems interesting to explore how far Corollary 2.2 is from implying BIBO stability of the system.

Now consider a nonsingular transformation for the state variable in (2.1)

$$\begin{bmatrix} \hat{x}^h(i,j) \\ \hat{x}^v(i,j) \end{bmatrix} \triangleq \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \triangleq T \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \tag{2.5}$$

which brings (2.1) to

$$\begin{bmatrix} \hat{x}^h(i+1,j) \\ \hat{x}^v(i,j+1) \end{bmatrix} = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix} \begin{bmatrix} \hat{x}^h(i,j) \\ \hat{x}^v(i,j) \end{bmatrix} + \hat{B}u(i,j)$$

$$\triangleq \hat{A} \begin{bmatrix} \hat{x}^h(i,j) \\ \hat{x}^v(i,j) \end{bmatrix} + \hat{B}u(i,j)$$

$$y(i,j) = \hat{C} \begin{bmatrix} \hat{x}^h(i,j) \\ \hat{x}^v(i,j) \end{bmatrix} \tag{2.6}$$

where $\hat{A} = TAT^{-1}$, $\hat{B} = TB$, and $\hat{C} = CT^{-1}$.

Note that $a(z^{-1}, w^{-1})$ in (2.3) is invariant under the transformation (2.5) and so is each term in (2.4). This fact leads to the following sufficient condition for BIBO stability of the system:

Theorem 2.2. The conditions

(i) A_1 and A_4 are stable,

(ii) A_1 and A_4 are diagonalizable and the transformation in (2.5) is chosen such that $\hat{A}_1$ and $\hat{A}_4$ in (2.6) are diagonal,

(iii) $\|\hat{A}_2\| \|\hat{A}_3\| < (1 - e^*)(1 - s^*)$ (2.7)

will guarantee the BIBO stability of system (2.1), where

$$e^* = \max_{1 \leqslant i \leqslant n} |\lambda(A_1)|, \quad s^* = \max_{1 \leqslant i \leqslant m} |\lambda(A_4)|$$

$\|\cdot\|$ is the induced norm defined as

$$\|A\| = \max_{1 \leqslant i \leqslant n} [\lambda_i(A^T A)]^{1/2}. \tag{2.8}$$

Proof: By 2) of Theorem 2.1 it is sufficient to show that the norm of $\hat{A}_4 + \hat{A}_3(zI_n - \hat{A}_1)^{-1}\hat{A}_2$ with $|z| = 1$ is less than one. Indeed, we can take

$$\hat{A}_1 = \text{diag}[e_1, e_2, \cdots, e_n]$$
$$\hat{A}_4 = \text{diag}[s_1, s_2, \cdots, s_m]$$

so that $\|\hat{A}_4\| = s^*$ and

$$\|(zI_n - A_1)^{-1}\| = \left\| \text{diag}\left[(z - e_1)^{-1}, \cdots, (z - e_n)^{-1}\right] \right\| \leqslant \frac{1}{1 - e^*}.$$

We thus have

$$\|\hat{A}_4 + \hat{A}_3(zI_n - \hat{A}_1)^{-1}\hat{A}_2\| \leqslant s^* + \frac{\|\hat{A}_2\| \|\hat{A}_3\|}{1 - e^*} < 1$$

The last inequality is due to (2.7). Q.E.D.

A square matrix N is said to be normal if $N^T N = N N^T$. As is well known, a normal matrix can be diagonalized by using an orthonormal coordinate transformation. We thus have

Corollary 2.4. Assume that

(i) A_1 and A_4 are stable,

(ii) A_1 and A_4 are normal,

(iii) $\|A_2\| \|A_3\| < (1 - e^*)(1 - s^*)$

then the system (2.1) is BIBO stable.

Proof: We note that for any orthonormal matrix the induced norm defined in (2.8) is equal to one.

Suppose the submatrices T_1 and T_2 in (2.5) are two orthonormal matrices such that $\hat{A}_1$ and $\hat{A}_2$ are diagonal matrices, then

$$\|\hat{A}_2\| = \|T_1 A_2 T_2^T\| \leqslant \|T_1\| \|A_2\| \|T_2^T\| = \|A_2\|$$
$$\|\hat{A}_3\| = \|T_2 A_3 T_1^T\| \leqslant \|T_2\| \|A_3\| \|T_1^T\| = \|A_3\|.$$

We thus have

$$\|\hat{A}_4 + \hat{A}_3(zI_n - \hat{A}_1)^{-1}\hat{A}_1\| \leqslant s^* + \frac{\|A_2\| \|A_3\|}{1 - e^*} < 1.$$

Q.E.D.

From this observation, the restriction for the norms of A_2 and A_3 would lead to stability of the system. However, the magnitude of $\|A_2\| \|A_3\|$ seems not yet to be an essential thing. For instance, using 2) of Theorem 2.1 we can check that a system with

$$A = \left[\begin{array}{cc:c} 0.5 & 0 & 1 \\ 0 & 0.5 & 5 \\ \hdashline -9.97 & 2 & 0.7 \end{array}\right]$$

is BIBO stable, but

$$\|A_2\| \|A_3\| \approx 51.9 \gg (1 - e^*)(1 - s^*) = 0.15.$$

Furthermore, we note that the eigenvalues of A are 0.4, 0.5, and 0.8 which are quite near the eigenvalues of A_1 and A_4, respectively, even though $\|A_2\| \|A_3\|$ is quite large. With this object in view, recall that fact that in case $A_2 = 0$ or $A_3 = 0$ the spectra of A is just the union of the spectra of A_1 and the spectra of A_4 and then the stabilities of A_1 and A_4 are a characteristic for the BIBO stability of the system. Therefore, the deviation of the spectra of A from the union of the spectra of A_1 and A_4 because of the existence of A_2 and A_3 may be an essential one for the BIBO stability of a 2-D system. In the next section, we consider special cases of this.

III. Further Analysis

By 4) of Theorem 2.1, the system is BIBO stable if and only if A and A_4 are stable and

$$\left|wI_m - \left[A_4 + A_3(zI_n - A_1)^{-1}A_2\right]\right| \neq 0 \quad \text{with } |w| = |z| = 1. \tag{3.1}$$

Rewriting (3.1) as

$$\left|(w-z)I_m + \left\{zI_m - \left[A_4 + A_3(zI_n - A_1)^{-1}A_2\right]\right\}\right| \neq 0 \quad \text{with } |w| = |z| = 1 \tag{3.2}$$

and noting

$$\left|zI_m - \left[A_4 + A_3(zI_n - A_1)^{-1}A_2\right]\right| = \frac{\begin{vmatrix} zI_n - A_1 & -A_2 \\ -A_3 & zI_m - A_4 \end{vmatrix}}{|zI_n - A_1|} = \frac{\prod_{i=1}^{n+m}(z-\gamma_i)}{\prod_{i=1}^{n}(z-\alpha_i)} \tag{3.3}$$

where $\{\gamma_i,\ 1 \leqslant i \leqslant n+m\}$ and $\{\alpha_i,\ 1 \leqslant i \leqslant n\}$ are the eigenvalues of A and A_1, respectively, we can express condition (3.2), for the special case $m=1$, in the following way:

$$z - \frac{\prod_{i=1}^{n+1}(z-\gamma_i)}{\prod_{i=1}^{n}(z-\alpha_i)} \neq w \quad \text{with } |w| = |z| = 1 \tag{3.4}$$

which means that the complex variable on the left side of (3.4) is not on the unit circle. Therefore, condition (3.2) is equivalent to

$$\left| z - \frac{\prod_{i=1}^{n+1}(z-\gamma_i)}{\prod_{i=1}^{n}(z-\alpha_i)} \right| \neq 1 \quad \text{with } |z| = 1 \tag{3.5}$$

where $|\cdot|$ represents the modulus of the involved complex variable. We thus have

Theorem 3.1

1) If $m=1$, system (2.1) is BIBO stable if and only if
   (i) A and A_1 are stable,
   (ii)
$$\xi \triangleq \left| z - \frac{\prod_{i=1}^{n+1}(z-\gamma_i)}{\prod_{i=1}^{n}(z-\alpha_i)} \right| \neq 1 \quad \text{with } |z| = 1$$

or,

2) If $n=1$, system (2.1) is BIBO stable if and only if
   (i) A and A_4 are stable,
   (ii)
$$\eta \triangleq \left| w - \frac{\prod_{i=1}^{m+1}(w-\gamma_i)}{\prod_{i=1}^{m}(w-\beta_i)} \right| \neq 1 \quad \text{with } |w| = 1$$

where $\{\beta_i,\ 1 \leqslant i \leqslant m\}$ are the eigenvalues of A_4.

It should be noted that if $A_2 = 0$ or $A_3 = 0$ in (2.1), then for $m=1$ we have $\gamma_i = \alpha_i$ $(i = 1,2,\cdots,n)$ and $\gamma_{n+1} = \beta$, which implies the equivalence of the condition (ii) in Theorem 3.1, 1), and the stability of A_4. Also, for $n=1$, we have $\gamma_i = \beta_i$ $(i = 1,2,\cdots,m)$ and $\gamma_{m+1} = \alpha_1$ which means the equivalence of the condition 2)(ii) in Theorem 3.1, and the stability of A_1.

Now, we denote the unit circle in a complex plane by $e^{j\theta}$ $(0 \leqslant \theta \leqslant 2\pi)$, and set

$$\xi^* = \max_{0 \leqslant \theta \leqslant 2\pi} \left| e^{j\theta} - \frac{\prod_{j=1}^{n+1}(e^{j\theta}-\gamma_i)}{\prod_{i=1}^{n}(e^{j\theta}-\alpha_i)} \right|$$

$$\xi_* = \min_{0 \leqslant \theta \leqslant 2\pi} \left| e^{j\theta} - \frac{\prod_{i=1}^{n+1}(e^{j\theta}-\gamma_i)}{\prod_{i=1}^{n}(e^{j\theta}-\alpha_i)} \right|$$

$$\eta^* = \max_{0 \leqslant \theta \leqslant 2\pi} \left| e^{j\theta} - \frac{\prod_{i=1}^{m+1}(e^{j\theta}-\gamma_i)}{\prod_{i=1}^{m}(e^{j\theta}-\beta_i)} \right|$$

$$\eta_* = \min_{0 \leqslant \theta \leqslant 2\pi} \left| e^{j\theta} - \frac{\prod_{i=1}^{m+1}(e^{j\theta}-\gamma_i)}{\prod_{i=1}^{m}(e^{j\theta}-\beta_i)} \right|.$$

These extremes can be found, at least theoretically, with ordinary calculus because the real-valued functions of the real variable ξ and η are continuously differentiable on a closed interval. Using these extrema the previous theorem can be restated in the following way:

Corollary 3.1.

1) If $m=1$, system (2.1) is BIBO stable if and only if
   (i) A and A_1 are stable,
   (ii) $\xi_* > 1$, or, $\xi^* < 1$,

or,

2) If $n=1$, system (2.1) is BIBO stable if and only if
   (i) A and A_4 are stable,
   (ii) $\eta_* > 1$, or, $\eta^* < 1$.

It is, however, difficult to determine these extremes, especially in case of large n or m. In order to obtain checkable conditions, we rearrange the functions ξ and η as follows:

$$\xi = \left| z - \frac{\prod_{i=1}^{n+1}(z-\gamma_i)}{\prod_{i=1}^{n}(z-\alpha_i)} \right| = \frac{\left| z\prod_{i=1}^{n}(z-\alpha_i) - \prod_{i=1}^{n+1}(z-\gamma_i) \right|}{\left| \prod_{i=1}^{n}(z-\alpha_i) \right|}$$

$$= |a| \frac{\prod_{i=1}^{k}|z-\delta_i|}{\prod_{i=1}^{n'}|z-\alpha_i|}$$

and

$$\eta=\left|w-\frac{\prod_{i=1}^{m+1}(w-\gamma_i)}{\prod_{i=1}^{m}(w-\beta_i)}\right|=|b|\frac{\prod_{i=1}^{l}|w-\tau_i|}{\prod_{i=1}^{m'}|w-\beta_i|}$$

where $a\neq 0$, $b\neq 0$, $\{\delta_i,\ 1\leqslant i\leqslant k\}$ and $\{\tau_i,\ 1\leqslant i\leqslant l\}$ are some complex constants and $k\leqslant n'\leqslant n$, $l\leqslant m'\leqslant m$.

For any specified arrangement of the sets $\{\alpha_i\}$, $\{\beta_i\}$, $\{\delta_i\}$, $\{\tau_i\}$, we always have

$$\begin{aligned}\xi^*&=\max_{0\leqslant\theta\leqslant 2\pi}|a|\frac{\prod_{i=1}^{k}|e^{j\theta}-\delta_i|}{\prod_{i=1}^{n'}|e^{j\theta}-\alpha_i|}\\&\leqslant|a|\left[\prod_{i=1}^{k}\max_{0\leqslant\theta\leqslant 2\pi}\left|\frac{e^{j\theta}-\delta_i}{e^{i\theta}-\alpha_i}\right|\prod_{i=k+1}^{n'}\max_{0\leqslant\theta\leqslant 2\pi}\frac{1}{|e^{i\theta}-\alpha_i|}\right]\\&=|a|\left[\prod_{i=1}^{k}\max_{0\leqslant\theta\leqslant 2\pi}\left|\frac{e^{j\theta}-\delta_i}{e^{j\theta}-\alpha_i}\right|\prod_{i=k+1}^{n'}\frac{1}{1-|\alpha_i|}\right]\end{aligned}$$

$$\begin{aligned}\xi_*&=\min_{0\leqslant\theta\leqslant 2\pi}|a|\frac{\prod_{i=1}^{k}|e^{j\theta}-\delta_i|}{\prod_{i=1}^{n'}|e^{j\theta}-\alpha_i|}\\&\geqslant|a|\left[\prod_{i=1}^{k}\min_{0\leqslant\theta\leqslant 2\pi}\left|\frac{e^{j\theta}-\delta_i}{e^{j\theta}-\alpha_i}\right|\prod_{i=k+1}^{n'}\min_{0\leqslant\theta\leqslant 2\pi}\frac{1}{|e^{j\theta}-\alpha_i|}\right]\\&=|a|\left[\prod_{i=1}^{k}\min_{0\leqslant\theta\leqslant 2\pi}\left|\frac{e^{j\theta}-\delta_i}{e^{j\theta}-\alpha_i}\right|\prod_{i=k+1}^{n'}\frac{1}{1+|\alpha_i|}\right]\end{aligned}$$

and in a similar fashion relations can be found for η^* and η_*. We note that for the function, for example,

$$\left|\frac{e^{j\theta}-\delta_i}{e^{j\theta}-\alpha_i}\right|$$

its extrema in the closed interval $[0,2\pi]$ can be determined easily. We also note that these relations hold for any ordered set $\{\alpha_i\}$, $\{\beta_i\}$, $\{\delta_i\}$, and $\{\tau_i\}$. Therefore, it is reasonable to define

$$\xi^{**}=\min_{\text{ordered }\{\alpha_i\}}\left[\prod_{i=1}^{k}\max_{\theta}\left|\frac{e^{j\theta}-\delta_i}{e^{j\theta}-\alpha_i}\right|\prod_{i=k+1}^{n'}\frac{1}{1-|\alpha_i|}\right]\tag{3.6}$$

$$\xi_{**}=\max_{\text{ordered }\{\alpha_i\}}\left[\prod_{i=1}^{k}\min_{\theta}\left|\frac{e^{j\theta}-\delta_i}{e^{j\theta}-\alpha_i}\right|\prod_{i=k+1}^{n'}\frac{1}{1+|\alpha_i|}\right]\tag{3.7}$$

$$\eta^{**}=\min_{\text{ordered }\{\beta_i\}}\left[\prod_{i=1}^{l}\max_{\theta}\left|\frac{e^{j\theta}-\tau_i}{e^{j\theta}-\beta_i}\right|\prod_{i=l+1}^{m'}\frac{1}{1-|\beta_i|}\right]\tag{3.8}$$

$$\eta_{**}=\max_{\text{ordered }\{\beta_i\}}\left[\prod_{i=1}^{l}\min_{\theta}\left|\frac{e^{j\theta}-\tau_i}{e^{j\theta}-\beta_i}\right|\prod_{i=l+1}^{m'}\frac{1}{1+|\beta_i|}\right].\tag{3.9}$$

We now have the following obvious relations:

$$\xi^{**}\geqslant\frac{\xi^*}{|a|}\qquad\xi_{**}\leqslant\frac{\xi_*}{|a|}$$

$$\eta^{**}\geqslant\frac{\eta^*}{|b|}\quad\text{and}\quad\eta_{**}\leqslant\frac{\eta_*}{|b|}$$

which together with Corollary 3.1 lead to a checkable sufficient condition for the special cases $m=1$ or $n=1$ as follows:

Corollary 3.2.

1) If $m=1$, system (2.1) is BIBO stable if
   (i) A and A_1 are stable,
   (ii) $\xi^{**}<1/|a|$, or $\xi_{**}>1/|a|$, where ξ^{**} and ξ_{**} are given by (3.6) and (3.7), respectively, or,

2) If $n=1$, system (2.1) is BIBO stable if
   (i) A and A_4 are stable,
   (ii) $\eta^{**}<1/|b|$, or $\eta_{**}>1/|b|$ where η^{**} and η_{**} are given by (3.8) and (3.9), respectively.

Example 3.1.

Consider a 2-D system with

$$A=\left[\begin{array}{cc:c}0.5&0&1\\0&0.5&5\\\hdashline -9.97&2&0.7\end{array}\right]$$

As can be seen from the results at the end of Section II, this system is BIBO stable. Now we use the results given in this section to verify this fact.

First of all, we know that the spectra of A and A_1 are $\{0.4, 0.5, 0.8\}$ and $\{0.5, 0.5\}$, respectively, which implies the stability of A and A_1. Moreover, we have

$$\begin{aligned}\xi&=\left|z-\frac{(z-0.4)(z-0.5)(z-0.8)}{(z-0.5)^2}\right|\\&=0.7\left|\frac{z-0.46}{z-0.5}\right|\leqslant 0.7+0.7\frac{0.04}{|z-0.5|}\\&\leqslant 0.7+0.7\frac{0.04}{0.5}=0.756\end{aligned}$$

which implies $\xi^*<1$ and Corollary 3.1 leads to the BIBO stability of this system.

IV. Stabilization of 2-D Systems by State Feedback or Output Feedback

For an unstable 2-D system (2.1), one may ask if there exists a suitable state feedback

$$\begin{aligned}u(i,j)&=K\begin{bmatrix}x^h(i,j)\\x^v(i,j)\end{bmatrix}\\&=[K_1\quad K_2]\begin{bmatrix}x^h(i,j)\\x^v(i,j)\end{bmatrix}\end{aligned}\tag{4.1}$$

where K, K_1, K_2 are $p\times(n+m)$, $p\times n$, $p\times m$ matrices, respectively, such that the closed-loop 2-D system

$$\begin{bmatrix}x^h(i+1,j)\\x^v(i,j+1)\end{bmatrix}=(A+BK)\begin{bmatrix}x^h(i,j)\\x^v(i,j)\end{bmatrix}\tag{4.2}$$

is BIBO stable where

$$B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

and

$$A+BK=\begin{bmatrix} A_1+B_1K_1 & A_2+B_1K_2 \\ A_3+B_2K_1 & A_4+B_2K_2 \end{bmatrix}. \tag{4.3}$$

We call the system (2.1) stabilizable by state feedback whenever such a matrix K exists.

Similarly, one may pose the question of whether there exists a suitable output feedback

$$\begin{aligned} u(i,j) &= Ky(i,j) \\ &= KC\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \\ &= [KC_1 \quad KC_2]\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \end{aligned} \tag{4.4}$$

where K, C_1, C_2 are $p \times r$, $r \times n$, $r \times m$ matrices, respectively, such that the resulting closed-loop 2-D system

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1+B_1KC_1 & A_2+B_1KC_2 \\ A_3+B_2KC_1 & A_4+B_2KC_2 \end{bmatrix}\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \tag{4.5}$$

is BIBO stable.

The system is said to be stabilizable by output feedback whenever such a matrix K exists.

As an application of Corollary 2.2, we have

Theorem 4.1. The following three conditions are necessary for stabilizing a 2-D system by state feedback:

(S1) (A, B) as an 1-D system is stabilizable by state feedback,

(S2) (A_1, B_1) is stabilizable by state feedback,

(S3) (A_4, B_2) is stabilizable by state feedback.

Similarly, the following three conditions are necessary for stabilizing a 2-D system by output feedback:

(01) (A, B, C) as an 1-D system is stabilizable by output feedback,

(02) (A_1, B_1, C_1) is stabilizable by output feedback,

(03) (A_4, B_2, C_2) is stabilizable by output feedback.

This theorem provides some possibilities to verify the unstabilizability by using the corresponding results from the 1-D system theory.

Furthermore, Corollary 2.3 is related to the stabilization issue, which has partly been covered by a recent paper (Hinamoto *et al.* [11]).

Theorem 4.2. The system (2.1) is stabilizable by state feedback (4.1) if either there exists a $p \times n$ matrix K_1 such that $A_3+B_2K_1=0$ with $A_1+B_1K_1$ stable and (A_4, B_2) is stabilizable by state feedback or, there exists a $p \times m$ matrix K_2 such that $A_2+B_1K_2=0$ with $A_4+B_2K_2$ stable and (A_1, B_1) is stabilizable by state feedback. Similarly, the system (2.1) is stabilizable by output feedback (4.4) if either there exists a $p \times r$ matrix K such that $A_2+B_1KC_2=0$ with $A_1+B_1KC_1$ and $A_4+B_2KC_2$ stable, or there exists a $p \times r$ matrix K such that $A_3+B_2KC_1=0$ with $A_1+B_1KC_1$ and $A_4+B_2KC_2$ stable.

A discussion of these sufficient conditions as given in Theorem 4.2 for stabilizing a 2-D system by state feedback can be found in Hinamoto *et al.* [11]. Here we note that a necessary and sufficient condition for the equation $B_1KC_2 = -A_2$ to have a solution is that (see Rao and Mitra [12, ch. 2])

$$B_1B_1^+A_2C_2^+C_2=A_2 \tag{4.6}$$

where B_1^+ and C_2^+ are the generalized inverses of B_1 and C_2, respectively, and then the general solution is

$$K=-B_1^+A_2C_2^+ + \Gamma_1 - B_1^+B_1\Gamma_1C_2C_2^+ \tag{4.7}$$

where Γ_1 is an arbitrary $p \times r$ matrix. Thus $A_1+B_1KC_1$ and $A_4+B_2KC_2$ become $(A_1-B_1B_1^+A_2C_2^+C_1)+B_1\Gamma_1(I_r-C_2C_2^+)C_1$ and $(A_4-B_2B_1^+A_2C_2^+C_2)+B_2(I_p-B_1^+B_1)\Gamma_1C_2$, respectively. A similar analysis may be carried out for the equation $A_3+B_2KC_1=0$. We thus have

Corollary 4.1. The system (2.1) is stabilizable by output feedback (4.4) if, either, (4.6) holds and the systems $(A_1 - B_1B_1^+A_2C_2^+C_1,\ B_1,\ (I_\gamma - C_2C_2^+)C_1)$ and $(A_4 - B_2 B_1^+A_2C_2^+C_2,\ B_2(I_p-B_1^+B_1),\ C_2)$ can be stabilized by same output feedback matrix Γ_1, then the desired feedback matrix K in (4.4) could be obtained from (4.7); or,

$$B_2B_2^+A_3C_1^+C_1=A_3 \tag{4.8}$$

and the system $(A_1-B_1B_2^+A_3C_1^+C_1,\ B_1(I_p-B_2^+B_2),\ C_1)$, and $(A_4-B_2B_2^+A_3C_1^+C_2,\ B_2,\ (I_r-C_1C_1^+)C_2)$ can be stabilized by same output feedback matrix Γ_2, then the desired feedback matrix K in (4.4) could be taken as

$$K=-B_2^+A_3C_1^+ + \Gamma_2 - B_2^+B_2\Gamma_2C_1C_1^+. \tag{4.9}$$

Concerning the stabilization of a 2-D system, it should be noted that Theorem 2.1 is also useful. For instance, 2) and 3) of Theorem 2.1 lead to

Corollary 4.2. The system (2.1) is stabilizable by state feedback if and only if there exist two matrices K_1 and K_2 such that either

1) (i) $A_1+B_1K_1$ is stable,
   (ii) $A_4+B_2K_2+(A_3+B_2K_1)[zI_n-(A_1+B_1K_1)]^{-1}(A_2+B_1K_2)$ with $|z^{-1}| \leqslant 1$ is stable; or

2) (i) $A_4+B_2K_2$ is stable,
   (ii) $A_1+B_1K_1+(A_2+B_1k_2)[wI_m-(A_4+B_2K_2)]^{-1}(A_3+B_2K_1)$ with $|w^{-1}| \leqslant 1$ is stable.

Thus stabilizing a 2-D system can be reduced to stabilizing a 1-D constant system and then stabilizing another 1-D system with a complex parameter as well. Some recent work which is closely related to stabilizing a linear system depending on parameters have appeared (see, e.g., [13]). In this paper, however, we would like to use another point of view.

To do this, rewrite condition 1)(ii) in Corollary 4.2 as

(ii) $F_1(z^{-1})+G_1(z^{-1})K_2$ with $|z^{-1}| \leqslant 1$ is stable where

$$F_1(z^{-1})=A_4+P(z^{-1})A_2 \tag{4.10}$$

$$G_1(z^{-1})=B_2+p(z^{-1})B_1 \tag{4.11}$$

and

$$P(z^{-1}) = \frac{z^{-1}(A_3 + B_2 K_1)\,\mathrm{adj}\left[I_n - z^{-1}(A_1 + B_1 K_1)\right]}{\det\left[I_n - z^{-1}(A_1 + B_1 K_1)\right]}.$$

Let

$$R = \left\{ \frac{b(z^{-1})}{a(z^{-1})} \middle| a(z^{-1}) \neq 0, \quad \text{for } |z^{-1}| \leq 1 \right\}$$

where $a(z^{-1})$ and $b(z^{-1})$ are polynomials in z^{-1} with real coefficients, then R is a principal integral domain [14], and for $F \in R^{m\times m}$, $G \in R^{m\times p}$, if (F, G) is R-reachable, i.e., every $x \in R^m$ is an R-linear combination of the columns of $G, FG, \cdots, F^{m-1}G$, then for every $p_1, \cdots, p_m \in R$, there exists $K_2 \in R^{p\times m}$ such that

$$\det\left[wI_m - (F + GK_2)\right] = \prod_{i=1}^{m} (w - p_i) \quad [14].$$

These observations and Corollary 4.1 lead to

Theorem 4.3

1) The 2-D system (2.1) will be stabilizable by state feedback with the gain matrix $K = (K_1, K_2(z^{-1}))$ where K_1 is a constant $p \times n$ matrix and $K_2 \in R^{p\times m}$, if

(i) (A_1, B_1) is stabilizable by constant state feedback,

(ii) For some K_1 stabilizing (A_1, B_1), the pair (F_1, G_1) defined in (4.10) and (4.11) are R-reachable.

Similarly,

2) The system (2.1) will also be stabilizable by state feedback with $K = (K_1(w^{-1}), K_2)$ where $K_1(w^{-1}) \in R^{p\times n}$ and K_2 constant matrix, if

(i) (A_4, B_2) is stabilizable by constant state feedback,

(ii) For some K_2 stabilizing (A_4, B_2) the pair (F_2, G_2) are R-reachable, where

$$F_2(w^{-1}) = A_1 + Q(w^{-1})A_3$$

$$G_2(w^{-1}) = B_1 + Q(w^{-1})B_2$$

$$Q(w^{-1}) = \frac{w^{-1}(A_2 + B_1 K_2)\,\mathrm{Adj}\left[I_m - w^{-1}(A_4 + B_2 K_2)\right]}{\det\left[I_m - w^{-1}(A_4 + B_2 K_2)\right]}.$$

V. Conclusions

The stability considerations for the 2-D systems in the state-space version presented in this paper indicate that in general the stability of A, A_1, and A_4 is not sufficient to guarantee BIBO stability. The deviation or distance between the spectra set of A and the union of the spectra set of A_1 and A_4 because of the appearance of A_2 and A_3 might be important. In the special case $m = 1$ or $n = 1$, the real-valued functions ξ and η defined in Section III may be viewed as such a measurement.

Moreover, the results obtained in Section IV show that stabilizing a 2-D system can be reduced to considering the same question for a 1-D constant subsystem and then stabilizing a 1-D system depending on a parameter (which makes it possible to use some recent results in algebraic system theory [15]).

References

[1] D. Goodman, "Some stability properties of two-dimensional linear shift-invariant digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-24, pp. 201–208, 1977.

[2] J. L. Shanks, S. Treitel, and J. H. Justice, "Stability and synthesis of two-dimensional recursive filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-20, pp. 115–128, 1972.

[3] T. S. Huang, "Stability of two-dimensional recursive filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-20, pp. 158–163, 1972.

[4] M. G. Strintzis, "Tests of stability of multidimensional filters," *IEEE Trans. Circuits Syst.*, vol. CAS-24, pp. 432–437, 1977.

[5] R. A. DeCarlo, J. Murray, and R. Saeks, "Multivariable Nyquist theory," *Int. J. Contr.*, vol. 25, pp. 657–675, 1977.

[6] Ph. Delsarte, Y. V. Genin, and Y. G. Kamp," A simple proof of Rudin's Multivariable Stability Theorem," *IEEE Acoust. Speech, Signal Processing*, vol. ASSP-28, pp. 701–704, 1980.

[7] R. P. Roesser, "A discrete state-space model for linear image processing," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 1–10, 1975.

[8] M. Morf, B. C. Lévy, and S. Y. Kung, "New results in 2-D systems theory, Part I: 2-D polynomial matrices, factorization, and coprimeness," *Proc. IEEE*, vol. 65, pp. 861–872, 1977.

[9] S. Y. Kung, B. C. Lévy, M. Morf, and T. Kailath, "New results in 2-D systems theory, Part II: 2-D state-space model—realization and the notions of controllability, observability, and minimality," *Proc. IEEE*, vol. 65, pp. 945–961, 1977.

[10] F. M. Boland and D. H. Owens, "Linear multipass processes: a two-dimensional interpretation," *Proc. Inst. Elect. Eng.*, vol. 127, pt. D., 189–193, 1980.

[11] T. Hinamoto, F. W. Fairman, and J. Shimonishi, "Stabilization of 2D filter using 2D observers," *Int. J. Syst. Sci.*, 1982.

[12] C. R. Rao and S. K. Mitra, *Generalized Inverse of Matrices and Its Applications*. New York: Wiley, 1971.

[13] M. L. J. Hautus and E. D. Sontag, "An approach to detectability and observer," Dept. of Math, Memo COSOR 70-08, Eindhoven Univ. of Tech., Eindhoven, The Netherlands, 1980.

[14] E. D. Sontag, "Linear Systems over Commutative rings: a survey," Richerche di Automatica, 7, 1–34, 1976.

[15] E. B. Lee, S. H. Zak, and S. D. Brierley, "Stabilization of Generalized Linear Systems via the Algebraic Riccati Equation," submitted to IEEE Trans. on Automat. Contr. (1982)

[16] A. F. Humes and E. I. Jury, "Stability tests for two dimensional linear multivariable digital filters," *Int. J. Control*, vol. 26, 225–234, 1977.

[17] P. Agathoklis and Mansour, "Sufficient conditions for instability of two and higher dimensional discrete systems," *IEEE Trans. Circuits Syst.*, vol. CAS-29, 486–488, 1982.

Wu-Sheng Lu received the equivalent of the B.S. and M.S. degrees, both in mathematics, from Fudan University and East China Normal University, Shanghai, People's Republic of China, in 1964 and 1980, respectively. He is now a Ph.D. candidate in the Control Science and Dynamical Systems Center, University of Minnesota.

Concurrently he works as both a Teaching Assistant and a Research Assistant. His present research is in the area of multidimensional system theory, optimal control, and the numerical aspects of linear system theory.

E. Bruce Lee (M'77–SM'82–F'83) received the B.S. and M.S. degrees in mechanical engineering from the University of North Dakota and the Ph.D. degree from the University of Minnesota in 1960.

From 1956 to 1963 he was employed by Honeywell Inc., to develop digital inertial sensors and control system synthesis techniques. Since 1963 he has been employed by the University of Minnesota, where he is currently Professor of Electrical Engineering and Associate Director of the Center for Control Sciences and Dynamical Systems.

Dr. Lee is a member of Sigma Xi and SIAM.